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A Paley-Wiener theorem for the inverse Fourier transform on some homogeneous spaces

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Abstract. We formulate and prove a version of Paley-Wiener theorem for the inverse Fourier transforms on noncompact Riemannian symmetric spaces and Heisenberg groups. The main ingredient in the proof is the Gutzmer's formula.

1. Introduction

The classical Paley-Wiener theorem for the Euclidean Fourier transform characterises compactly supported functions on $\mathbf{R}^n$ in terms of holomorphic properties of their Fourier transforms. Analogues of Paley-Wiener theorem have been proved in the context of Fourier transforms on Lie groups. One such result is a theorem of Gangolli [5] for the spherical Fourier transform on noncompact Riemannian symmetric spaces. However, there are no satisfactory results available in certain cases. One such example is the case of the Fourier transform on the Heisenberg group $\mathbf{H}^n$. Here the Fourier transform is operator valued, parametrised by non-zero real numbers. When a function f on $\mathbf{H}^n$ is compactly supported it is not possible to extend the Fourier transform $\hat{f}(\lambda)$ as an operator valued entire function. There are some versions of Paley-Wiener theorem for the Heisenberg group which treat the central and non-central variables separately, see [1] and [12]. The situation of general nilpotent Lie groups is much more difficult.

In 2000, Pasquale [10] considered the problem of characterising functions on a non-compact symmetric space $X = G/K$ whose spherical Fourier transforms are compactly supported. When G is a complex semisimple Lie group or of rank one she showed that K-biinvariant functions whose spherical Fourier transforms are compactly supported can be extended to the complexification of X as meromorphic functions leading to a Paley-Wiener theorem for the inverse spherical Fourier transform. One of the main results of this paper is such a

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theorem for Helgason Fourier transform of general functions on X. The main ingredient in the proof (which also motivates the formulation) is Gutzmer's formula proved by Faraut [3] for functions holomorphic in a domain, called the complex crown, contained in the complexification of X. Our result is similar in spirit to the characterisation of the image of the heat kernel transform studied by Krötz et al [9].

Instead of treating $\mathbf{H}^n$ as a nilpotent Lie group we may consider it as a homogeneous space of a bigger group, namely the Heisenberg motion group G_n which is the semidirect product of $\mathbf{H}^n$ with the unitary group $U(n)$. Thus $\mathbf{H}^n = G_n/U(n)$ and we treat functions on $\mathbf{H}^n$ as right $U(n)$-invariant functions on G_n. With this view point the Fourier transform of f on $\mathbf{H}^n$ is considered as a function of two variables λ and k. Here λ is a nonzero real number and k is a non-negative integer. For each such pair there is a unitary representation of G_n denoted by ρ_k^λ and we consider the operators $\rho_k^\lambda(f)$ as parametrised by the point $(\lambda, (2k+n)|\lambda|)$ from the Heisenberg fan which is the spectrum of the sublaplacian. Our Paley-Wiener theorem for the Heisenberg group characterises functions for which $\rho_k^\lambda(f)$ is supported in $|\lambda| \le a$ and $(2k+n)|\lambda| \le b^2$. The proof requires an analogue of Gutzmer's formula for the Heisenberg motion group which has been proved recently [15].

The theorem of Krötz et al [9] and our Paley-Wiener theorem both involve a certain peseudo-differential shift operator D. As was shown in [9] the operator is inevitable in characterising the image of the heat kernel transform. When the group G is complex the operator D is simple (multiplication by a Jacobian factor) but otherwise it is quite complicated. Interestingly enough our Paley-Wiener theorem for the Heisenberg group also involves a similar operator $\mathscr{D}$. The operator D has the effect of replacing the elementary spherical function φ_λ by the Weyl symmetrised exponential ψ_λ. The same is true of $\mathscr{D}$. It, in effect, changes the Laguerre functions φ_k^λ into Bessel functions. Without these operators, certain orbital integrals are not entire functions of exponential type.

It is worthwhile to see how our version of Paley-Wiener theorem looks like for the Euclidean Fourier transform. Let f be a Schwartz class function on $\mathbf{R}^n$ and consider the Fourier transform $\hat{f}$. When $\hat{f}$ is supported in $|\xi| \le a$ f extends to $\mathbf{C}^n$ as an entire function. Let $G = M(n)$ be the Euclidean motion group acting on $\mathbf{R}^n$ which has a natural extension to $\mathbf{C}^n$. Then it is easy to see that the following Gutzmer's formula is valid:

$$\int_G |f(g.z)|^2 dg = c_n \int_0^\infty \int_{S^{n-1}} |\hat{f}(\lambda\omega)|^2 \varphi_\lambda(2iy)\lambda^{n-1}\, d\omega d\lambda$$

where $\varphi_\lambda(iy) = (\lambda|y|)^{-n/2+1} J_{n/2-1}(i\lambda|y|)$ and $z = x+iy$. From the above it is clear that the orbital integral $\int_G |f(g.z)|^2 dg$ satisfies the estimate

$$\int_G |f(g.z)|^2 dg \le Ce^{2a|y|}.$$

Conversely, if a Schwartz function f extends to $\mathbf{C}^n$ as an entire function and the orbital integral satisfies the above estimate then $\hat{f}$ is supported in $|\xi| \le a$. This follows easily from the Gutzmer's formula.

