On Paley–Wiener properties of the Heisenberg group

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1. Introduction

Given a function f on R, consider its Fourier transform defined by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x.$$

The classical Paley–Wiener theorem characterizes compactly-supported functions in terms of their Fourier transforms.

Theorem 1.1. The function f is compactly supported if and only if its Fourier transform $\hat{f}$ originally defined on R extends to C as an entire function of exponential type.

An immediate consequence of this theorem is the following uncertainty principle. If f is compactly supported, then $\hat{f}$ cannot have compact support unless $f = 0$. Let us call this a Paley–Wiener property of the Fourier transform on the real line. More refined versions of this property are known. A result of M. Benedicks[1] says that if the sets $A = \{x: f(x) \neq 0\}$ and $B = \{\xi: \hat{f}(\xi) \neq 0\}$ both have finite Lebesgue measure, then $f = 0$. These results involve the size of the supports of f and $\hat{f}$. There are also versions of the Paley–Wiener property that involve decay of the function at infinity. For example, if f is compactly supported, then its Fourier transform cannot have any decay of the form $|\hat{f}(\xi)| \leq Ce^{-a|\xi|^2}$ unless $f = 0$. For, then f can be extended as a holomorphic function on the strip containing the real line. A much more refined version of this property is the theorem of Hardy.

Theorem 1.2. Suppose we have $|f(x)| \leq Ce^{-a|x|^2}$ and $|\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}$. Then $f = 0$ whenever $ab > \frac{1}{4}$.

A search for similar Paley–Wiener properties of the Fourier transform on general locally compact unimodular groups has been going on for sometime. Scott and Sitaram[2] proved among other things, a Paley–Wiener property for the Fourier transform on the Heisenberg group. They conjectured that an analogue of Benedicks' theorem holds for all connected, simply connected nilpotent Lie groups. That is, if f is a compactly-supported function on such a group G, then its Fourier transform $\pi(f)$ cannot vanish on a set of positive Plancherel measure unless $f = 0$. Moss[3] and Park[4] proved this conjecture for certain special classes of nilpotent Lie groups and recently several proofs of the conjecture were obtained by Lipsman and Rosenberg[5], Arnal and Ludwig[6] and Garimella[7].

The problem of proving exact analogues of the Paley–Wiener theorem for the group Fourier transform has been taken up by several authors. In the case of the Heisenberg group, such theorems have been proved by Kumahara[8], Ando[9] and Thangavelu[10] and for step two nilpotent groups Thangavelu has proved a Paley–Wiener theorem for the modified Fourier transform (see ref. 11). It will be interesting to prove such Paley–Wiener theorems for general nilpotent Lie groups and also to find their relation to the Paley–Wiener properties.

In the conjecture of Scott and Sitaram, it is assumed that the group Fourier transform $\pi(f)$ vanishes on a set of positive Plancherel measure. For nilpotent Lie groups the group Fourier transform is operator valued. Thus the requirement that $\pi(f) = 0$ imposes very strong restrictions on the function. To see how strong this condition is, let us take the case of the Heisenberg group H^n. In this case the Fourier transform is parametrized by nonzero reals λ and for each λ the operator $\pi_\lambda(f)$ acts on $L^2(R^n)$. The condition $\pi_\lambda(f) = 0$ translates into $f^\lambda(z) = 0$ for all $z \in C^n$, where f^λ is the inverse Fourier transform of f in the central variable. Undoubtedly this is a very strong assumption to make.

It is therefore desirable to replace the assumption $\pi(f) = 0$ by a weaker one. To certain extent this problem on the Heisenberg group has been treated by Sitaram *et al.*[12]. Our aim in this article is to elaborate on this and prove an analogue of Hardy's theorem for the Heisenberg group.

2. Paley–Wiener properties of Hilbert–Schmidt operators

As we have already noted, the Fourier transform on nilpotent Lie groups is operator-valued. If π is an irreducible unitary representation of a nilpotent Lie group G

and if f is a square integrable function on G, then $\pi(f)$ will be a Hilbert–Schmidt operator acting on a suitable Hilbert space. So, we need to study Paley–Wiener theorems and Paley–Wiener properties of such operators in order to study the same for the group Fourier transform. In this section we formulate analogues of the Paley–Wiener theorem and Hardy's theorem for operators. For the sake of simplicity we consider operators acting on $L^2(R)$.

