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Littlewood-Paley-Stein Theory on $\mathbb{C}^n$ and Weyl Multipliers

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1. Introduction

On $\mathbb{C}^n$ consider the $2n$ linear differential operators

$$Z_j = \partial_j + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \bar{\partial}_j - \frac{1}{4}z_j, \qquad j = 1, 2, \ldots, n. \tag{1.1}$$

Together with the identity they generate a Lie algebra h^n which is isomorphic to the $2n + 1$ dimensional Heisenberg algebra. The only non trivial commutation relations are

$$[Z_j, \bar{Z}_j] = -\frac{1}{2}I, \qquad j = 1, 2, \ldots, n. \tag{1.2}$$

The operator L defined by

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

is nonnegative, self-adjoint and elliptic. Hence it generates a diffusion semigroup T^t. Following Stein [8], we study g and g^* functions associated to this semigroup and apply the results to prove a multiplier theorem for the Weyl transform.

The above operators in (1.1) generate a family of 'twisted translations' and using them we can define 'twisted convolution' of functions on $\mathbb{C}^n$. It turns out that the operator g is defined by a twisted convolution operator whose kernel is of Calderón-Zygmund type but it takes values in a Hilbert space. Also twisted convolution operators with Calderón-Zygmund kernels can be thought of as ordinary convolutions with the kernel having an oscillatory factor. Such singular integral operators, called oscillatory singular integrals have been studied by Ricci and Stein [5] and Chanillo and Christ [1]. We study the g functions using oscillaroty singular integrals whose kernels are taking values in a Hilbert space.

As an application of the LPS theory we prove a multiplier theorem for the Weyl transform. The Weyl transform, which we denote by τ takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. It enjoys most of the properties of the ordinary Fourier transform. In analogy with the definition of Fourier multipliers, we can define Weyl multipliers. In [4] Mauceri has studied Weyl multipliers and has given sufficient conditions on an operator $M \in B(L^2(\mathbb{R}^n))$ so that it will be an L^p multiplier for the Weyl transform. In this paper we are concerned only with multipliers of the form $\phi(H)$ where H is the Hermite operator.

The multiplier theorem of Mauceri follows form a modified version of the Calderón-Zygmund theory in the general setting of homogeneous spaces developed by Coifman and Weiss [2]. On the other hand we follow the method used by Stein in his proof of Hormander-Mihlin multiplier theorem for the Fourier transform. The same method was successfully employed by Strichartz [9] to prove a multiplier theorem for the Spherical Harmonic expansions. Recently the author [11] used the LPS theory for the Hermite semigroup to prove a multiplier theorem for the Hermite expansions.

The plan of the paper is as follows. In the next section we briefly review the relevant facts about twisted convolution and the Weyl transform and state the main results of the paper. In Section 3 we apply Ricci-Stein theory of oscillatory singular integrals, after making necessary modifications, to study the functions g and g_k^*. Finally, in Section 4 we prove a version of the multiplier theorem for the Weyl transform.

2. Preliminaries and Main Results

Let

$$\omega(z, v) = \exp\left(-\frac{i}{2}\operatorname{Im}(z, \bar{v})\right)$$

and let $dv\,d\bar{v}$ stand for the Lebesgue measure on $\mathbb{C}^n$. Then the product

$$f \times g(z) = \int_{\mathbb{C}^n} f(z - v) g(v) \bar{\omega}(z, v)\, dv\, d\bar{v} \tag{2.1}$$

is called the twisted convolution of the functions f and g on $\mathbb{C}^n$. It is well known [3] that ω satisfies the cocycle identities

(*a*) $\omega(z, 0) = \omega(z, z) = \omega(0, z) = 1$
(*b*) $\omega(z + v, u)\omega(z, u) = \omega(z, v + u)\omega(v, u)$

and that there exists an irreducible projective representation W of $\mathbb{C}^n$ into a separable Hilbert space H_W such that

$$W(z + v) = \omega(z, v) W(z) W(v).$$

Given a function f in $L^1(\mathbb{C}^n)$ its Weyl transform $\tau(f)$ is a bounded operator on H_W defined by

$$\tau(f) = \int_{\mathbb{C}^n} f(z) W(z)\, dz\, d\bar{z}. \tag{2.2}$$

The Weyl transform enjoys many of the properties of the ordinary Fourier transform. Indeed we have an analogue of the Fourier inversion formula:

$$f(z) = (2\pi)^{-n} \operatorname{tr}(W(z)^* \tau(f)) \tag{2.3}$$

and the Plancherel formula:

$$\|f\|_2^2 = (2\pi)^{-n} \|\tau(f)\|_{HS}^2 \tag{2.4}$$

where tr is the canonical semifinite trace on the algebra of bounded operators on H_W and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm. Moreover, the Weyl transform of the twisted convolution is the product of the Weyl transforms,

$$\tau(f \times g) = \tau(f)\tau(g). \tag{2.5}$$

Let h^n be the Lie algebra generated by the following differential operators and the identity:

$$Z_j = \partial_j + \frac{1}{4}\bar{z}_j, \qquad \bar{Z}_j = \bar{\partial}_j - \frac{1}{4} z_j, \qquad j = 1, \dots, n. \tag{2.6}$$

