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Degenerations of the moduli spaces of vector bundles on curves II (Generalized Gieseker moduli spaces)

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Abstract. Let X_0 be a projective curve whose singularity is one ordinary double point. We construct a birational model $G(n,d)$ of the moduli space $U(n,d)$ of stable torsion free sheaves in the case $(n,d)=1$, such that $G(n,d)$ has normal crossing singularities and behaves well under specialization i.e. if a smooth projective curve specializes to X_0, then the moduli space of stable vector bundles of rank n and degree d on X specializes to $G(n,d)$. This generalizes an earlier work of Gieseker in the rank two case.

Keywords. Torsion free sheaves; Gieseker functor; moduli.

1. Introduction

Let X_0 be an irreducible projective curve whose singularity is one ordinary double point and arithmetic genus $g \geq 2$. Then one has the moduli space $U(n,d)$ of stable torsion free sheaves of rank n and degree d, which is projective if $(n,d)=1$. Further it is reduced, its singularities are known and it has good specialization properties when $(n,d)=1$ i.e. if a smooth projective curve specializes to X_0, then the corresponding moduli space of the smooth curve specializes to $U(n,d)$ (cf. [10], [11], these constructions and properties hold more generally when X_0 has only double point singularities. It need not be irreducible). Let $U(n,d)^0$ denote the open subscheme of $U(n,d)$, corresponding to vector bundles on X_0 (i.e. locally free torsion free sheaves on X_0). Then $U(n,d)$ is a compactification of $U(n,d)^0$ and $U(n,d)\backslash U(n,d)^0$ is the singular locus of $U(n,d)$ when $(n,d)=1$. One knows that these singularities are *not* normal crossings.

Gieseker has constructed a compactification G of $G^0 = U(2,1)^0$ (i.e. for the case of rank 2 and degree 1) such that the singularities of G are (analytic) normal crossings and it has good specialization properties (cf. [5]). The points of $G\backslash G^0$ consist of vector bundles E on curves which are semi-stably equivalent to X_0, more precisely they are curves of the form X_k with a morphism $\pi : X_k \longrightarrow X_0$ such that π is an isomorphism over $X_0\backslash\{p\}$ and $\pi^{-1}(p)$ is a chain R of projective lines (cf. Definition–Notation 2).

In this paper, we give a generalisation of Gieseker's construction for arbitrary rank. Gieseker's construction is based on m-Hilbert stability i.e. stability (in the GIT sense) of points of a Hilbert scheme corresponding to imbeddings of curves in Grassmannians. This approach is quite natural; however generalizing it to arbitrary rank seems complicated. Our method is different. It consists in establishing a relationship of the Gieseker moduli with the moduli of torsion free sheaves. This allows us to deduce the construction of these new moduli spaces from those of torsion free sheaves.

Let $(n,d)=1$. We construct a projective variety $G(n,d)$ which is a compactification of $G(n,d)^0=U(n,d)^0$ such that the singularities of $G(n,d)$ are (analytic) normal crossings (cf. Theorems 1 and 2). The points of $G(n,d)\setminus G(n,d)^0$ are again suitable vector bundles E on curves of the form X_k (modulo an equivalence relation, cf. Def. 3). We also get a canonical morphism

$$\pi_*: G(n,d)\longrightarrow U(n,d), \quad \text{defined by}$$
$$E\mapsto \pi_*(E) \ (\pi \text{ canonical morphism } X_k\longrightarrow X_0).$$

A crucial point in our construction is that a point of $G(n,d)$ which is a vector bundle E on X_k, is completely characterized by the following properties:

(1) the restriction of E to every component of R (which is $\simeq \mathbb{P}^1$) is of the form

$$\oplus\mathcal{O}(a_i), \ a_i\geq 0 \text{ and at least one } a_i>0$$

or

(1′) the global sections of $E|_R$ give a closed immersion of R into a Grassmannian.

(By tensorisation by a line bundle from X_0, (1) $\Longleftrightarrow$ (1′), in fact by this tensorisation (1) $\Longrightarrow$ the global sections of E define a closed immersion of X_k into a Grassmannian (cf. Proposition 4))

(2) $\pi_*(E)$ is torsion free and stable.

In fact, we were led to this, among other things, by the observation that Gieseker's list of vector bundles (cf. p. 176, [5]) in rank two and degree one is characterized by the above properties.

Now π_* can be defined at the functorial level

$$\pi_*: \mathcal{G}\longrightarrow \mathcal{U}$$

or in more concrete terms we have to define families of objects in $G(n,d)$ parametrized by schemes T and check that π_* defines families of objects in $U(n,d)$ parametrized by T (cf. Proposition 7 and Lemma 4). Another crucial point in our construction is that π_* is proper (cf. Proposition 10). This comes to proving that a morphism (related to π_*) from a total family representing $G(n,d)$ into one representing $U(n,d)$ is proper. Once this is done the construction of the moduli space $G(n,d)$ results from that of $U(n,d)$ and standard geometric invariant theory. Total families representing $G(n,d)$ already figure in Gieseker's work (based on (1)′ above, cf. [5]). They are open subsets of Hilbert schemes associated to imbeddings of X_k in Grassmannians.

To prove the specialization properties, we have only to carry over our construction when $\mathcal{X}\longrightarrow S$ is a proper, flat family of curves ($S=\operatorname{Spec} A$, A discrete valuation ring) such that the generic fibre over S is smooth and the closed fibre $\simeq X_0$. The fact that $G(n,d)$ has only normal crossing singularities is essentially in Gieseker (cf. [5]).

Our construction seems to work in a far more general context, say a family of *stable* curves (in the sense of Deligne–Mumford, cf. [3]).

Further, if $(n,d)\neq 1$, it should also be possible to construct the generalized semi-stable Gieseker moduli spaces, which we have only briefly sketched (cf. Remark 6).

An interesting fact which we will take up later is about the fibres of the morphism π_*. They happen to be the wonderful compactifications of the projective group (in the sense of De Concini–Procesi [2], see Remark 9).

We believe that our set up gives a proper understanding of Gieseker's work (cf. [5]). The generalized Gieseker moduli spaces should be considered as solutions of the moduli problem associated to the following objects over X_0:

$$\{(\pi, E), \pi \text{ a proper map } X' \longrightarrow X_0 \text{ which is an isomorphism over } X_0 \setminus \{p\}$$
$$\text{and } E \text{ a vector bundle on } X' \text{ such that } \pi_*(E) \text{ is torsion free}\}.$$

We have of course to fix the "invariants" for the moduli. In this context it seems interesting to ask for generalisations when X_0 is replaced by a higher dimensional variety (say even a smooth surface).