The plan of the paper is as follows. In the next section we treat the Helgason Fourier transform on non-compact Riemannian symmetric spaces. In Section 3 we consider the inverse Fourier transform on the Heisenberg group.

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2. Noncompact Riemannian symmetric spaces

In this section we formulate and prove a Paley-Wiener theorem for the inverse Helgason Fourier transform on a Riemannian symmetric space of non-compact type. We follow the standard notations; in fact we closely follow Krötz et al in setting up the notation and we refer to the same for any undefined term. Let $X = G/K$ be a homogeneous space where G is a semisimple Lie group and K a maximal compact subgroup. Considering the Iwasawa decomposition $G = KAN$ we let M be the centraliser of A in K. We define $B = K/M$ and consider the Helgason Fourier transform

$$\hat{f}(\lambda, b) = \int_X f(x) e^{(-\lambda+\rho, A(x,b))}\, dx$$

where $\lambda \in i\mathbf{a}^*$ and $b \in B$. Some remarks about the notation are in order. We let $\mathbf{a}$ stand for the Lie algebra of the group A. The Iwasawa decomposition $G = KAN$ leads to the map $g \to H(g)$ which is determined by the requirement that $g = k \exp H(g) n$ with $k \in K$ and $n \in N$. We then define $A(g) = -H(g^{-1})$ and $A(x,b) = A(k^{-1}g)$ if $x = gK$ and $b = kM$. The inversion formula valid for suitable functions reads as follows:

$$f(x) = w^{-1} \int_{i\mathbf{a}^*} \left(\int_B \hat{f}(\lambda, b) e^{(\lambda+\rho, A(x,b))}\, db \right) |c(\lambda)|^{-2} d\lambda.$$

Here w is the order of the Weyl group.

For every $\lambda \in \mathbf{a}_{\mathbf{C}}^*$, $b \in B$ the function $x \to e^{(\lambda+\rho, A(x,b))}$ has a holomorphic extension to a domain Ξ in the complexification $X_{\mathbf{C}}$. This domain, called the complex crown of X is defined as follows. Let $\mathbf{g}$ be the Lie algebra of G with the Cartan decomposition $\mathbf{g} = \mathbf{k} + \mathbf{p}$. Let $\mathbf{a}$ be a Cartan subspace with Σ the associated system of restricted roots. The complex crown Ξ is a G-invariant domain in $X_{\mathbf{C}} = G_{\mathbf{C}}/K_{\mathbf{C}}$ defined by $\Xi = G\exp(i\Omega).x_0$, $x_0 = eK$ where $\Omega = \{H \in \mathbf{a} : |\alpha(H)| < \frac{\pi}{2}, \alpha \in \Sigma\}$. (see Akhiezer-Gindikin [1] and Krötz-Stanton [7]). It follows that a function $f \in L^2(X)$ whose Fourier transform $\hat{f}(\lambda, b)$ has compact support admits a holomorphic extension to Ξ which is given by

$$f(z) = w^{-1}\int_{i\mathbf{a}^*}\left(\int_B \hat{f}(\lambda, b)e^{(\lambda+\rho, A(z,b))}\,db\right)|c(\lambda)|^{-2}d\lambda.$$

Let φ_λ, $\lambda \in i\mathbf{a}^*$ be the spherical functions on G given by the integral

$$\varphi_\lambda(g) = \int_B e^{(\lambda+\rho, A(x,b))}\,db.$$

In [7] Krötz-Stanton proved that for $\lambda \in i\mathbf{a}^*$ the function $H \to \varphi_\lambda(\exp iH)$ admits a holomorphic continuation to the tube domain $\mathbf{a} + 2i\Omega$. In order to state Gutzmer's formula and formulate a Paley-Wiener theorem we need to recall the definition of orbital integrals developed by Gindikin et al [6]. For a function h on Ξ suitably decreasing at the boundary and $Y \in 2\Omega$ we define

$$O_h(iY) = \int_G h\left(g\exp\left(\frac{i}{2}Y\right).x_0\right)dg.$$

Let $G(\Xi)$ be the space of all holomorphic functions on the complex crown. Then in [3] Faraut has established the following formula known as Gutzmer's formula.