Here $\partial_j = \partial/\partial z_j$ and $\bar{\partial}_j = \partial/\partial \bar{z}_j$. Let U^n stand for the universal enveloping algebra of h^n. Let us take $H_W = L^2(\mathbb{R}^n)$ and consider the Schrödinger representation defined by

$$W(z)\phi(\xi) = \exp\left\{ i\left(x, \frac{1}{2} y + \xi \right) \right\} \phi(\xi + y) \tag{2.6$'$}$$

where $z = x + iy \in \mathbb{C}^n$. The representation W extends to a representation of the enveloping algebra denoted by dW. From (2.6) it follows that for every

f in the Schwartz class

$$\tau(Z_j f) = \tau(f)\,dW(Z_j) = i\tau(f)A_j^* \tag{2.7}$$

$$\tau(\bar{Z}_j f) = \tau(f)\,dW(\bar{Z}_j) = i\tau(f)A_j \tag{2.8}$$

where A_j and A_j^* are the 'annihilation' and creation operators defined by

$$A_j = \partial/\partial\xi_j + \xi_j, \qquad j = 1, 2, \ldots, n \tag{2.9}$$

$$A_j^* = -\partial/\partial\xi_j + \xi_j, \qquad j = 1, 2, \ldots, n \tag{2.10}$$

Let

$$H = \frac{1}{2}\sum_{j=1}^{n}(A_jA_j^* + A_j^*A_j) = \sum_{j=1}^{n}(-\partial^2/\partial\xi_j^2 + \xi_j^2)$$

be the Hermite operator. Let $\{\Phi_\alpha\}$ be the family of n-dimensional Hermite functions which form an othonormal basis for $L^2(\mathbb{R}^n)$ and let P_N be the projection onto the eigenspace spanned by $\{\Phi_\alpha : |\alpha| = N\}$. Then H has the spectral resolution

$$H = \sum_{N=0}^{\infty}(2N+n)P_N. \tag{2.11}$$

If we let

$$L = -\frac{1}{2}\sum_{j=1}^{n}(Z_j\bar{Z}_j + \bar{Z}_jZ_j)$$

then it follows that $\tau(Lf) = \tau(f)H$ or more generally

$$\tau(\phi(L)f) = \tau(f)\phi(H). \tag{2.12}$$

Let T^t be the semigroup generated by the operator L. Then following Stein [8] we define the following Littlewood-Paley g and g^* functions.

$$g(f,z)^2 = \int_0^\infty t|\partial_t T^t f(z)|^2\,dt \tag{2.13}$$

$$g_k(f,z)^2 = \int_0^\infty t^{2k-1}|\partial_t^k T^t f(z)|^2\,dt \tag{2.14}$$

$$g_k^*(f,z)^2 = \int_0^\infty\int_{\mathbb{C}^n} t^{1-n}(1 + t^{-1}|z-v|^2)^{-k}|\partial_t T^t f(v)|^2\,dv\,d\bar{v}\,dt \tag{2.15}$$

It is easy to verify that $g(f,z) \leqslant c g_k(f,z)$, $k \geqslant 1$. The following theorem is our main result on the boundedness properties of g and g^* functions.

Theorem 2.1.

(i) $C_1\|f\|_p \leqslant \|g(f)\|_p \leqslant C_2\|f\|_p, \quad 1 < p < \infty.$

(ii) $\|g_k^*(f)\|_p \leqslant C\|f\|_p, \quad p > 2$ *provided* $k > n$.

The proof of this theorem will be given in the next section. We then want to apply this theorem to prove a multiplier theorem of the Weyl transform.

Recall that the Weyl transform takes functions on $\mathbb{C}^n$ into bounded operators on $L^2(\mathbb{R}^n)$. In analogy with the definition of Fourier multipliers, we say that a bounded operator M on $L^2(\mathbb{R}^n)$ is a Weyl multiplier on $L^p(\mathbb{C}^n)$ if the operator T_M initially defined on $L^1 \cap L^p$ by

$$\tau(T_M f) = \tau(f)M \tag{2.16}$$

extends to a bounded operator on $L^p(\mathbb{C}^n)$. Sufficient conditions on the operator M have been obtained by Mauceri [4] so that T_M is a bounded operator on $L^p(\mathbb{C}^n)$. In this paper we consider only multipliers M of the form $\phi(H)$ where H is the Hermite operator.