Let T be a Hilbert–Schmidt operator acting on $L^2(R)$. We define the Fourier transform of T in the following way. Consider the projective representation $\pi(z)$ of C acting on $L^2(R)$ given by

$$\pi(z)\varphi(\xi) = \mathrm{e}^{\mathrm{i}(x\xi + \frac{1}{2}xy)}\varphi(\xi + y),$$

where $\varphi \in L^2(R)$ and $z = x + iy$. Using this we define

$$\hat{T}(\xi) = \pi(\xi' + i\xi'')T\pi(-\xi' - i\xi''),$$

where $\xi = (\xi', \xi'') \in R^2$ and call this Fourier transform of the operator T. Now we use the fact that every Hilbert–Schmidt operator T on $L^2(R)$ is the Weyl transform of a function $f \in L^2(C)$. That is to say, there is $f \in L^2(C)$ such that $T = W(f)$, where $W(f)$ is the Weyl transform of f given by

$$W(f) = \int_C f(z)\pi(z)\mathrm{d}z.$$

If f and T are related as above, we call $\hat{T}(\xi)$ the Fourier–Weyl transform of the function f on C. For this transform we have the following analogue of the Paley–Wiener theorem.

Theorem 2.1. Let $f \in L^2(C)$ and let $T = W(f)$. Then f is supported in $|z| \leq B$ if and only if its Fourier–Weyl transform $\hat{T}(\xi)$ extends to C^2 as an entire function satisfying the estimate $\|\hat{T}(\zeta)\| \leq Ce^{B|Im\zeta|}$ where $\|.\|$ is the Hilbert–Schmidt operator norm.

When the function f on R is supported in $|x| \leq B$, it follows from the definition of the Fourier transform that the derivatives of $\hat{f}$ satisfy the estimates $|D^k\hat{f}(\xi)| \leq C_kB^k$ and this is equivalent to saying that $\hat{f}$ extends to an entire function of exponential type. Similarly, in the above theorem we can state the conditions in terms of certain derivations of the operator T. In order to define these derivations, we use the annihilation and creation operators of quantum mechanics: $A = (\mathrm{d}/\mathrm{d}x) + x$ and $A^* = -(\mathrm{d}/\mathrm{d}x) + x$. We define δT and $\bar{\delta} T$ by $\delta T = TA - AT$ and $\bar{\delta} T = A^*T - TA^*$. In terms of these derivations the above theorem takes the following form.

Theorem 2.2. Let f and T be as in the previous theorem. Then f is supported in $|z| \leq B$ if and only if $\|\delta^j\bar{\delta}^kT\| \leq CB^{j+k}$ for all j and k.

The above derivations δT and $\bar{\delta} T$ take particularly simple form when the operator T is diagonal in the Hermite basis. Let

$$H = \frac{1}{2}(AA^* + A^*A) = -\Delta + |x|^2$$

be the Hermite operator and consider its spectral decomposition $H = \sum_{k=0}^{\infty}(2k+1)p_k$. When the operator T is of the form

$$T = \varphi(H) = \sum_{k=0}^{\infty}\varphi(2k+1)P_k,$$

then it can be verified that

$$\delta^j\bar{\delta}^kT = \sum C_{jkr}(A^*)^{k+r-j}A^r(\Delta_-^r\Delta_+^k\varphi)(H),$$

where the sum is extended over all non-negative integers r such that $0 \leq r \leq j \leq k + r$. Here Δ_- and Δ_+ are the backward and forward finite difference operators. Thus the conditions on the derivatives of T translate into conditions on the derivatives of the function φ. Note that in this particular case the sequence $\varphi(2k + 1)$ gives the singular numbers of the operator T.

We now look for a condition on the operator T that is analogous to $|\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}$. In the case of operators of the form $T = \varphi(H)$, the natural condition is that $|\varphi(2k + 1)| \leq Ce^{-b(2k+1)}$. In the general case we state the condition in terms of the operator e^{tH} which is bounded when $t \leq 0$ but unbounded when $t > 0$. This is a densely-defined operator and the operator analogue of the condition $|\hat{f}(\xi)| \leq Ce^{-b|\xi|^2}$ is that Te^{bH} is Hilbert–Schmidt. Note that when $T = \varphi(H)$ this condition translates into the exponential decay of $\varphi(2k + 1)$. We now have the following Paley–Wiener property of the Weyl transform.