THEOREM 2.1. *Let $f \in G(\Xi)$ be such that for all $H \in \Omega$,*

$$\int_G |f(g\exp(iH).x_0)|^2 dg \le M$$

for some constant M. Then for all $H \in \Omega$ we have

$$\int_G |f(g\exp(iH).x_0)|^2 dg = \int_{i\mathbf{a}^*}\left(\int_B |\hat{f}(\lambda, b)|^2 db\right)\varphi_\lambda(\exp(2iH))|c(\lambda)|^{-2}d\lambda.$$

In [9] Krötz et al have used this Gutzmer's formula to characterise the image of the heat kernel transform. Let $k_t(x)$ stand for the heat kernel associated to the Laplace-Beltrami operator on X which is a K-biinvariant function given by the integral

$$k_t(x) = \int_{i\mathbf{a}^*} e^{-t(|\lambda|^2+|\rho|^2)}\varphi_\lambda(x)|c(\lambda)|^{-2}d\lambda.$$

It is clear that k_t has a holomorphic extension to the complex crown. If $f \in L^2(X)$ the function $H_t f(x) = f * k_t(x)$ which solves the heat equation with initial condition f also extends to Ξ as a holomorphic function. Let imH_t stand for the image of the above transform, $f(x) \to H_t f(z)$ called the heat kernel transform. For the Euclidean Laplacian the corresponding image turned out to be a weighted Bergman space; the same is true for compact symmetric spaces. However, in [9] Krötz et al proved that imH_t is not a weighted Bergman space. Instead they obtained the following characterisation.

In order to state their result we need to set up some more notation. Let W be the Weyl group and consider the Weyl symmetrised exponential function

$$\psi_\lambda(Z) = \sum_{w \in W} e^{(\lambda, wZ)}, \qquad Z \in \mathbf{a}_{\mathbf{C}},\ \lambda \in i\mathbf{a}^*.$$

If a holomorphic function h on the tube domain $\mathbf{a} + 2i\Omega$ has the representation

$$h(Z) = \int_{i\mathbf{a}^*} g(\lambda)\varphi_\lambda(\exp(Z).x_0)|c(\lambda)|^{-2} d\lambda$$

then we define

$$Dh(Z) = \int_{i\mathbf{a}^*} g(\lambda)\psi_\lambda(Z)|c(\lambda)|^{-2} d\lambda.$$

Under some conditions on g this is well defined, see [9]. It is known that D is a pseudo-differential shift operator which has a simpler form when the group G is complex. It can be expressed in terms of Abel transform and Fourier multipliers. Using this operator D the following characterisation of the heat kernel transform was obtained in [9].

Theorem 2.2. *A function $F \in G(\Xi)$ belongs to imH_t if and only if*

$$\int_{\mathbf{a}} DO_{|F|^2}(iY) w_t(Y) dY < \infty$$

where w_t is given in terms of the Euclidean heat kernel as

$$w_t(Y) = |W|^{-1}(2\pi t)^{-n/2} e^{2t|\rho|^2} e^{-|Y|^2/2t}.$$

This theorem is an easy consequence of the Gutzmer's formula. Our Paley-Wiener theorem is very similar in spirit to the above theorem.

Theorem 2.3. *Let f be a function in $L^2(X)$. Then the Helgason Fourier transform $\hat{f}(\lambda, b)$ is supported in $|\lambda| \le R$ if and only if f has a holomorphic extension $F \in G(\Xi)$ which satisfies the estimate*

$$DO_{|F|^2}(iY) \le Ce^{2R|Y|}$$

for some constant C independent of Y.

Proof. First assume that $\hat{f}(\lambda, b)$ is compactly supported in $|\lambda| \leq R$. From the inversion formula for the Helgason Fourier transform it is clear that f can be holomorphically extended to Ξ. If F is the extension then by Plancherel theorem it follows that $F \in imH_t$ for all $t > 0$. Moreover, Gutzmer's formula can be applied and we get

$$DO_{|F|^2}(iY) = \int_{i\mathbf{a}^*} \left(\int_B |\hat{f}(\lambda, b)|^2 db \right) \psi_\lambda(2iY) |c(\lambda)|^{-2} d\lambda.$$

This gives the estimate,

$$DO_{|F|^2}(iY) \leq C\|f\|_2^2 e^{2R|Y|}$$

as $\hat{f}(\lambda, b)$ is supported in $|\lambda| \leq R$ and $|\psi_\lambda(iY)| \leq Ce^{|\lambda||Y|}$.

Conversely, assume that F satisfies the hypothesis of the theorem. Then it is easy to see that $F \in imH_t$ for every $t > 0$. More precisely, for every $t > 0$ we have

$$\int_{\mathbf{a}} DO_{|F|^2}(iY) w_t(Y) dY \leq Ce^{2t|\rho|^2} P(R, t) e^{2tR^2}$$

where P is some polynomial. Consider the integral

$$\begin{aligned} &\int_{|\lambda| \geq R+\varepsilon} \left(\int_B |\hat{f}(\lambda, b)|^2 db \right) |c(\lambda)|^{-2} d\lambda \\ &\qquad \leq e^{-2t(R+\varepsilon)^2} \int_{i\mathbf{a}^*} \left(\int_B |\hat{f}(\lambda, b)|^2 db \right) e^{2t|\lambda|^2} |c(\lambda)|^{-2} d\lambda. \end{aligned}$$

By the above and Gutzmer's formula, we get the estimate

$$\int_{|\lambda| \geq R+\varepsilon} \left(\int_B |\hat{f}(\lambda, b)|^2 db \right) |c(\lambda)|^{-2} d\lambda \leq Ce^{-2t(R+\varepsilon)^2} P(R, t) e^{2tR^2}.$$

By letting t tend to infinity we conclude that $\hat{f}(\lambda, b)$ vanishes almost everywhere for $|\lambda| \geq R + \varepsilon$. As ε is arbitrary $\hat{f}(\lambda, b)$ is supported in $|\lambda| \leq R$ proving the theorem.