To state our result on the multiplier theorem for Weyl transform we introduce the following forward and backward difference operators

$$\Delta_+ \phi(N) = \phi(N+1) - \phi(N)$$
$$\Delta_- \phi(N) = \phi(N) - \phi(N-1)$$

Theorem 2.2. *Suppose that the function ϕ satisfies*

$$|\Delta_-^k \Delta_+^m \phi(N)| \leqslant CN^{-(k+m)} \tag{2.17}$$

with k, m positive integers such that $k + m = 0, 1, \ldots, \nu$, where $\nu = n + 1$ when n is odd and $\nu = n + 2$ when n is even. Then $\phi(H)$ is a Weyl multiplier on $L^p(\mathbb{C}^n)$, $1 < p < \infty$.

This theorem will be proved in Section 4. A good reference for the Weyl transform is [5].

3. Oscillatory Integral and LPS Theory

The aim of this section is to prove Theorem 2.1 on the boundedness of g and g_k^* functions. That will be done by first studying oscillatory singular integrals whose kernel takes values in a Hilbert space. To see how oscillatory singular integrals enter the picture let us analyze the operator $f \to \partial_t T^t f$ more closely.

In view of the equation (2.12) if we take the Weyl transform of $\partial_t T^t f$ we get

$$\tau(\partial_t T^t f) = \tau(f)\partial_t(e^{-tH}) \tag{3.1}$$

In other words, $\partial_t T^t f$ is given by a twisted convolution

$$\partial_t T^t f(z) = f \times k_t(z) \tag{3.2}$$

where $k_t(z) = \tau^{-1}(\partial_t e^{-tH})$ is the inverse Weyl transform of $\partial_t e^{-tH}$. Since $\partial_t e^{-tH}$ has the spectral resolution

$$\partial_t e^{-tH} = -\sum_{N=0}^{\infty} (2N+n) e^{-(2N+n)t} P_N \tag{3.3}$$

it is easy to calculate the kernel k_t explicitly. Indeed, Peetre [5] has shown that

$$\tau^{-1}(P_N) = (2\pi)^{-n} e^{-(1/4)|z|^2} L_N^{n-1}\left(\frac{1}{2}|z|^2\right) \tag{3.4}$$

where L_N^{n-1} are the Laguerre polynomials of degree N and type $n-1$.

Recall that for $\alpha > -1$, Laguerre polynomials $L_k^\alpha(x)$ are defined by the equation

$$e^{-x} x^\alpha L_k^\alpha(x) = \frac{1}{k!} \frac{d^k}{dx^k} (e^{-x} x^{k+\alpha}). \tag{3.5}$$

The Laguerre polynomials also satisfy the following generating function relation

$$\sum_{k=0}^{\infty} L_k^\alpha(x) r^k = (1-r)^{-\alpha-1} e^{-xr/(1-r)} \tag{3.6}$$

In view of the relations (3.3), (3.4) and (3.6) we see that

$$k_t(z) = (2\pi)^{-n} \partial_t \{(\sinh t)^{-n} e^{-1/4|z|^2 \coth t}\}. \tag{3.7}$$

Writing out the twisted convolution $f \times k_t$ we have

$$\partial_t T^t f(z) = \int_{\mathbb{C}^n} f(v) e^{-i/2 \operatorname{Im}(z\bar{v})} k_t(z-v) f(v)\, dv\, d\bar{v} \tag{3.8}$$

or equivalently

$$\partial_t T^t f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)} k_t(x-y) f(y)\, dy \tag{3.9}$$

where $P(x, y)$ is a real valued polynomial in x and y.

The kernel $k_t(x)$ can be considered as taking values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. The following lemma shows that it is a Calderón-Zygmund kernel.

Lemma 3.1. *The kernel k_t satisfies*

(i) $\|k_t(x)\| \leqslant C|x|^{-2n}$
(ii) $\|\nabla k_t(x)\| \leqslant C|x|^{-2n-1}$

where $\|\cdot\|$ stand for the norm of $L^2(\mathbb{R}^+, t\,dt)$.

PROOF. Since

$$k_t(x) = (2\pi)^{-n}\partial_t\{(\sinh t)^{-n}e^{-1/4|x|^2\coth t}\}$$

it is easy to see that the following estimate holds

$$|k_t(x)| \leqslant Ct^{-n-1}(1 + t^{-1}|x|^2)^{-n-1}.$$

From this it follows immediately that

$$\begin{aligned}\|k_t(x)\|^2 &\leqslant C\int_0^\infty t^{-2n-1}(1 + t^{-1}|x|^2)^{-2n-2}\,dt\\ &\leqslant C|x|^{-4n}\int_0^\infty t^{-2n-1}(1 + t^{-1})^{-2n-2}\,dt\\ &= C|x|^{-4n}\end{aligned}$$

as the t integral is convergent. This proves (i). The proof of the second estimate is similar. Any x_j derivative will in effect bring down a factor of $t^{-1/2}$ which accounts for the extra factor $|x|^{-1}$. The details are omitted.