Our set up seems also to give a tool for a systematic investigation of compactifications of the moduli spaces of principal bundles with reductive structure groups, on curves with singularities (ordinary double points).

We would also like to remark that attempts to generalize our earlier work (cf. [8]) to rank ≥ 3, by taking suitable reducible curves and showing that the moduli spaces of stable (or semi-stable) torsion free sheaves on these curves, would have normal crossings as singularities, do not seem to succeed.

After completing this paper, we came to know of a preprint of Teixidor i Bigas (cf. [15]), which is related to our work.

2. Vector bundles over the curves X_k

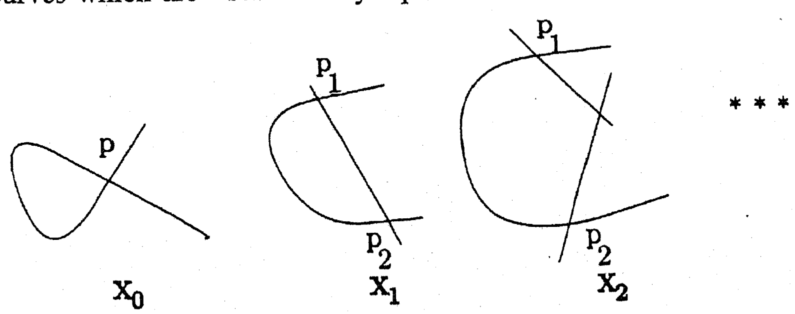
We work over an algebraic field k, which, for simplicity, we can assume to be the field of complex numbers.

DEFINITION-NOTATION 1

We call a scheme R, a chain of projective lines if $R = \cup_{i=1}^m R_i$, $R_i \simeq \mathbb{P}^1$, $R_i \cap R_j$ (for distinct i, j) is a single point if $|i - j| = 1$ and otherwise empty. We call m the length of R. Let E be a vector bundle of rank n on R. Then one knows that $E|_{R_i} \simeq \oplus_{j=1}^n \mathcal{O}(a_{ij})$, $a_{ij} \in \mathbb{Z}$. We say E is positive (≥ 0) if $a_{ij} \geq 0$, for all i and j. We say E is strictly positive (> 0) if it is positive and for every i, there is a j such that $a_{ij} > 0$. We say E is standard if $1 \geq a_{ij} \geq 0$ for all i,j and strictly standard if it is moreover strictly positive. If E is strictly standard, then $E_i = E|_{R_i} = L \oplus M$, where L is a direct sum of $\mathcal{O}(1)$'s and M is trivial. Then L is canonically defined and called the canonical sub-bundle of E_i and $E_i/L \simeq M$ is called the canonical quotient bundle of E_i.

DEFINITION-NOTATION 2

Let X_0 denote an irreducible projective curve which has just one ordinary double point 'p' as singular point. Let $\pi : X \longrightarrow X_0$ be the normalisation of X_0 and $\pi^{-1}(p) = \{p_1, p_2\}$. Let X_k be the curves which are "semi-stably equivalent to X_0":



i.e. X is a component of X_k $(k \geq 1)$ and if π denotes the canonical morphism $\pi : X_k \longrightarrow X_0$, $\pi^{-1}(p)$ is a chain R of projective lines of length k, passing through p_1 and p_2.

Let Z be a projective scheme with an ample line bundle $\mathcal{O}_Z(1)$ and E a vector bundle of rank n on Z. Then we see that if $H^0(E)$ generates E, through the evaluation map $H^0(E) \longrightarrow E_z$ (fibre of E at $z \in Z$) we get a canonical morphism

$$\phi_E = \phi : Z \longrightarrow \mathrm{Gr}(H^0(E), n) \text{ (Grassmannian of } n \text{ dim. quotients of } H^0(E))$$

such that the inverse image by ϕ of the canonical tautological quotient bundle on $\mathrm{Gr}(H^0(E), n)$ is isomorphic to E and conversely if this holds, H^0 generates E. If we choose further a basis of $H^0(E)$, we get in fact a canonical morphism of Z into $\mathrm{Gr}(m,n)$, the Grassmannian of rank n quotients of the standard vector space of rank m ($m =$ rank $H^0(E)$). Suppose then that ϕ is such a morphism. Then ϕ is injective, if given $z_1, z_2 \in Z$ $(z_1 \neq z_2)$, there exist n sections $s_1, \ldots, s_n$ of E such that $(s_1 \wedge \cdots \wedge s_n)(z_1) = 0$ and $(s_1 \wedge \cdots \wedge s_n)(z_2) \neq 0$. Another sufficient condition for injectivity is that for the exact sequence

$$0 \longrightarrow I_{z_1,z_2}E \longrightarrow E \longrightarrow E_{z_1} \oplus E_{z_2} \longrightarrow 0 \quad (I_{z_1,z_2} \text{ ideal sheaf of } z_1, z_2)$$

the induced sequence

$$0 \longrightarrow H^0(I_{z_1,z_2}E) \longrightarrow H^0(E) \longrightarrow H^0(E_{z_1} \oplus E_{z_2}) (= E_{z_1} \oplus E_{z_2}) \longrightarrow 0 \tag{1}$$

is exact.

To give the differential of ϕ, note that we have the following commutative diagram:

$$\begin{array}{ccccccccl} 0 & \longrightarrow & H^0(I_zE) & \longrightarrow & H^0(E) & \longrightarrow & E_z & \longrightarrow & 0, \text{ exact} \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & (I_z/I_z^2) \otimes E & \longrightarrow & E/I_z^2E & \longrightarrow & E_z & \longrightarrow & 0, \text{ exact.} \end{array} \tag{2}$$

Then for every linear form $l : I_z/I_z^2 \longrightarrow k$ i.e. l belongs to the tangent space T_z of Z at z, we get a linear map $f_l : H^0(I_zE) \longrightarrow E_z$ i.e. $f_l \in \mathrm{Hom}(H^0(I_zE), E_z)$, which is the tangent space of $\mathrm{Gr}(H^0(E), n)$ at $\phi(z)$. We see that $d\phi(l) = f_l$. A sufficient condition for the injectivity of $d\phi$ at z is that the first vertical arrow in (2) is surjective and this is implied by the surjectivity of the second vertical arrow in (2).