We conclude this section with the following remarks. For each $t > 0$ the image imH_t is a Hilbert space with the norm

$$\|F\|_t^2 = \int_{\mathbf{a}} DO_{|F|^2}(iY) w_t(Y) dy.$$

As shown in [9] $\|F\|_t = \|f\|_2$ if $F = f * k_t$. Let Δ be the Laplace-Beltrami operator, taken to be non-negative so that $e^{-t\Delta} f = f * k_t$. Let us define $\mathbf{H}$ to be the intersection of all imH_t, $t > 0$. If $L_b^2(X)$ stand for the subspace of

$L^2(X)$ with compactly supported Helgason Fourier transforms then it is clear that f is the restriction of an $F \in \mathbf{H}$. The above theorem can be viewed as one characterising the image of $L_b^2(X)$ under the heat kernel transform.

3. Fourier transform on the Heisenberg group

In this section we consider the Heisenberg group as the homogeneous space $G_n/U(n)$ where G_n is the Heisenberg motion group. The general references for this section are the papers Krötz et al [8] and [15]. See also the monographs [4] and [13]. We take $\mathbf{H}^n$ to be $\mathbf{C}^n \times \mathbf{R}$ with group law $(z,t)(w,s) = \left(z+w, t+s+\frac{1}{2}\Im(z\cdot\overline{w})\right)$. More often we write (x,u,t) in place of (z,t) and the group law takes the form

$$(x,u,t)(x',u',t') = \left(x+x', u+u', t+t'+\frac{1}{2}(u\cdot x' - x\cdot u')\right)$$

where $x,u,x',u' \in \mathbf{R}^n$. For each non-zero $\lambda \in \mathbf{R}$ the Schrödinger representation π_λ of $\mathbf{H}^n$ is defined by

$$\pi_\lambda(x,u,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x\cdot\xi+(1/2)x\cdot u)}\varphi(\xi+u).$$

The group Fourier transform of $f \in L^1(\mathbf{H}^n)$ is defined by

$$\hat{f}(\lambda) = \int_{\mathbf{H}^n} f(z,t)\pi_\lambda(z,t)dzdt.$$

For inversion and Plancherel theorems see [13].

As mentioned in the introduction we would like to consider $\mathbf{H}^n$ as the homogeneous space $G_n/U(n)$ and rewrite the inversion formula in terms of certain representations of G_n. First let us recall some definitions. The unitary group $U(n)$ acts on the Heisenberg group as automorphisms, the action being defined by $\sigma(z,t) = (\sigma.z,t)$ where $\sigma \in U(n)$. The Heisenberg motion group G_n is the semidirect product of $U(n)$ and $\mathbf{H}^n$ with group law

$$(\sigma,z,t)(\tau,w,s) = (\sigma\tau,(z,t)(\sigma.w,s)).$$

Functions on $\mathbf{H}^n$ can be considered as right $U(n)$-invariant functions on G_n. As such the inversion formula for such functions on G_n will involve only certain class-one representations of G_n. We now proceed to describe the relevant representations.

Let Φ_α, $\alpha \in \mathbf{N}^n$ be the normalised Hermite functions on $\mathbf{R}^n$. Let $\Phi_\alpha^\lambda(x) = |\lambda|^{n/4}\Phi_\alpha(|\lambda|^{1/2}x)$ and define $E_{\alpha,\beta}^\lambda(z,t) = (\pi_\lambda(z,t)\Phi_\alpha^\lambda, \Phi_\beta^\lambda)$. For each $k \in \mathbf{N}$ and non-zero $\lambda \in \mathbf{R}$ let H_k^λ be the Hilbert space for which the functions $E_{\alpha,\beta}^\lambda$ with $\alpha,\beta \in \mathbf{N}^n$, $|\alpha| = k$ form an orthonormal basis. The inner product in H_k^λ is defined by

$$(F, G) = |\lambda|^n \int_{\mathbf{C}^n} F(z, 0)\overline{G(z, 0)}dz.$$

On this Hilbert space we define a representation ρ_k^λ of the Heisenberg motion group by

$$\rho_k^\lambda(\sigma, z, t)F(w, s) = F((\sigma, z, t)^{-1}(w, s)).$$

Then it is known that (see [15]) ρ_k^λ is an irreducible unitary representation of G_n. As $(G_n, U(n))$ is a Gelfand pair ρ_k^λ has a unique $U(n)$ fixed vector which is none other than the Laguerre function e_k^λ (see below).