Thus the operator $\partial_t T^t f$ can be considered as a oscillatory singular integral whose kernel takes values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. Having made this observation we proceed to prove Theorem 2.1. In doing so we closely follow Ricci-Stein [6]. As the first step we prove the following L^2 result.

Proposition 3.1. *For* $f \in L^2(\mathbb{C}^n)$,

$$\|g(f)\|_2 = \frac{1}{2}\|f\|_2.$$

PROOF. We have from the definition

$$\int_{\mathbb{C}^n} g(f,z)^2\,dz\,d\bar{z} = \int_0^\infty\int_{\mathbb{C}^n} t|\partial_t T^t f(x)|^2\,dz\,d\bar{z}\,dt.$$

To the inner integral we apply the Plancherel formula (2.4). The result is

$$\int_{\mathbb{C}^n} |\partial_t T^t f(z)|^2\,dz\,d\bar{z} = (2\pi)^{-n}\|\tau(\partial_t T^t f)\|_{HS}^2.$$

Since $\tau(\partial_t T^t f) = \tau(f)(\partial_t e^{-tH})$,

$$\|\tau(\partial_t T^t f)\|_{HS}^2 = \operatorname{tr}((\tau(f)(\partial_t e^{-tH}))^*(\tau(f)(\partial_t e^{-tH})))$$

As $\partial_t e^{-tH}$ is a self-adjoint operator and $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ we have

$$\|\tau(\partial_t T^t f)\|_{HS}^2 = \operatorname{tr}(\tau(f)^*\tau(f)H^2e^{-2tH}).$$

Since $\{\Phi_\alpha\}$ form an orthonormal basis for $L^2(\mathbb{R}^n)$,

$$\operatorname{tr}(\tau(f)^*\tau(f)H^2e^{-2tH}) = \sum_{\alpha\geq 0}(\Phi_\alpha, \tau(f)^*\tau(f)H^2e^{-2tH}\phi_\alpha)$$

which equals $\sum_{\alpha\geq 0}(2|\alpha|+n)^2e^{-2(2|\alpha|+n)t}(\Phi_\alpha, \tau(f)^*\tau(f)\Phi_\alpha)$. Integrating the last equation with respect to $t\,dt$ we get

$$\|g(f)\|_2^2 = (2\pi)^{-n}\sum_{\alpha\geq 0}\int_0^\infty (2|\alpha|+n)^2 te^{-2(2|\alpha|+n)t}\,dt\,(\Phi_\alpha, \tau(f)^*\tau(f)\Phi_\alpha)$$

$$= (2\pi)^{-n}\frac{1}{4}\operatorname{tr}(\tau(f)^*\tau(f)) = \frac{1}{4}\|f\|_2^2.$$

Hence we have proved

$$\|g(f)\|_2 = \frac{1}{2}\|f\|_2.$$

To prove that g is bounded on $L^p(\mathbb{C}^n)$ we split the operator $\partial_t T^t f$ into two parts. Let $\alpha(x)$ be a C_0^∞ function supported in $|x|\leqslant 1$ such that $\alpha(x)=1$ for $|x|\leqslant 3/4$. Let $\beta(x)=1-\alpha(x)$ and define

$$T_0 f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)}k_t(x-y)\alpha(x-y)f(y)\,dy \tag{3.10}$$

$$T_\infty f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)}k_t(x-y)\beta(x-y)f(y)\,dy \tag{3.11}$$

First we will take care of the local operator T_0. To do so, we first want to prove that the operator

$$\tilde{T}_0 f(x) = \int_{\mathbb{R}^{2n}} k_t(x-y)f(y)\alpha(x-y)\,dy \tag{3.12}$$

is bounded from $L^p(\mathbb{R}^{2n})$ to $L^p(\mathbb{R}^{2n}, L^2(\mathbb{R}^+, t\,dt))$. (Hereafter we simply say $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.) Observe that in view of Lemma 3.1 $\tilde{T}_0$ is a vector valued singular integral operator whose kernel satisfies the estimates

(i) $\|k_t(x)\alpha(x)\| \leqslant C|x|^{-2n}$,

(ii) $\|\nabla(k_t(x)\alpha(x))\| \leqslant C|x|^{-2n-1}$.

So if we know that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$ then we can apply the following theorem to conclude that $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.

Theorem 3.2. (Stein [7].) *Let $k(x)$ be a C^1 function away from the origin taking values in $B(H_1, H_2)$ where H_1 and H_2 are Hilbert spaces. Assume that*

(i) $\|k(x)\| \leqslant C|x|^{-n}$,

(ii) $\|\nabla k(x)\| \leqslant C|x|^{-n-1}$.

Let $f(x)$ takes values in H_1 and T be defined by

$$Tf(x) = \int_{\mathbb{R}^n} k(x-y)f(y)\,dy.$$

If T is bounded on $L^2(R^n)$ it is also bounded on $L^p(R^n)$, $1<p<\infty$.