We see also that for the bundle $E(k)$, k sufficiently large, all these sufficient conditions (also the one in (1) above) are satisfied for all $z \in Z$ so that the morphism $\phi_{E(k)}$ is a closed immersion.

PROPOSITION 1

Let R be a chain of projective lines and E a positive vector bundle of rank n on R. Then we have:

(i) *for $x \in R$, the evaluation map $H^0(R,E) \longrightarrow E_x$ is surjective,*
(ii) *if R_1 is a subchain (in the obvious sense) of projective lines of R, then the restriction map $H^0(R,E) \longrightarrow H^0(R_1, E)$ is surjective,*
(iii) $H^1(R,E) = 0$.

If moreover E is strictly positive, we have:

(iv) *Given $x, y \in R (x \neq y)$, $\exists\, s_1, \ldots, s_n \in H^0(R,E)$ such that $(s_1 \wedge \cdots \wedge s_n)(x) \neq 0$ and $(s_1 \wedge \cdots \wedge s_n)(y) = 0$.*

(v) *The canonical morphism*

$$\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$$

is a closed immersion.

Proof. The assertions are obvious when $R \simeq \mathbb{P}^1$. Then they are proved by induction on $l(R) =$ length of R. The proofs of (i) and (ii) are rather immediate. For (iii) we cut R into subchains R_1 and R_2 such that $R = R_1 \cup R_2$ and $R_1 \cap R_2$ reduces to a point q. Then we have the "patching" exact sequence (of sheaves)

$$0 \longrightarrow E \longrightarrow E|_{R_1} \oplus E|_{R_2} \xrightarrow{j} E_q \longrightarrow 0$$

where the map j is defined by $(s_1, s_2) \mapsto s_1(q) - s_2(q)$. By (ii) $H^0(E|_{R_1} \oplus E|_{R_2}) \longrightarrow E_q$ is surjective. Then since $H^1(E_q) = 0$, we see that

$$0 \longrightarrow H^1(E) \longrightarrow H^1(E|_{R_1}) \oplus H^1(E|_{R_2}) \longrightarrow 0$$

is exact. By the induction hypothesis the last term is zero. Hence $H^1(E) = 0$.

To prove (iv), we write $R = R_1 \cup R_2$, R_i subchains with $R_1 \cap R_2 = \{q\}$ and (in fact we can even assume $R_2 = \mathbb{P}^1$). Because of the induction hypothesis and (i), we see easily that if $x, y \in R_1$ or $x, y \in R_2$, we are done. Then we have only to consider the case $x \in R_1$, $y \in R_2$ and $x \neq q$. Then by the induction hypothesis, we can find $s'_1, \ldots, s'_n \in H^0(E|_{R_1})$ such that $(s'_1 \wedge \cdots \wedge s'_n)(x) \neq 0$ and $(s'_1 \wedge \cdots \wedge s'_n)(q) = 0$. Then $s'_1(q), \ldots, s'_n(q)$ are linearly dependent in E_q so that say

$$s'_1(q) = \sum_{i=2}^{n} a_i s'_i(q), \quad a_i \in k.$$

Then we can find $t_i \in H^0(E|_{R_2})$ such that $t_i(q) = s'_i(q)$ for $i \geq 2$ by (1). We set $t_1 = \sum_{i=2}^{n} a_i t_i$, $t_1 \in H^0(E|_{R_2})$. Then s'_i and t_i patch up to define s_i which have the required properties.

By (iv), the morphism ϕ is injective and hence to prove (v), we have only to show $d\phi$ is injective at all $x \in R$. Again we express $R = R_1 \cup R_2$ as above. Then by induction

$$\phi_i : R_i \longrightarrow \mathrm{Gr}(H^0(E|_{R_i}), n), \quad i = 1, 2$$

are closed immersions. Then only to show $(d\phi)_q$ is injective. Now by (ii), the canonical maps $H^0(E) \longrightarrow H^0(E|_{R_i})$, $i = 1, 2$, are surjective. Hence we get closed immersions

$$\mathrm{Gr}(H^0(E|_{R_i}), n) \hookrightarrow \mathrm{Gr}(H^0(E), n), \quad i = 1, 2.$$

Besides, we have the exact sequence

$$0 \longrightarrow H^0(E) \longrightarrow H^0(E|_{R_1}) \oplus H^0(E|_{R_2}) \longrightarrow E_q \longrightarrow 0.$$

This implies that the intersection

$$\mathrm{Gr}(H^0(E|_{R_1}), n) \cap \mathrm{Gr}(H^0(E|R_2), n) \quad \text{in } \mathrm{Gr}(H^0(E), n)$$

reduces to one point. It is not difficult to see that this intersection is transversal at this point. This implies that

$$(d\phi_1)|_q(T_{R_1,q}) \cap (d\phi_2)(T_{R_2,q}) = 0$$
($T_{R_i,q}$ is the tangent space to R_i at q)

from which it follows that $(d\phi)_q$ is injective. This proves Proposition 1.

Remark 1. Conversely, if E is a vector bundle on R such that the canonical morphism $\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$ is a closed immersion, we see that E is strictly positive.

PROPOSITION 2

Let E be a vector bundle of rank n on R (chain of projective lines) with $E|_{R_i} = \oplus_{j=1}^n \mathcal{O}(a_{ij})$ *(R_i-ith $\mathbb{P}^1$ component of R). Then we have*

(i) *(Riemann–Roch for E)*

$$\chi(E) = \sum_i \sum_j a_{ij} + n = \sum_i \deg E|_{R_i} + n = \deg E + n$$

where $\deg E = \sum_i \deg E|_{R_i}$ *(sometimes* $\deg E$ *is called the "total degree" of E).*

(ii) *if E is positive*

$$h^1(E) = 0 \quad \textit{and } h^0(E) = \deg E + n.$$

The proof of this proposition is left as an easy exercise.

PROPOSITION 3

Let X_k *be the curve together with the canonical morphism* $\pi : X_k \longrightarrow X_0$ *so that* $\pi^{-1}(p)$ *is a chain R of projective lines of length k. Let E be a vector bundle of rank n on* X_k *such that* $E|_R$ *is positive. Then we have*

(i) $\pi_* \mathcal{O}_{X_k} \simeq \mathcal{O}_{X_0}$.
(ii) $R^i \pi_*(E) = 0,\ i > 0$.
(iii) $H^i(X_k, E) \simeq H^i(X_0, \pi_*(E))$.
(iv) *If E is trivial on R, then* $\pi_*(E)$ *is a vector bundle (i.e. locally trivial) and* $E \simeq \pi^*(\pi_*(E))$.
(v) *If* $H^1(X, I_{p_1,p_2}E|_X) = 0$, *then* $H^1(X_k, E) = 0$ *so that in this case we have also* $H^1(X_0, \pi_*(E)) = 0$.