Given $f \in L^1(\mathbf{H}^n)$ we can define its group Fourier transform by

$$\rho_k^\lambda(f) = \int_{G_n} f(z, t)\rho_k^\lambda(\sigma, z, t)d\sigma dz dt$$

which is a bounded operator acting on H_k^λ. As shown in [15] we have

$$tr(\rho_k^\lambda(\sigma, z, t)^*\rho_k^\lambda(f)) = \frac{k!(n-1)!}{(k+n-1)!} f * e_k^\lambda(z, t)$$

where $e_k^\lambda(z, t) = e^{i\lambda t}\varphi_k^\lambda(z)$. Here

$$\varphi_k^\lambda(z) = L_k^{n-1}\left(\frac{1}{2}|\lambda|\,|z|^2\right)e^{-(1/4)|z|^2}$$

and L_k^{n-1} are Laguerre polynomials of type $(n-1)$. The inversion formula for a right $U(n)-$invariant function on G_n takes the form

$$f(z, t) = (2\pi)^{-n-1}\int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty} tr(\rho_k^\lambda(\sigma, z, t)^*\rho_k^\lambda(f))\frac{(k+n-1)!}{k!(n-1)!}\right)|\lambda|^n d\lambda.$$

Also the Plancherel theorem can be written as

$$\int_{\mathbf{H}^n} |f(z, t)|^2 dz dt = \int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}\|\rho_k^\lambda(f)\|_{HS}^2\frac{(k+n-1)!}{k!(n-1)!}\right)|\lambda|^n d\mu(\lambda)$$

where $d\mu(\lambda) = (2\pi)^{-n-1}|\lambda|^n d\lambda$.

THEOREM 3.1. *For every Schwartz class function f on $\mathbf{H}^n$ the following inversion formula holds:*

$$f(z, t) = \int_{-\infty}^{\infty}\left(\sum_{k=0}^{\infty}(\rho_k^\lambda(f)e_k^\lambda, \rho_k^\lambda(1, z, t)e_k^\lambda)\right)d\mu(\lambda)$$

where 1 stands for the identity matrix in $U(n)$.

From now on let us identify $\mathbf{H}^n$ with $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ and use the notation (x, u, t) rather than $(x + iu, t)$ to denote elements of $\mathbf{H}^n$. The action of $U(n)$ on $\mathbf{H}^n$ then takes the form $\sigma.(x, u, t) = (a.x - b.u, b.x + a.u, t)$ where a and b are the real and imaginary parts of σ. This action has a natural extension to $\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$ given by $\sigma.(z, w, \zeta) = (a.z - b.w, b.z + a.w, \zeta)$. With this definition we can extend the action of G_n on $\mathbf{H}^n$ to $\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$:

$$(a + ib, x', u', t')(z, w, \zeta) = (x', u', t')(a.z - b.w, b.z + a.w, \zeta).$$

This action is then extended to functions defined on $\mathbf{C}^n \times \mathbf{C}^n \times \mathbf{C}$:

$$\rho(g)f(z, w, \zeta) = f(g^{-1}.(z, w, \zeta)), \qquad g \in G_n.$$

We are now ready to prove Gutzmer's formula for the Heisenberg group. For a function f on $\mathbf{H}^n$ we let $f^\lambda(z)$ stand for the inverse Fourier transform of $f(z, t)$ in the t-variable. Suppose f is a Schwartz class function on $\mathbf{H}^n$ such that $f^\lambda = 0$ for all $|\lambda| > A$ and $\rho_k^\lambda(f) = 0$ for all λ, k such that $(2k + n)|\lambda| > B$. We say that the Fourier transform of f is compactly supported if this condition is satisfied for some A and B. Now the inversion formula

$$f(g.(x, u, \xi)) = \int_{-A}^{A} \sum_{(2k+n)|\lambda| \le B} (\rho_k^\lambda(f)e_k^\lambda, \rho_k^\lambda(g)\rho_k^\lambda(1, x, u, \xi)e_k^\lambda)\,d\mu(\lambda)$$

is valid for any $g \in G_n$. Moreover, as each of $\rho_k^\lambda(1, x, u, \xi)e_k^\lambda$ extends to $\mathbf{C}^{2n+1}$ as an entire function the same is true of $f(g.(x, u, \xi))$ and we have

$$f(g.(z, w, \zeta)) = \int_{-A}^{A} e^{\lambda\eta} \sum_{(2k+n)|\lambda| \le B} (\rho_k^\lambda(f)e_k^\lambda, \rho_k^\lambda(g)\rho_k^\lambda(1, z, w, \xi)e_k^\lambda)\,d\mu(\lambda)$$

where $\zeta = \xi + i\eta$. We then have the following Gutzmer's formula for the action of Heisenberg motion group on $\mathbf{C}^{2n+1}$ which is the complexification of $\mathbf{H}^n$.

In the following theorem we let $F *_\lambda G$ stand for the λ-twisted convolution defined by

$$F *_\lambda G(z) = \int_{\mathbf{C}^n} F(z - w)G(w)e^{(i/2)\lambda\Im(z\cdot\bar{w})}\,dw.$$

We refer to [13] for the connection between this convolution and the group convolution on $\mathbf{H}^n$.