To prove that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$ we need to know if T_0 is bounded on $L^2(\mathbb{R}^{2n})$ or not.

Lemma 3.2. *T_0 is bounded on $L^2(\mathbb{R}^{2n})$.*

PROOF. The operator $f \to \partial_t T^t f$ is bounded on $L^2(\mathbb{R}^{2n})$ and T_0 is a truncation of this operator. Hence T_0 is bounded on $L^2(\mathbb{R}^{2n})$. For a proof see the corresponding lemma in Ricci-Stein [6].

Lemma 3.3. *$\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.*

PROOF. As we have already mentioned we need only to prove that $\tilde{T}_0$ is bounded on L^2. We write

$$\begin{aligned}\tilde{T}_0 f(x) &= \int k_t(x-y)\alpha(x-y)f(y)\,dy \\ &= \int e^{iP(x,y)}k_t(x-y)\alpha(x-y)f(y)\,dy \\ &\quad - \int [e^{iP(x,y)}-1]k_t(x-y)\alpha(x-y)f(y)\,dy \\ &= T_0 f(x) + T_1 f(x).\end{aligned}$$

We will prove that

$$\int_{|x|\le 1} \|\tilde{T}_0 f(x)\|^2\,dx \leqslant C\int_{|y|\le 2} |f(y)|^2\,dy. \tag{3.13}$$

Since the kernel is supported in $|x-y| \leqslant 1$, when $|x| \leqslant 1$ only the points $|y| \leqslant 2$ matter. Since $P(y,y)=0$, $e^{iP(x,y)}-1 = O(|x-y|)$ and so the kernel of T_1 is integrable. Since T_0 is already known to be bounded on L^2 this proves (3.13). Since $\tilde{T}_0$ is translation invariant we also have

$$\int_{|x-h|\le 1} \|\tilde{T}_0 f(x)\|^2\,dx \leqslant C\int_{|y-h|\le 2} |f(y)|^2\,dy.$$

Integration with respect to h proves that $\tilde{T}_0$ is bounded on $L^2(\mathbb{R}^{2n})$. Hence by the previous Theorem 3.1, $\tilde{T}_0$ is bounded on $L^p(\mathbb{R}^{2n})$.

Now we use Lemma 3.3 to prove that T_0 is bounded on $L^p(\mathbb{R}^{2n})$.

Proposition 3.2. *T_0 is bounded on $L^p(\mathbb{R}^{2n})$, $1<p<\infty$.*

PROOF. The proof is similar to the corresponding theorem in Ricci-Stein [6]. We give the details for the sake of completeness. We use the same trick as in Lemma 3.3. We will first prove that

$$\int_{|x|\le 1} \|T_0 f(x)\|^p\,dx \leqslant C\int_{|y|\le 2} |f(y)|^p\,dy \tag{3.14}$$

$$T_0 f(x) = \int [e^{iP(x,y)} - 1] k_t(x-y)\alpha(x-y) f(y)\, dy + \int k_t(x-y)\alpha(x-y) f(y)\, dy.$$

As before the kernel of the first integral is integrable and the second integral is $\tilde{T}_0 f$. Hence we have (3.14).

It is easily verified that

$$P(x+h, y+h) = P(x,y) + P(x,h) + P(h,y)$$

and therefore

$$\int_{|x-h|\le 1} \| T_0 f(x) \|^p \, dx \leqslant C \int_{|y-h|\le 2} |f(y)|^p \, dy$$

is also true. Integration with respect to h proves the Proposition.

Having taken care of T_0 we now turn our attention to the study of T_∞. Again we repeat the arguments of Ricci-Stein [6] but this time we will not give the details.

Let us take a partition of unity

$$1 = \sum_{j=-\infty}^{\infty} \psi_0(2^{-j}x)$$

where ψ_0 is supported in $1/2 \leqslant |x| \leqslant 1$ and write

$$k_t(x)\beta(x) = \sum_{j=0}^{\infty} k_t(x)\beta(x)\psi_0(2^{-j}x) = \sum_{j=0}^{\infty} k_t^j(x).$$

Let

$$T_j f(x) = \int_{\mathbb{R}^{2n}} e^{iP(x,y)} k_t^j(x-y) f(y)\, dy. \tag{3.15}$$

For these operators we wish to prove that their norm as operators on $L^2(\mathbb{R}^{2n})$ satisfy

$$\| T_j \| \leqslant C 2^{-j\epsilon} \quad \text{for some} \quad \epsilon > 0. \tag{3.16}$$

To do this we consider $T_j^* T_j$ and estimate the kernel of $T_j^* T_j$. For that purpose we make the following observations regarding singular integral operators whose kernels take values in $B(H_1, H_2)$.