Proof. The proof of (i) is rather immediate and we leave it. Let V be an affine neighbourhood of p, $U = \pi^{-1}(V)$ and $V' = U \cap X$. Then V' is affine and we have the "patching" exact sequence

$$0 \longrightarrow E|_U \longrightarrow E|_{V'} \oplus E|_R \longrightarrow T (= E_{p_1} \oplus E_{p_2}) \longrightarrow 0. \qquad (*)$$

Since V' is affine, the canonical (restriction) map $E|_{V'} \longrightarrow E_{p_1} \oplus E_{p_2}$ is surjective so that we get

$$H^i(E|_U) \simeq H^i(E|_{V'}) \oplus H^i(E|_R), \quad i > 0.$$

The RHS is zero and the assertions (ii) and (iii) follow. When $E|_R$ is trivial, $H^0(E|_R)$ is of dimension n and we get canonical isomorphisms of E_{p_i} with $H^0(E|_R)$, which leads to a canonical isomorphism $\theta : E_{p_1} \longrightarrow E_{p_2}$. We see that $H^0(E|_U)$ identifies with the subspace of $H^0(E|_{V'})$ consisting of elements s such that $\theta \cdot s(p_1) = s(p_2)$. This shows that $\pi_*(E)$

$(\pi_*(E)|_V = H^0(E|_U))$ identifies with the vector bundle on X_0 defined by $E|_X$ on the normalisation X and the patching condition $\theta : E_{p_1} \longrightarrow E_{p_2}$ and then (iv) follows:

To prove (v) note that the hypothesis implies that the canonical map $H^0(E|_X) \longrightarrow E_{p_1} \oplus E_{p_2}$ is surjective and $H^1(E|_X) = 0$. Consider the patching exact sequence:

$$0 \longrightarrow E \longrightarrow E|_X \oplus E|_R \longrightarrow T \longrightarrow 0.$$

Again we have $H^1(E) \simeq H^1(E|_X) \oplus H^1(E|_R)$, which implies $H^1(E) = 0$. This proves the proposition.

Remark 2. Let E be a positive vector bundle of rank n on a chain R of projective lines. Then we have a morphism $\pi : R \longrightarrow R'$ which contracts all the R_i in R ($R_i \simeq \mathbb{P}^1$) such that $E|_{R_i}$ is trivial and we have $E \simeq \pi^*(\pi_*(E))$ with $\pi_*(E)$ a strictly positive vector bundle on R' (for example by the same type of argument as for (v) of the above Proposition). By (i) of Proposition 1, we get a canonical morphism $\phi : R \longrightarrow \mathrm{Gr}(H^0(E), n)$ and the image is again a chain of projective lines.

PROPOSITION 4

Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. If $F = E \otimes \pi^*(\mathcal{O}_{X_0}(l))$, then for $l \gg 0$ (more precisely if the conditions (a), (b), (c), (d) *in the proof below are satisfied), $H^0(F)$ generates F and the canonical morphism $\phi : X_k \longrightarrow \mathrm{Gr}$* $(H^0(F), n)$ is a closed immersion.

Further (for $l \gg 0$), $H^1(X_k, F) = 0$ so that by Prop. 3, we have

$$\begin{cases} H^0(X_k, F) \simeq H^0(X_0, \pi_*(F)), \text{ and} \\ H^i(X_k, F) = H^i(X_0, \pi_*(F)) = 0, \ i > 0. \end{cases}$$

Note that $E|_R \simeq F|_R$.

Proof. If $\pi^*(\mathcal{O}_{X_0}(1))$ were ample, this proposition is an easy consequence of the previous considerations, but this is not the case ($k \geq 1$). However, $\pi^*(\mathcal{O}_{X_0}(1)|_X$ is ample. Hence for $l \gg 0$, we see that $H^1(X, I_{p_1,p_2}F|_X) = 0$, so that by Prop. 3, $H^1(X_k, F) = 0$ and the last assertions of the proposition follow. Thus it remains only to show that ϕ is a closed immersion.

We can now suppose that if $l \gg 0$, the following conditions are satisfied:

(a) $H^1(X, I_{p_1,p_2}F|_X) = 0$, which implies that the canonical map $H^0(X, F|_X) \longrightarrow F_{p_1} \oplus F_{p_2}$ is surjective,

(b) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow I_{p_1,p_2}F|_X / I^2_{p_1,p_2}F|_X$$

is surjective,

(c) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow F|_X / I_x^2 F|_X$$

is surjective for $x \in X \setminus \{p_1, p_2\}$,

(d) The canonical map

$$H^0(X, I_{p_1,p_2}F|_X) \longrightarrow F_{x_1} \oplus F_{x_2}$$

is surjective for $x_1, x_2 \in X \setminus \{p_1, p_2\}$, $x_1 \neq x_2$. We shall now see that these properties imply that ϕ is a closed immersion.

By (a) we see that the canonical restriction map

$$H^0(X_k, F) \longrightarrow H^0(R, F|_R) \qquad \text{(i)}$$

is surjective. However the canonical map

$$H^0(X_k, F) \longrightarrow H^0(X, F|_X)$$

need not be surjective. If this were the case, the proof of the proposition would be straightforward. We have to do a little work to circumvent this minor difficulty.

We first observe that the image of the map $H^0(X_k, F) \longrightarrow H^0(X, F|_X)$ contains $H^0(X, I_{p_1, p_2} F|_X)$ i.e. sections of $F|_X$ vanishing at p_1, p_2. By this remark and (d), we see that

$$H^0(X_k, F) \longrightarrow F_{x_1} \oplus F_{x_2}, \quad x_1, x_2 \in X \setminus \{p_1, p_2\}, \quad x_1 \neq x_2$$

is surjective. Besides, because of (i) above and the fact that the canonical morphism

$$R \longrightarrow \mathrm{Gr}(H^0(F|_R), n) \qquad \text{(ii)}$$

is a closed immersion, we see that $H^0(X_k, F)$ generates F and the canonical morphism

$$\phi : X_k \longrightarrow \mathrm{Gr}(H^0(X_k, F), n)$$

is in fact injective. In a similar manner, we see that (c) implies that the canonical map

$$H^0(X_k, F) \longrightarrow F/I_x^2 F, \quad x \in X \setminus \{p_1, p_2\}$$

is surjective. By the remarks on imbeddings into Grassmannians, these observations imply that $(d\phi)$ is injective at all $x \in X_k \setminus \{p_1, p_2\}$. Thus to complete the proof of the proposition, we have only to show that $(d\phi)$ is injective at p_1, p_2 respectively.