Theorem 3.2. *Let f be Schwartz function whose Fourier transform is compactly supported in the above sense. Then f extends to $\mathbf{C}^{2n+1}$ as an entire function and we have the following identity:*

$$\int_{G_n} |f(g.(z,w,\zeta))|^2 dg$$

$$= \int_{-\infty}^{\infty} e^{2\lambda\eta} e^{-\lambda(u\cdot y - v\cdot x)} \left(\sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv) \right) d\mu(\lambda)$$

where $\|f^\lambda *_\lambda \varphi_k^\lambda\|_2$ *is the* $L^2(\mathbf{C}^n)$ *norm of* $f^\lambda *_\lambda \varphi_k^\lambda$.

For a proof of this theorem we refer to [15] where the formula was proved under a slightly different condition. In fact, the above formula holds good as long as the right hand side expression is finite. This can be proved by means of a density argument.

We now consider the heat kernel transform on the Heisenberg group. Let $\mathscr{L}$ be the sublaplacian on the Heisenberg group and let $\Delta = \mathscr{L} - \partial_t^2$ be the full Laplacian. Let $q_t(x,u,\xi)$ be the heat kernel associated to Δ. Then its Fourier transform (in the central variable) $q_t^\lambda(x,u)$ is explicitly known, see [13]. From the expression it follows that $q_t(x,u,\xi)$ can be extended to $\mathbf{C}^{2n+1}$ as an entire function. The same is true of $f * q_t(x,u,\xi)$ for any $f \in L^2(\mathbf{H}^n)$. In [8] the authors studied the problem of characterising the image of this heat kernel transform as a space of entire functions on the complexification $\mathbf{H}^n_{\mathbf{C}}$ which is just $\mathbf{C}^{2n+1}$. They showed that the image is not a weighted Bergman space but it can be written as a direct integral of twisted Bergman spaces. They also showed that it is the direct sum of two Bergman spaces defined in terms of signed weight functions. Here using Gutzmer's formula we prove another characterisation similar to the one obtained on Riemannian symmetric spaces.

Given functions $m(k,\lambda)$ defined on $\mathbf{N} \times \mathbf{R}$ we consider functions of the form

$$h(iy, iv, i\eta) = \int_{-\infty}^{\infty} e^{\lambda\eta} \sum_{k=0}^{\infty} m(k,\lambda) \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv) d\mu(\lambda).$$

When $m(k,\lambda)$ is compactly supported in the sense that it is supported in $|\lambda| \le A$, $(2k+n)|\lambda| \le B$ the above function is well defined and extends to $\mathbf{C}^{2n+1}$ as an entire function. Let $j_{n-1}(s) = s^{-n+1} J_{n-1}(s)$ and define an operator $\mathscr{D}$ by

$$\mathscr{D}h(iy, iv, i\eta) = \int_{-\infty}^{\infty} e^{\lambda\eta} \sum_{k=0}^{\infty} m(k,\lambda) j_{n-1}(i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2}) d\mu(\lambda)$$

whenever h is given as above. Note that $\mathscr{D}h$ is also an entire function on $\mathbf{C}^{2n+1}$. Let $p_t(y,v,\xi)$ be the Euclidean heat kernel on $\mathbf{R}^{2n+1}$. Let us write

$$O_{|f|^2}(iy, iv, i\eta) = \int_{G_n} |f(g.(iy, iv, i\eta))|^2 dg$$

and call it the orbital integral of $|f|^2$. We now state and prove the following theorem on the image of the heat kernel transform.

Theorem 3.3. *An entire function F on $\mathbf{C}^{2n+1}$ belongs to the image of the heat kernel transform on $L^2(\mathbf{H}^n)$ if and only if*

$$\int_{\mathbf{R}^{2n+1}} \mathscr{D}O_{|F|^2}(iy, iv, i\eta) p_{t/2}(y, v, \eta) dy dv d\eta < \infty.$$

The above is in fact a constant multiple of the $L^2(\mathbf{H}^n)$ norm of $F(x, u, \xi)$.

Proof. Suppose $F = f * q_t$ for some $f \in L^2(\mathbf{H}^n)$. Then F extends to $\mathbf{C}^{2n+1}$ as an entire function. If $m(k, \lambda) = \|\rho_k^\lambda(F)\|_{HS} = e^{-t\lambda^2} e^{-(2k+n)|\lambda| t} \|\rho_k^\lambda(f)\|_{HS}$ then the function

$$\int_{-\infty}^{\infty} e^{2\lambda\eta} \sum_{k=0}^{\infty} |m(k, \lambda)|^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv) d\mu(\lambda)$$

and hence by Gutzmer's formula $\mathscr{D}O_{|F|^2}(iy, iv, i\eta)$ is given by

$$\int_{-\infty}^{\infty} e^{2\lambda\eta} \sum_{k=0}^{\infty} e^{-2t\lambda^2} e^{-2(2k+n)|\lambda| t} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2$$
$$\times j_{n-1}(2i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2}) d\mu(\lambda).$$

Integrating this against $p_{t/2}(y, v, \eta)$ and noting that

$$\int_{\mathbf{R}^{2n+1}} p_{t/2}(y, v, \eta) j_{n-1}(i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2}) dy dv d\eta = e^{2t\lambda^2} e^{2(2k+n)|\lambda| t}$$

we obtain

$$\int_{\mathbf{R}^{2n+1}} \mathscr{D}O_{|F|^2}(iy, iv, i\eta) p_{t/2}(y, v, \eta) dy dv d\eta = c_n \int_{\mathbf{H}^n} |f(x, u, \xi)|^2 dx du d\xi.$$

This proves one half of the theorem. The other half is proved by noting that all the steps are reversible.