So, let $k(x,y)$ take values in $B(H_1, H_2)$ and $f(x)$ take values in H_1. Consider the operator

$$Tf(x) = \int k(x,y) f(x). \tag{3.17}$$

Assume that T is bounded from $L^2(\mathbb{R}^n, H_1)$ into $L^2(\mathbb{R}^n, H_2)$. This operator T has a formal adjoint T^* which maps $L^2(\mathbb{R}^n, H_2)$ into $L^2(\mathbb{R}^n, H_1)$. Let $k^*(x,y)$

denote the adjoint of the operator $k(x, y)$ which belongs to $B(H_2, H_1)$. Then it is easily verified that T^* is given by

$$T^*f(x) = \int k^*(y, x) f(y)\, dy. \tag{3.18}$$

From this it follows that

$$T^*Tf(x) = \iint k^*(y, x) k(y, z) f(z)\, dz\, dy.$$

Thus the kernel $G(x, z)$ of T^*T is

$$G(x, z) = \int k^*(y, x) k(y, z)\, dy$$

which is a bounded operator on H_1.

Let us specialize these observations to our operator T_j. The kernel of T_j is $e^{iP(x,y)} k_t^j(x - y)$ which takes values in the Hilbert space $L^2(\mathbb{R}^+, t\,dt)$. By taking $H_1 = \mathbb{C}$ and $H_2 = L^2(\mathbb{R}^+, t\,dt)$ we can assume that the kernel belongs to $B(H_1, H_2)$. The action of $e^{iP(x,y)} \times k_t^j(x - y)$ on $\mathbb{C}$ is simply given by $\lambda \to e^{iP(x,y)} k_t^j(x - y)\lambda$. Call this operator $s_j(x, y)$. The adjoint of $s_j(x, y)$ is then given by

$$s_j^*(x, y) g = \int_0^\infty e^{-iP(x,y)} k_t^j(x - y) g(t) t\, dt.$$

Therefore, the kernel $L_j(x, z)$ of $T_j^* T_j$ is

$$L_j(x, z) = \iint_0^\infty k_t^j(y - x) k_t^j(y - z) e^{i(P(y,z) - P(y,x))}\, t\, dt\, dy.$$

Thus the kernel of $T_j^* T_j$ is a scalar valued function which acts on H_1 by multiplication.

Having calculated the kernel of $T_j^* T_j$ now we can simply repeat the arguments of Ricci-Stein. Since k_t is a Calderón-Zygmund kernel the proof given in [6] goes through without any change. This completes the proof that T_∞ is bounded on L^p.

Now it is time to complete the proof of Theorem 3.1. Just now we showed that $g(f)$ is bounded on $L^p(\mathbb{C}^n)$. Since

$$\| g(f) \|_2 = \frac{1}{2} \| f \|_2,$$

the reverse inequality can be proved as in Stein [7]. Deduction of (ii) from (i) is routine and we refer to Stein [7] for details.

4. Multiplier Theorem for the Weyl Transform

In this section we will prove Theorem 3.2. In view of Theorem 3.1 it is enough to prove that

$$g_{k+1}(F, z) \leqslant C g_k^*(f, z) \tag{4.1}$$

where $F(z) = T_\phi f(z)$ for $k = n + 1$. Recall that the operator T_ϕ is defined by means of the Weyl transform as $\tau(T_\phi f) = \tau(f)\phi(H)$. Assume that the function ϕ satisfies the conditions stated in Theorem 3.2. Without loss of generality we further assume that $\phi(2N + n) = 0$ for $N \leqslant n + 1$.

Let $u(z, t) = T^t f(z)$ and $U(z, t) = T^t F(z)$. Then it is easily verified that

$$U(z, t + s) = u(z, s) \times M(t, z) \tag{4.2}$$

where $M(t, z)$ is the function defined by

$$M(t, z) = (2\pi)^{-n} \sum_{N=0}^{\infty} \phi(2N + n) e^{-(2N+n)t} e^{-1/4|z|^2} L_N^{n-1}\left(\frac{1}{2}|z|^2\right). \tag{4.3}$$

Taking one derivative with respect to s, k derivatives with respect to t and setting $t = s$ we get

$$\partial_t^{k+1} U(z, 2t) = (\partial_t T^t f) \times \partial_t^k M. \tag{4.4}$$

The following lemma translates the conditions on ϕ into properties of the function M.