We have a canonical map

$$H^0(X_k, I_{X_k, p_1} F) \longrightarrow (I_{X_k, p_1}/I^2_{X_k, p_1}) \otimes E_{p_1}.$$

We have

$$(I_{X_k, p_1}/I^2_{X_k, p_1}) \otimes F_{p_1} \simeq (I_{X, p_1}/I^2_{X, p_1} \oplus I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1}.$$

Hence we get canonical linear maps

$$H^0(X_k, I_{X_k, p_1}, F) \xrightarrow{j_1} (I_{X, p_1}/I^2_{X, p_1}) \otimes F_{p_1}$$

$$H^0(X_k, I_{X_k, p_1} F) \xrightarrow{j_2} (I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1}.$$

Now $(I_{X, p_1}/I^2_{X, p_1})$ (resp. $I_{R, p_1}/I^2_{R, p_1}$) are 1-dimensional and therefore the RHS of the above maps can be identified with F_{p_1}. To prove injectivity of $d\phi$ at p_1, we have only to show $j_1 \neq 0$ and $j_2 \neq 0$ and they are not linearly dependent.

We have the following commutative diagram

$$\begin{array}{lll} H^0(X_k, I_{X_k, p_1} F) & \xrightarrow{j_2} & (I_{R, p_1}/I^2_{R, p_1}) \otimes F_{p_1} \\ \quad f_2 \downarrow & \nearrow g_2 & \\ H^0(R, I_{R, p_1} F|_R) & & \end{array}$$

By (i) above, f_2 is surjective and by (ii), $g_2 \neq 0$. Hence $j_2 \neq 0$. We see that

$$H^0(X, I_{p_1, p_2}F|_X) \subset \ker f_2 \subset \ker j_2. \tag{iii}$$

Similarly for j_1, we have a commutative diagram

$$\begin{array}{l} H^0(X_k, I_{X_k, p_1}F) \xrightarrow{j_1} (I_{X, p_1}/I^2_{X, p_1}) \otimes F_{p_1} \\ \quad f_1 \downarrow \qquad \nearrow g_1 \\ H^0(X, I_{X, p_1}F|_X) \end{array}.$$

Now the image of f_1 contains $H^0(X, I_{p_1, p_2}F|_X)$ and by (b) g_1 restricted to this subspace is surjective. This implies that $j_1 \neq 0$. We can identify $H^0(X, I_{p_1, p_2}F|_X)$ as also a subspace of $H^0(X_k, I_{X_k, p_1}F)$ and then f_1 is an isomorphism restricted to these spaces. Then we see that there are elements of $H^0(X, I_{p_1, p_2}F|_X)$ which are not in the kernel of j_1. Then by (iii), $\ker j_1 \neq \ker j_2$ which implies that j_1 and j_2 are linearly independent. This completes the proof of the proposition.

Consider the vector bundles E on X_k such that $E|_R$ are strictly positive. Our next aim is to characterize those E such that $\pi_*(E)$ are torsion free $(\pi : X_k \to X_0)$. This characterization involves only properties of $E|_R$.

Lemma 1. *Let E be a strictly standard vector bundle on* $\mathbb{P}^1$ *(see Def.* 1) *and* $x, y \in \mathbb{P}^1$, $x \neq y$. *Let* L_x *be a linear subspace of* E_x. *Let V be the linear subspace of* $H^0(E)$ *consisting of sections s such that* $s(x) \in L_x$. *Then we have the following:*

(i) *The canonical evaluation map* $V \to L_x$ *is surjective.*
(ii) *Let* L_y *be the image of V in* E_y. *Then* $L_y \supset K_y$, *K being the canonical subbundle of E. (cf. Def.* 1). *Further*

$$\dim(L_y/K_y) = \dim(\textit{Image of } L_x \textit{ in } V_x/K_x).$$

(iii) *Let* $Q =$ *the canonical quotient bundle* E/K *of E. Then we have a well-defined subbundle F of E such that* $K \subset F$ *and* $F_x/K_x =$ *Image of* L_x *in* $Q_x = E_x/K_x$. *Further,* $F_y = L_y$. *In fact the image of* L_x *in* Q_x *defines a subbundle* Q' *of Q such that* $Q'_x =$ *this image and F is the inverse image of* Q' *by the canonical homomorphism* $E \to Q$.

Proof. Recall that K is a direct sum of $\mathcal{O}(1)$ and $Q = E/K$ is trivial. The statement (iii) essentially gives the proof which is left as an easy exercise.

Remark 3. Let R be a chain of projective lines and E a strictly standard vector bundle on R. We denote by K_i the standard subbundle of $E_i = E|_{R_i}$ (R_i are the $\mathbb{P}^1$-components of R). We denote by q_i the point $R_i \cap R_{i+1}$. Let $p_1 = q_0 (\neq q_1)$ be a point on R_1 and $p_2 = q_k$ $(\neq q_{k-1})$ be a point on R_k.

Let S_i be the subchain $S_i = R_1 \cup \cdots \cup R_i$. Then by iterating the method in Lemma 1, given a linear subspace L of $E_{p_1} = E_{q_0}$, we get a linear subspace L_i of E_{q_i} such that if V_i is the subspace consisting of $s \in H^0(S_i, E|_{S_i})$ such that $s(q_0) \in L$, then the evaluation map $V_i \to L$ is surjective and the image of V_i in E_{q_i} is L_i.

In particular, if we take $L = (0)$, we denote the subspace L_k of $E_{q_k} = E_{p_2}$ by M.