We now state and prove a Paley-Wiener theorem for the inverse Fourier transform on the Heisenberg group.

Theorem 3.4. *Let $f \in L^2(\mathbf{H}^n)$. The Fourier transform $\rho_k^\lambda(f)$ of f is compactly supported in $|\lambda| \le A$, $(2k+n)|\lambda| \le B$ if and only if f has an entire extension F to $\mathbf{C}^{2n+1}$ which satisfies the estimate*

$$\mathscr{D}O_{|F|^2}(iy, iv, i\eta) \le C e^{2A|\eta|} e^{2\sqrt{B}(|y|^2 + |v|^2)^{1/2}}$$

for all $(y, v, \eta) \in \mathbf{R}^{2n+1}$.

PROOF. First assume that $\rho_k^\lambda(f)$ is compactly supported in $|\lambda| \le A$, $(2k+n)|\lambda| \le B$. As we have seen in the proof of Gutzmer's formula f extends to an entire function F and $\mathscr{D}O_{|F|^2}(iy, iv, i\eta)$ is given by

$$\int_{-A}^{A} e^{2\lambda\eta} \sum_{(2k+n)|\lambda| \le B} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 j_{n-1}(2i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2}) d\mu(\lambda).$$

As $j_{n-1}(is) \le Ce^s$ the above gives the estimate

$$\mathscr{D}O_{|F|^2}(iy, iv, i\eta) \le Ce^{2A|\eta|} e^{2\sqrt{B}(|y|^2+|v|^2)^{1/2}} \|f\|_2^2.$$

This proves the sufficiency part of the theorem.

To prove the necessity, assume that F satisfies the hypothesis of the theorem. First of all the Euclidean Paley-Wiener theorem for the central variable shows that $f^\lambda = 0$ for all $|\lambda| > A$. The hypothesis then implies that F belongs to the image of the heat kernel transform $f \to f * q_t$ for any $t > 0$ and also

$$\int_{\mathbf{R}^{2n+1}} \mathscr{D}O_{|F|^2}(iy, iv, i\eta) p_{t/2}(y, v, 0) p_{s/2}(0, 0, \eta) dy dv d\eta \le Ce^{2sA^2} e^{2tB}.$$

By Gutzmer's formula this means that

$$\int_{-\infty}^{\infty} e^{2s\lambda^2} \sum_{k=0}^{\infty} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 e^{2(2k+n)|\lambda| t}\, d\mu(\lambda) \le Ce^{2sA^2} e^{2tB}.$$

As this is true for every t proceeding as in the case of symmetric spaces we can show that $\rho_k^\lambda(f)$ is supported in $(2k+n)|\lambda| \le B$. This completes the proof of the theorem.

The following remarks on the operator $\mathscr{D}$ are in order. It has the effect of changing $\varphi_k^\lambda(iy, iv)$ into $j_{n-1}(i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2})$. Notice that φ_k^λ are the spherical functions associated to the k-th ray of the Heisenberg fan whereas j_{n-1} is the spherical function associated to the limiting ray. Moreover, from the asymptotic formula of Hilb's type for Laguerre functions (see Theorem 8.22.4 in Szegö [11]) we see that $\varphi_k^\lambda(iy, iv)$ is approximated by $j_{n-1}(i\sqrt{(2k+n)|\lambda|}(|y|^2 + |v|^2)^{1/2})$. Further study of the operator $\mathscr{D}$ is worth considering.

In view of the above remarks the above theorem is not completely satisfactory. We can prove another version of Paley-Wiener theorem if we make use of the characterisation of the image of $L^2(\mathbf{H}^n)$ under the heat kernel transform obtained in [8]. There the authors have shown that the image is the direct sum of two Bergman spaces each of which is defined in terms of certain weight function which takes both positive and negative values. To be more precise these weight functions denoted by W_t^+ and W_t^- are defined by the equations

$$\int_{-\infty}^{\infty} e^{2\lambda\eta} W_t^+(iy, iv, \eta)d\eta = e^{2t\lambda^2} p_{2t}^{\lambda}(2y, 2v)$$

for $\lambda > 0$, and a similar equation for W_t^- valid for $\lambda < 0$. In the above p_t^λ is the heat kernel associated to the special Hermite operator and given explicitly by

$$p_t^\lambda(z, w) = c_n \left(\frac{\lambda}{\sinh(\lambda t)}\right)^n e^{-(\lambda/4)\coth(\lambda t)(z^2+w^2)}.$$

The existence of such weight functions have been proved in [8].

We now have the following theorem. We consider only functions f for which f^λ is supported in $\lambda > 0$. A similar result is true for functions f for which f^λ is supported in $\lambda < 0$.