Lemma 4.1. *Under the assumptions stated in Theorem* 3.2 *the following estimates are true.*

(i) $|\partial_t^k M(t, z)| \leqslant C t^{-n-k}$,

(ii) $\int_{\mathbb{C}^n} |z|^{2k} |\partial_t^k M(t, z)|^2 \, dz \, d\bar{z} \leqslant C t^{-n-k}$ *if k is even,*

(iii) $\int_{\mathbb{C}^n} |z|^{2k+2} |\partial_t^k M(t, z)|^2 \, dz \, d\bar{z} \leqslant C t^{-n-k+1}$ *if k is odd.*

Assuming the lemma for a moment we will complete the proof of Theorem 3.2. We want to prove (4.1) when $k = n + 1$. Recall that

$$g_{k+1}(F, z)^2 = \int_0^\infty t^{2k+1} |\partial_t^{k+1} T^t F(z)|^2 \, dt.$$

From equation (4.4) we have

$$\begin{aligned} |\partial_t^{k+1} T^{2t} F(z)| &\leqslant \int_{\mathbb{C}^n} |\partial_t T^t f(v)| \, |\partial_t^k M(t, z - v)| \, dv \, d\bar{v} \\ &= \int_{|z-v| \leq t^{1/2}} + \int_{|z-v| > t^{1/2}} = A_t(z) + B_t(z). \end{aligned}$$

Applying Schwarz inequality and using (i) of Lemma 4.1 we get

$$\begin{aligned} A_t(z)^2 &\leqslant \int_{|z-v|\leq t^{1/2}} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \int_{|z-v|\leq t^{1/2}} |\partial_t^k M(t, z-v)|^2 \, dv \, d\bar{v} \\ &\leqslant C \int_{|z-v|\leq t^{1/2}} t^{-n-2k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \\ &\leqslant Ct^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \end{aligned}$$

When k is even another application of Schwarz inequality together with (ii) gives

$$\begin{aligned} B_t(z)^2 &\leqslant \int_{|z-v|>t^{1/2}} |z-v|^{-2k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \\ &\quad \int_{|z-v|>t^{1/2}} |z-v|^{2k} |\partial_t^k M(t, z-v)|^2 \, dv \, d\bar{v} \\ &\leqslant Ct^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v}. \end{aligned}$$

When k is odd we use (iii):

$$\begin{aligned} B_t(z)^2 &\leqslant \int_{|z-v|>t^{1/2}} |z-v|^{-2k-2} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \\ &\quad \int_{|z-v|>t^{1/2}} |z-v|^{2k+2} |\partial_t^k M(t, z-v)|^2 \, d\bar{v} \, dv \\ &\leqslant Ct^{-1} t^{-n-k+1} \int_{|z-v|>t^{1/2}} |z-v|^{-2k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v} \\ &\leqslant Ct^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v}. \end{aligned}$$

Hence

$$|\partial_t^{k+1} T^t F(z)|^2 \leqslant Ct^{-n-2k} \int_{\mathbb{C}^n} (1 + t^{-1}|z-v|^2)^{-k} |\partial_t T^t f(v)|^2 \, dv \, d\bar{v}.$$

Multiplying by t^{2k+1} and integrating with respect to t we get

$$g_{k+1}(F, z)^2 \leqslant C g_k^*(f, z)^2.$$

Hence Theorem 3.2 is proved when $p > 2$. But it is easy to see that the adjoint of T_ϕ is also a multiplier of the same form and hence the Theorem 3.2 is completely proved.

Before going to the proof of Lemma 4.1 let us collect some facts about Laguerre polynomials which will be needed in the proof. If we set

$$\phi_n^\alpha(x) = r_n^{-1/2} e^{-x/2} L_n^\alpha(x) x^{\alpha/2}$$

where

$$r_n = \binom{n+\alpha}{n}$$

then $\{\phi_n^\alpha\}$ forms an orthonormal system for $L^2(0, \infty)$. We also observe that $r_n \sim n^\alpha$ as $n \to \infty$. The normalized Laguerre functions $\{\phi_k^\alpha\}$ satisfy the following generating function relation

$$(4.5)\quad \sum_{k=0}^{\infty} \phi_k^\alpha(x)\phi_k^\alpha(y)r^k$$

$$= \Gamma(\alpha+1)(1+r)^{-1}(-r)^{-\alpha/2}e^{-(x+y)/2}J_\alpha(2(-xyr)^{1/2}(1-r)^{-1})$$

where J_α is the Bessel function of order α. We also need the following recursion relation satisfied by the Laguerre polynomials

$$(4.6)\qquad kL_k^\alpha(x) = (-x+2k+\alpha-1)L_{k-1}^\alpha(x) - (k+\alpha-1)L_{k-2}^\alpha(x).$$

Finally the following asymptotic properties of the Bessel function are also needed

$$(4.7)\qquad |J_\alpha(z)| \leqslant C|z|^\alpha, \qquad |z| \leqslant 1$$

$$(4.8)\qquad |J_\alpha(iz)| \leqslant Cz^{-1/2}e^z, \qquad z \geqslant 1.$$

A good reference for all the above facts is Szego [10].