We have the following:

Lemma 2. *We keep the notations as in Remark* 3. *Then the following are equivalent:*

(a) $\dim M = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_k$,
(b) *if* $s \in H^0(R, E)$ *and s vanishes at* p_1, p_2, *then s vanishes identically.*

Proof. We first observe that

$$\text{(a)} \iff \dim L_i = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_i, 1 \leq i \leq k.$$

where L_i are the spaces defined as above for $L = (0)$. By induction, we can assume that the lemma holds for $S_i, i \leq k-1$. Suppose that (a) holds. Let $s \in H^0(R, E)$ be such that s vanishes at p_1 and p_2. Then $s(q_{k-1}) \in L_{k-1}$. But then by (a) we see that $(K_k)_{q_{k-1}} \cap L_{k-1} = (0)$. On the other hand, since $s(q_k) = 0$, we see that $s(q_{k-1}) \in (K_k)_{q_{k-1}}$. Hence $s(q_{k-1}) = 0$. This implies that s vanishes identically on R_k as well as on S_{k-1} by the induction hypothesis i.e. s vanishes identically on R.

Suppose now (b) holds. Then if (a) does not hold, we see that there is a j such that

$$\dim L_{j-1} = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_{j-1},$$
$$\text{and } (K_j)_{q_{j-1}} \cap L_{j-1} \neq (0).$$

We see that without loss of generality, we can suppose that $j = k$ i.e. we have

$$\dim L_{k-1} = \operatorname{rk} K_1 + \cdots + \operatorname{rk} K_{k-1},$$
$$\text{and } (K_k)_{q_{k-1}} \cap L_{k-1} \neq (0).$$

Then if $x \in (K_k)_{q_{k-1}} \cap L_{k-1}, x \neq 0$, we have a section s' of E on S_{k-1} such that $x = s'(q_{k-1})$ and $s'(q_0)$(i.e. $s'(p_1)) = 0$. On the other hand we have a section s'' of E on R_k such that $s''(q_{k-1}) = x$ and $s''(q_k) = 0$. Then s'' and s' patch up to define a section s of E on R such that s vanishes at p_1 and p_2 and s not identically zero. This gives a contradiction and the lemma follows.

Lemma 3. *Let R be a chain of projective lines and E a vector bundle on R such that $E|_{R_i} \simeq \oplus \mathcal{O}(a_{ij})$. If some $a_{ij} \geq 2$, say $a_{1j} \geq 2$ (without loss of generality). Then there is a section $s \in H^0(R, E)$ such that $s(p_1) = 0 = s(p_2)$ and s is not identically zero.*

Proof. There is a section θ of $\mathcal{O}(a_{1j})$ such that θ vanishes at q_0 and q_1 and θ not identically zero. Hence there is a section s' of $E|_{R_1}$ vanishing at q_0 and q_1 and which is not identically zero. We can obviously extend s' to a nonzero section s of E on R such that s vanishes at p_1 and p_2. This proves the lemma.

DEFINITION 3

Let F be a torsion free sheaf on X_0. Then one knows (cf. [8], [10]) that locally at the singular point p, F is of the form

$$F \simeq \bigoplus_{i=1}^{a} m \oplus \bigoplus_{i=1}^{b} \mathcal{O},$$

m being the maximal ideal of $\mathcal{O} = \mathcal{O}_{X,p}$. We refer to a as the type of F at p.

PROPOSITION 5

Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. Then we have the following:

(A) *$\pi_*(E)$ is torsion free on X_0 ($\pi : X_k \to X_0$) if and only if a global section s of $E|_R$ vanishing at p_1 and p_2 vanishes identically. Thus by Lemmas 1, 2 and 3, we see that $\pi_*(E)$*

is torsion free if and only if

(1) $E|_R$ *is strictly standard, and*
(2) *the condition (a) of Lemma* 2 *holds.*

Note that (2) implies in particular that

(i) $k \leq n$, in fact
(ii) $k \leq \deg E|_R = \sum_i \deg E|_{R_i} \leq n$
(iii) if $k \geq 2$ and $R_i \cap R_j \neq \emptyset$, the intersection of the linear subspaces of $E|_{R_i \cap R_j}$, determined by the canonical sub-bundles of $E|_{R_i}$ and $E|_{R_j}$, is zero.

(B) *If* $\pi_*(E)$ *is torsion free, its type at 'p' is* $\deg E|_R$.

Proof. We have the following exact sequence of $\mathcal{O}_{X_k}$-modules

$$0 \to I_X E \to E \to E|_X \to 0,$$

I_X – ideal sheaf of X. Note that $I_X E$ can be identified with $I_{p_1,p_2}E|_R$ – the sheaf of sections of $E|_R$ vanishing at p_1, p_2. Then we have the exact sequence

$$0 \to \pi_*(I_{p_1,p_2}E|R) \to \pi_*(E) \to \pi_*(E|_X).$$

Now $\pi_*(E|_X)$ is torsion free on X_0 and it is clear that $\pi_*(I_{p_1,p_2}E|R)$ is a torsion sheaf, in fact its support is at p. Hence it follows that the torsion subsheaf of $\pi_*(E)$ is precisely $\pi_*(I_{p_1,p_2}E|R)$. It is clear that $\pi_*(I_{p_1,p_2}E|R)$ is the sheaf determined by the k vector space $H^0(R, I_{p_1,p_2}E|R)$ considered as an $\mathcal{O}_{X_0,p}$-module. From these remarks the assertion (A) follows.

To prove (B) consider the exact sequence

$$0 \to I_R E \to E \to E|_R \to 0.$$

This gives the following exact sequence of $\mathcal{O}_{X_0,p}$-modules

$$0 \to \pi_*(I_{p_1,p_2}E|_X)_{(p)} \to \pi_*(E)_{(p)} \to \pi_*(E|_R)_{(p)} \to 0,$$

where the suffix 'p' indicates taking stalks of the sheaves at p (e.g. $\pi_*(E)_{(p)}$ is the stalk of $\pi_*(E)$ at p, and hence an $\mathcal{O}_{X_0,p}$-module).

We claim that

$$m_{X_0,p}(\pi_*(E)_{(p)}) = \pi_*(I_{p_1,p_2}E|_X)_{(p)}, \qquad (*)$$

where $m_{X_0,p}$ is the maximal ideal of $\mathcal{O}_{X_0,p}$. We shall now show that $(*)$ implies the assertion (B).