Theorem 2.3. Let $f \in L^2(C)$ be compactly supported. If for some $b > 0$ the operator $W(f)e^{bH}$ is Hilbert–Schmidt then $f = 0$.

The proof of the above theorem uses some properties of special Hermite expansions and an elliptic regularity theorem of Kotake and Narasimhan. The condition on the operator T does not directly imply exponential decay of its singular numbers. However, it does imply exponential decay of the singular numbers of certain Fourier coefficients of the operator T. We now briefly recall how the Fourier coefficients of an operator are defined. In defining the Fourier transform of an operator, we have made use of the Schrödinger representation of the Heisenberg group. To define the Fourier coefficients we make use of the so-called metaplectic representation.

For each real θ there exists a unitary operator $\mu(\theta)$ acting on the Hilbert space $L^2(R)$ such that $\pi(e^{i\theta}z) = \mu(\theta)\pi(z)\mu(\theta)^*$. Using this we can define the operator valued function $T(\theta) = \mu(\theta)T\mu(\theta)^*$. Then, the Fourier coefficients T_m of the operator T are defined by

$$T_m = \int_0^{2\pi} T(\theta)e^{-im\theta}d\theta.$$

With this definition we can show that the condition $\|Te^{bH}\| < \infty$ translates into exponential decay of the singular numbers of the Fourier coefficients T_m. In terms of the Fourier coefficients and the Fourier transform we have the following Paley–Wiener property.

Theorem 2.4. Let T be a Hilbert–Schmidt operator on $L^2(R)$. If for every integer m the singular numbers of T_m are exponentially decaying and the Fourier transform $\hat{T}_m$ extends to entire functions of exponential type, then $T = 0$.

If $T = W(f)$ and $T_m = W(f_m)$ then f and f_m are related by

$$f_m(z) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}z)e^{-im\theta}d\theta.$$

The condition on the singular numbers of T_m implies that f_m are real analytic whereas the conditions on $\hat{T}_m$ show that f_m are compactly supported. Hence $f_m = 0$ for all m, thus proving the theorem. We can now assume exponential decay of f and prove an analogue of Hardy's theorem for the Weyl transform.

Theorem 2.5. Let f be a function on C which verifies $|f(z)| \le Ce^{-a|z|^2}$ and let $W(f)e^{bH}$ be bounded for some a and b positive. Then $f = 0$ whenever $a(\tanh(b/2)) > \frac{1}{4}$.

This theorem is proved by appealing to the Hardy's theorem for the Fourier transform on R. If $T = W(f)$ and $T_m = W(f_m)$ then we have $|f_m(z)| \le Ce^{-a|z|^2}$. We will show that $|\hat{f}_m(z)| \le Ce^{-(\tanh(b/2))|z|^2}$. The theorem will then follow immediately from Hardy's theorem. In order to get estimates on the Fourier transforms of f_m, we first observe that the condition $\|Te^{bH}\| < \infty$ implies that $\|T_me^{bH}\| < \infty$. This follows from the fact that the unitary operators $\mu(\theta)$ all commute with the Hermite operator H. Now we can write down the special Hermite expansion of f_m which reduces to a Laguerre expansion owing to the fact that the functions f_m are homogeneous. Then it is easy to calculate the singular numbers of T_m which are expressible in terms of the Laguerre coefficients of f_m. The condition $\|T_me^{bH}\| < \infty$ then translates into exponential decay of the Laguerre coefficients of f_m. As the Laguerre functions are eigenfunctions of the symplectic Fourier transform, we get estimates on the Laguerre coefficients of the Fourier transform of f_m. Then we use a generating function identity for Laguerre functions and get the required exponential decay of $\hat{f}_m$. We refer to ref. 13 for the details of the proof.