THEOREM 3.5. *Let $f \in L^2(\mathbf{H}^n)$ be such that f^λ is supported in $\lambda > 0$. If $\rho_k^\lambda(f)$ is compactly supported then f extends to $\mathbf{C}^{2n+1}$ as an entire function $F(z, w, \zeta)$ which is of exponential type in the last variable and satisfies the following condition: for some constants $B, C > 0$ we have*

$$\left|\int_{-\infty}^{\infty} O_{|F|^2}(iy, iv, i\eta) W_t^+(2iy, 2iv, i\eta)d\eta\right| \leq Ct^{-2n}e^{2tB}$$

for all $t > 0$. Conversely, if $F(z, w, \zeta)$ is entire, of exponential type in ζ and satisfies the slightly stronger estimate

$$\left|\int_{-\infty}^{\infty} O_{|F|^2}(iy, iv, i\eta) W_t^+(iy, iv, i\eta)d\eta\right| \leq C_N t^{-2n}(1 + |y|^2 + |v|^2)^{-N} e^{2tB}$$

for some $N > n$ and for all $t > 0$ then $\rho_k^\lambda(f)$ is compactly supported.

PROOF. If $\rho_k^\lambda(f)$ is supported in $0 < \lambda \leq \alpha$ and $(2k + n)|\lambda| \leq \beta$ then by Gutzmers' formula and the defining relation for W_t^+ we have

$$\int_{-\infty}^{\infty} O_{|F|^2}(iy, iv, i\eta) W_t^+(2iy, 2iv, i\eta)d\eta$$
$$= \int_0^\alpha e^{2t\lambda^2} \sum_{(2k+n)|\lambda| \leq \beta} \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^\lambda(2iy, 2iv) p_{2t}^\lambda(4y, 4v)d\mu(\lambda).$$

Now the function $\varphi_k^\lambda(z, w)$ belongs to the twisted Bergman spaces $\mathscr{B}_t^\lambda$ studied in [8] for any $t > 0$. The reproducing kernel K_t^λ for these spaces are given in terms of p_{2t}^λ (see [8]): more precisely

$$K_t^\lambda((z, w), (a, b)) = p_{2t}^\lambda(z - \bar{a}, w - \bar{b}) e^{-(i/2)\lambda(w\cdot\bar{a} - z\cdot\bar{b})}.$$

As evaluations are continuous it follows that

$$|\varphi_k^\lambda(2iy,2iv)| \le K_t^\lambda((2iy,2iv),(2iy,2iv))\|\varphi_k^\lambda\|.$$

Since $\|\varphi_k^\lambda\| = C\frac{(k+n-1)!}{k!(n-1)!}e^{2(2k+n)|\lambda|t}$ it follows that

$$\frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(2iy,2iv)p_{2t}^\lambda(4y,4v) \le C\left(\frac{\lambda}{\sinh(2t\lambda)}\right)^{2n} e^{2(2k+n)|\lambda|t}.$$

Using this estimate in the above and noting that $\frac{\lambda}{\sinh\lambda}$ is a decreasing function we obtain

$$\int_{-\infty}^{\infty} O_{|F|^2}(iy,iv,i\eta)W_t^+(2iy,2iv,i\eta)d\eta \le Ct^{-2n}e^{t\alpha^2}e^{2t\beta}\|f\|_2^2.$$

This proves one half of the theorem with $B = \alpha^2+\beta$.

Now for the converse. The hypothesis on F means that

$$\int_0^\infty e^{2t\lambda^2}\sum_{k=0}^\infty \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2\frac{k!(n-1)!}{(k+n-1)!}\varphi_k^\lambda(2iy,2iv)p_{2t}^\lambda(2y,2v)d\mu(\lambda)$$
$$\le C_N t^{-2n}(1+|y|^2+|v|^2)^{-N}e^{2tB}.$$

Integrating with respect to $dydv$ and noting that

$$\int_{\mathbf{R}^{2n}}\varphi_k^\lambda(iy,iv)p_t^\lambda(y,v)dydv = c_n\frac{(k+n-1)!}{k!(n-1)!}e^{(2k+n)|\lambda|t}$$

(see Lemma 6.3 in [15]) we have

$$\int_0^\infty e^{2t\lambda^2}\sum_{k=0}^\infty \|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 e^{2(2k+n)|\lambda|t}\,d\mu(\lambda) \le C_N t^{-2n}e^{2tB}.$$

This gives for ant $C>0$

$$e^{2tC}\int_0^\infty \sum_{(2k+n)|\lambda|>C}\|f^\lambda *_\lambda \varphi_k^\lambda\|_2^2 d\mu(\lambda) \le C_N e^{2tB}$$

for all $t\ge 1$. Taking $C>B$ and letting t tend to infinity we conclude that $\rho_k^\lambda(f)$ is supported in $(2k+n)|\lambda|\le B$. This completes the proof.

In the above theorem the necessary and sufficient conditions are different. For the necessary condition involves pointwise estimate on the spherical functions φ_k^λ whereas for the sufficiency we have used an integral condition on the same functions. We therefore, may not hope to get the same condition as both necessary and sufficient.

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