Now $(*)$ implies that

$$\pi_*(E)_{(p)}/m_{X_0,p}(\pi_*(E)_{(p)}) \simeq \pi_*(E|_R)_{(p)}.$$

Now $\pi_*(E|_R)_{(p)}$ is annihilated by $m_{X_0,p}$ and in fact the k vector space $H^0(R, E|_R)$ considered as an $\mathcal{O}_{X_0,p}$-module. We have seen (cf. Proposition 2) that

$$\dim H^0(R, E|_R) = \deg E|_R + n.$$

Now if F is a torsion free $\mathcal{O}_{X_0,p}$-module of type a i.e. $F = \oplus_1^a m_{X_0,p} \oplus \oplus_1^{n-a} \mathcal{O}_{X_0,p}$, then

$$\dim F/m_{X_0,p}F = a + n,$$

since $\dim m_{X_0,p}/m^2_{X_0,p} = 2$. Now the assertion (B) follows. Thus it remains to prove the claim $(*)$ above. Choosing a trivialisation of $E|_X$ in a neighbourhood of p_1, p_2, we can consider $\pi_*(E)_{(p)}$ as the trivial module of rank n over the semi-local ring $\mathcal{O}_{X,p_1,p_2}$ of X at p_1, p_2. Then $\pi_*(I_{p_1,p_2}E|_X)_{(p)}$ is precisely its submodule vanishing at p_1, p_2. Then it suffices to prove the claim for $n=1$ and then the claim is just the statement that the radical of $\mathcal{O}_{X,p_1,p_2}$ identifies with $m_{X_0,p}$. This completes the proof of the proposition.

Remark 4. Let E be a vector bundle of rank n on X_k such that $E|_R$ is strictly positive. Then we have seen in the proof of (B) above, that if $\pi_*(E)$ is torsion free we have

$$m_{X_0,p}(\pi_*(E)_{(p)}) = \pi_*(I_{p_1,p_2}E|_X)_{(p)}.$$

This is equivalent to saying that

$$I_{X_0,p}(\pi_*(E)) = \pi_*(I_{p_1,p_2}E|_X) \qquad (*)$$

as on the R.H.S. π can be taken as the normalisation map $X \to X_0$ and then as π is an isomorphism over points outside p ($I_{X_0,p}$ is the ideal sheaf defined by p). Now we claim the following:

(i) $\pi_*(E)$ determines $E|_X$ i.e. if E_1 and E_2 on X_k (possibly for two different X_k with $E_i|_R$ strictly positive) are such that $\pi_*(E_i)$ are torsion free and $\pi_*(E_1) \simeq \pi_*(E_2)$, then $E_1|_X \simeq E_2|_X$.

(ii) if we have a family of vector bundles $\{E\}$ (on X_k's with $E|_R$ strictly positive) such that $\{\pi_*(E)\}$ is a bounded family of torsion free sheaves on X_0, then for $\ell \gg 0$ (independent of E) $\{E|_X\}$ is a bounded family and we have the properties of Proposition 4. i.e. for $F = E \otimes \pi^*(\mathcal{O}_{X_0}(\ell)), H^0(F)$ generates F and the canonical morphism $\phi : X_k \to \mathrm{Gr}(H^0(F), n)$ is a closed immersion. Besides $H^1(X_k, F) = 0$ and

(a) $H^0(X_k, F) \simeq H^0(X_0, \pi_*(F))$
(b) $H^i(X_k, F) = H^i(X_0, \pi_*(F)) = 0,\ i \geq 1.$

To prove these claims, we require

(iii) if $\pi : X \to X_0$ is the normalisation map, the functor π_* : (vector bundles on X) $\to$ (torsion free sheaves on X_0) is faithful i.e.

$$\mathrm{Hom}(V_1, V_2) \simeq \mathrm{Hom}(\pi_*(V_1), \pi_*(V_2));$$

in particular $V_1 \simeq V_2 \Longleftrightarrow \pi_*(V_1) \simeq \pi_*(V_2)$.

Let us assume (iii). Let E_i $(i=1,2)$ be as in (i) above. Then by $(*)$ if $\pi_*(E_1) \simeq \pi_*(E_2)$, we see that

$$I_{p_1,p_2}E_1|_X \simeq I_{p_1,p_2}E_2|_X,$$

multiplying (i.e. tensoring) by the line bundle $I^{-1}_{p_1,p_2}$, we deduce that $E_1|_X \simeq E_2|_X$. This proves the claim (i) above.

Now if $F = E \otimes \pi^*(\mathcal{O}_{X_0}(\ell))$ as in (ii) above, we see that

$$\pi_*(F) = \pi_*(E)(\ell),$$

so that $\pi_*(F)$ is torsion free and $F|_R \simeq E|_R$ is strictly positive. Then we have as in $(*)$

$$I_{X_0,p}(\pi_*(F)) = \pi_*(I_{p_1,p_2}F|_X).$$

If $\{\pi_*(E)\}$ is a bounded family, we can suppose that for $\ell \gg 0$ the LHS is generated by global sections, its H^1 is zero and without loss of generality that its dimension is independent of F. Hence if we set $W = I_{p_1,p_2}F|_X$, then for the family of vector bundles $\{W\}$ on X, we can suppose that $H^0(\pi_*(W))$ generates $\pi_*(W)$ (here $\pi : X \to X_0$ is the normalisation map), $\dim H^0(\pi_*(W))$ is independent of W and $H^1(\pi_*(W)) = 0$.

Now $H^0(\pi_*(W)) \simeq H^0(W)$ ($\pi : X \to X_0$ is an affine morphism) and we see that

$$H^0(\pi_*(W)) \text{ generates } \pi_*(W) \Rightarrow H^0(W) \text{ generates } W$$

(for

$$\pi_*(W)/m_{X_0,p}\pi_*(W) \simeq W_{p_1} \oplus W_{p_2},$$

W_{p_i} are the fibers of the vector bundle W at p_i). Hence for the family $\{W\}$, every W is the quotient of a trivial bundle, whose rank is independent of W and our hypothesis implies that $H^1(W) = 0$, so that the degree of W is independent of W. Hence one knows that $\{W\}$ is a bounded family (by the theory of Quot schemes). Thus we see that $\{I_{p_1,p_2}F|_X\}$ is a bounded family which implies that $\{F|_X\}$ and hence $\{E|_X\}$ is a bounded family. Then by the same arguments as in Proposition 4, we see that the claim (ii) follows.

Thus it remains to prove the above claim (iii). We see this claim is local with respect to X_0. If $A = \mathcal{O}_{X_0,p}$ and B is the semi-local ring of X at p_1, p_2, then B is the integral closure of A. The V_i in (iii) corresponds to the free B module B^n and the assertion (iii) reduces to showing that an A module isomorphism $B^n \to B^n$ is in fact a B module isomorphism. Such an A module isomorphism is an $(n \times n)$ matrix with entries in $\mathrm{Hom}_A(B,B)$ and thus it suffices to show that $\mathrm{Hom}_A(B,B) \simeq B$ (multiplication by elements of B). This is easy and well-known. This proves (iii).