Let us start by proving (i). We write

$$\rho = \frac{1}{2}|z|^2.$$

Since ϕ is a bounded function and

$$\binom{N+n-1}{N} \sim N^{n-1}$$

we have

$$|\partial_t^k M(t,z)| \leqslant C \sum_{N=0}^{\infty} (2N+n)^{k+(n-1)/2} e^{-(2N+n)t}\left(\frac{N!}{(N+n-1)!}\right)^{1/2} L_N^{n-1}(\rho)e^{-\rho/2}.$$

Applying Schwarz inequality

$$|\partial_t^k M(t,z)|^2 \leqslant C\left\{\sum_{N=0}^{\infty} (2N+n)^{2k+n-1}e^{-(2N+n)t}\right\}$$

$$\times\left\{\sum_{N=0}^{\infty} e^{-(2N+n)t}\left(\frac{N!}{(N+n-1)!}\right)^{1/2} L_N^{n-1}(\rho)^2 e^{-\rho}\right\}.$$

The first term is $(-1)^{2k+n-1}$ times the $2k+n-1$ derivative of $e^{-nt}(1-e^{-2t})^{-1}$ and hence is bounded by constant times t^{-2k-n}. In view of (4.5) we have the formula

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$
$$= \frac{1}{2}\Gamma(n)(-1)^{-(n-1)/2}(\sinh t)^{-1} e^{-\rho \coth t} \rho^{-(n-1)} J_{n-1}(i\rho \operatorname{cosech} t).$$

When $\rho \operatorname{cosech} t \leqslant 1$, in view of (4.7) we get

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$
$$\leqslant C(\sinh t)^{-1} \rho^{-(n-1)} \rho^{n-1} (\operatorname{cosech} t)^{n-1} \leqslant C t^{-n}$$

On the other hand when $\rho \operatorname{cosech} t > 1$ we use (4.8) to get the estimate

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho}$$
$$\leqslant C(\sinh t)^{-1} e^{-\rho \coth t} \rho^{-(n-1)} \rho^{-1/2} (\operatorname{cosech} t)^{-1/2} e^{\rho \operatorname{cosech} t}$$
$$= C(\sinh t)^{-1/2} \rho^{-n+1/2} e^{-\rho \tanh t/2}.$$

Since $\rho > \sinh t$ we get

$$\sum_{N=0}^{\infty} e^{-(2N+n)t} \frac{N!}{(N+n-1)!} L_N^{n-1}(\rho) L_N^{n-1}(\rho) e^{-\rho} \leqslant C(\sinh t)^{-n} e^{-\rho \tanh t/2} \leqslant C t^{-n}.$$

Hence $|\partial_t^k M(t, z)|^2 \leqslant C t^{-2k-2n}$ and (i) is proved.

To prove (ii) we make use of the recursion relation (4.6). We write it in the following form

$$\rho L_N^{n-1}(\rho) = (2N+n) L_N^{n-1}(\rho) - (N+1) L_{N+1}^{n-1}(\rho) - (N+n-1) L_{N-1}^{n-1}(\rho).$$

Let us set

$$\psi(N) = (2N+n)^k e^{-(2N+n)t} \phi(2N+n).$$

In view of the recursion relation, as $\phi(2N+n) = 0$ for $N \leqslant n+1$,

$$\rho \partial_t^k M(t, z) = \sum [(2N+n)\psi(N) - N\psi(N-1) - (N+n)\psi(N+1)] L_N^{n-1}(\rho) e^{-\rho/2}$$

In terms of the operators Δ_+ and Δ_- we can write

$$\rho \partial_t^k M(t, z) = -\sum [N \Delta_- \Delta_+ \psi(N) + n \Delta_- \psi(N)] L_N^{n-1}(\rho) e^{-\rho/2}.$$

Under the assumptions made on ϕ we observe that the effect of multiplying $\partial_t^k M$ by ρ is essentially to change $\psi(N)$ into $N^{-1}\psi(N)$. Now we can iterate this process. Since k is even in the present case after applying $\rho(k/2)$ times we get

$$\rho^{k/2}\partial_t^k M(t,z) = \sum \psi_k(N) L_N^{n-1}(\rho) e^{-\rho/2}$$

where we have an estimate of the form

$$|\psi_k(N)| \leqslant C(2N+n)^{k/2} e^{-(2N+n)t}.$$

Therefore,

$$\int_{\mathbb{C}^n} |z|^{2k} |\partial_t^k M(t,z)|^2 \, dz \, d\bar{z} = C \int_0^\infty |\rho^{k/2} \partial_t^k M(t,z)|^2 \rho^{n-1} \, d\rho$$
$$= C \int_0^\infty \left| \sum \psi_k(N) e^{-\rho/2} \rho^{(n-1)/2} L_N^{n-1}(\rho) \right|^2 d\rho$$

since $\{\phi_N^{n-1}(\rho)\}$ form an orthonormal system the last integral is equal to

$$\sum \psi_k(N)^2 \binom{N+n-1}{n} \leqslant C \sum (2N+n)^{k+n-1} e^{-(2N+n)t} \leqslant C t^{-k-n}.$$

This proves (ii). The proof of (iii) is similar. Hence the lemma.

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