3. Paley–Wiener properties of the Heisenberg group

In this section we use the results of the previous section to formulate some Paley–Wiener properties of the Heisenberg group H^n. For the sake of simplicity we assume $n = 1$ but whatever we say for this case has analogues for the higher dimensional case as well. First we recall the relevant facts from the representations of the Heisenberg group. This group, which we denote by G is just $C \times R$ equipped with the group law

$$(z,t)(w,s) = (z+w, t+s+\frac{1}{2}Im(z.\overline{w})).$$

There are two kinds of representations of G: one dimensional and infinite dimensional. In the definition of the group Fourier transform, only the latter is involved. These unitary representations are parametrized by non-zero reals and all of them are realized on the same Hilbert space, namely $L^2(R)$. Thus, for each non-zero $\lambda \in R$ and $(z, t) \in G$ we have the unitary operator $\pi_\lambda(z, t)$ whose action on $L^2(R)$ is given by

$$\pi_\lambda(z,t)\varphi(\xi) = e^{i\lambda t}e^{i\lambda(x\xi+\frac{1}{2}xy)}\varphi(\xi+y),$$

where $\varphi \in L^2(R)$. For a nice introduction to the representation theory of the Heisenberg group we refer to ref. 14.

Given a function f on G the group Fourier transform of f is defined by

$$\hat{f}(\lambda) = \int_G f(z,t)\pi_\lambda(z,t)dzdt.$$

Thus for each $\lambda \ne 0$, $\hat{f}(\lambda)$ is an operator on $L^2(R)$. It can be shown that the Fourier transform initially defined on $L^1(G)$ can be defined for L^2 functions as well and for such functions $\hat{f}(\lambda)$ is a Hilbert–Schmidt operator. The simplest version of the Paley–Wiener property of G is given in the following theorem (see ref. 2).

Theorem 3.1. Let $f \in L^1(G)$ be compactly supported as a function of t. Then, $\hat{f}(\lambda)$ as an operator valued function of λ cannot vanish on a set of positive measure unless $f = 0$.

From the definition of the Fourier transform it follows that

$$\hat{f}(\lambda) = \int_G f^\lambda(z)\pi_\lambda(z,0)dz,$$

where f^λ is the inverse Fourier transform of f in the t variable. The vanishing of $\hat{f}(\lambda)$ means that $f^\lambda = 0$ as a function on C and if it is so on a set of positive measure then by Benedick's theorem for the Fourier transform $f = 0$. It is clear that this is just a version of the Paley–

Wiener property which takes into account only the support properties of f as a function of t.

Combining the results of the previous section, we obtain several stronger versions of the Paley–Wiener property.

Theorem 3.2. Let $f \in G$ be compactly supported and let $E \subset R$ be a set of positive Lebesgue measure. Suppose that for every non-zero $\lambda \in E$ there is $b(\lambda) > 0$ such that $\| \hat{f}(\lambda)e^{b(\lambda)H} \|$ is finite. Then $f = 0$.

We also have the following version of Hardy's theorem for the Fourier transform on the Heisenberg group.

Theorem 3.3. Let f be compactly supported as a function of t and satisfy the estimate $|f(z, t)| \leq Ce^{-a|z|^2}$ as a function of z. Let $E \subset R$ be a set of positive measure with the property that for every non-zero $\lambda \in E$ there is $b(\lambda) > 0$ such that $\hat{f}(\lambda)e^{b(\lambda)H}$ is Hilbert–Schmidt. If $a(\tanh(b(\lambda)/2)) > 1/4$ then $f = 0$.

Combining the results of the previous section with the results of Lipsman–Rosenburg[5], Arnal–Ludwig[6] and Garimella[7] we can obtain various versions of the Paley–Wiener property for general nilpotent Lie groups. There is also an exact analogue of Paley–Wiener theorem for the Fourier transform on the Heisenberg group. Let us write T_λ in place of $\hat{f}(\lambda)$.

Theorem 3.4. Let $f \in L^2(G)$. Then f is compactly supported in the z variable if and only if for each non-zero λ the Fourier–Weyl transform $\hat{T}_\lambda$ extends to an entire function of exponential type.

We also have another version of the Paley–Wiener theorem for the modified Fourier transform on G which respects both the variables z and t. We refer to ref. 11 for formulation and proof of this result where it is proved in the more general set up of step two nilpotent Lie groups. It is an interesting open problem to formulate such a Paley–Wiener theorem for general nilpotent Lie groups.

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