3. The moduli space

We shall hereafter assume that the arithmetic genus g of X_0 is ≥ 2.

DEFINITION 4

(i) Let E be a vector bundle on X_k such that $E|_R$ is strictly positive. It is said to be stable if $\pi_*(E)$ ($\pi : X_k \to X_0$) is a stable (torsion free) sheaf on X_0 (cf. [10]). Note that it has then all the nice properties stated in Proposition 5, in particular $E|_R$ is standard, $k \leq n$, etc.

(ii) We call two vector bundles E_1, E_2 on X_k to be equivalent if $E_1 \simeq g^*(E_2)$, where g is an automorphism of X_k, which is identity on the component X (g could move points on R).

(iii) We set

$$G(n,d)_k = \begin{cases} \text{equivalence classes of stable vector bundles} \\ \text{(in the sense of (i) and (ii)) on } X_k \text{ of rank } n \\ \text{and degree } d \end{cases}$$

$$G(n,d) = \coprod_{k \leq n} G(n,d)_k \text{ (disjoint sum).}$$

Note that $G(n,d)_0$ is the set of isomorphism classes of stable vector bundles of rank n and degree d on X_0.

We shall see that if n and d are coprime, $G(n,d)$ is a projective variety with a birational morphism onto the projective variety $U(n,d)$ of stable torsion free sheaves of rank n and degree d on X_0 (cf. Theorems 1 and 2) and it has all the good properties like specialisation stated in the introduction. One can also define semi-stability and its moduli space but it has to be done in a more subtle manner.

Remark 5. Let L be a line bundle on X_0. If E is a vector bundle on X_k, then since

$$\pi_*(E \otimes \pi^*(L)) = \pi_*(E) \otimes L$$

we note that $\pi_*(E)$ is torsion free (resp. stable) if and only if $\pi_*(E \otimes \pi^*(L))$ is torsion free (resp. stable). Then by (ii) of Remark 3 (since stable torsion free sheaves on X_0 of rank n and degree d form a bounded family), we can find $\ell \gg 0$ such that for any $E \in G(n,d)$ (or rather a vector bundle represented by an element of $G(n,d)$), $F = \pi_*(E \otimes \pi^*(\mathcal{O}_{X_0}(\ell))$ is generated by its global sections, the canonical morphism

$$\phi_F : X_k \to \mathrm{Gr}(H^0(F), n)$$

is a closed immersion and the properties (ii) (a), (b) of Remark 3 are also satisfied. We see that $m = \dim(H^0(F))$ and $e = \deg(F)$ are independent of F. We obtain a bijection

$$G(n,d)_k \xrightarrow{\sim} G(n,e)_k,$$
$$E \mapsto F = \pi_*(E \otimes \pi^*(\mathcal{O}_{X_0}(\ell)).$$

Thus without loss of generality, we can suppose that if E is a vector bundle representing an element of $G(n,d)_k$ i.e. a stable vector bundle on X_k, $H^0(E)$ generates E, the canonical morphism

$$\phi_E : X_k \to \mathrm{Gr}(H^0(E), n)$$

is a closed immersion and properties (ii) (a), (b) of Remark 4 are satisfied.

If we choose a basis of $H^0(E)$, $\mathrm{Gr}(H^0(E), n)$ can be identified with the standard Grassmannian $\mathrm{Gr}(m,n)$ ($m = \dim H^0(E)$), and ϕ_E identified with a morphism (we denote it again by ϕ_E)

$$\phi_E : X_k \to \mathrm{Gr}(m,n).$$

Now $PGL(m)$ operates canonically on $\mathrm{Gr}(m,n)$ and also on $X_0 \times \mathrm{Gr}(m,n)$ by taking the identity action on X_0. Now ϕ_E gives rise to a closed immersion

$$\psi_E : X_k \hookrightarrow X_0 \times \mathrm{Gr}(m,n), \quad \psi_E = (\pi, \phi_E).$$

Let E_1, E_2 be stable vector bundles of rank n and degree d on X_k and ψ_{E_1}, ψ_{E_2} the imbeddings into $X_0 \times \mathrm{Gr}(m,n)$ (choosing basis of $H^0(E_1), H^0(E_2)$ respectively). Then the important remark is the easily seen observation:

$$E_1 \sim E_2 \text{ (equivalence relation defining } G(n,d))$$
$$\iff g(\mathrm{Im}\,\psi_{E_1}) = (\mathrm{Im}\,\psi_{E_2}), \quad g \in PGL(m).$$

We observe also that the Hilbert polynomial P_1 of $\mathrm{Im}\psi_E$ remains the same for all stable vector bundles E of rank n and degree d on X_k. Thus $\mathrm{Im}\psi_E \in \mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$ (we choose some polarisation on $X_0 \times \mathrm{Gr}(m,n)$). Note that the action of $PGL(m)$ on $X_0 \times \mathrm{Gr}$ (m,n) induces a canonical action of $PGL(m)$ on $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. The foregoing discussion thus shows that $G(n,d)$ can be identified (set theoretically) as the set of $PGL(m)$ orbits of a certain $PGL(m)$ stable subset of $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$.

We observe that given ψ_E, E is expressed canonically as a quotient of the trivial rank m vector bundle:

$$\mathcal{O}_{X_k}^m \to E, \quad H^0(\mathcal{O}_{X_k}^m) \simeq H^0(E), \quad H^1(E) = 0.$$

This representation is the pull-back by ϕ_E of the tautological rank n bundle on $\mathrm{Gr}(m,n)$ expressed as a quotient of the trivial rank m bundle. Then by (ii) (a) of Remark 4, $\pi_*(E)$ is a quotient of the trivial vector bundle of rank m on X_0

$$\mathcal{O}_{X_0}^m \to \pi_*(E) \quad \text{and} \quad H^0(\mathcal{O}_{X_0}^m) \simeq H^0(\pi_*(E)). \qquad (*)$$

Let P_2 be the Hilbert polynomial of the stable torsion free sheaf on X_0 of rank n and degree d. Let $Q(m, P_2)$ be the Quot scheme of quotients with Hilbert polynomial P_2 of the trivial vector bundle of rank m on X_0. Let R be the $PGL(m)$ stable open subset of $Q(m, P_2)$ of quotients $\mathcal{O}_{X_0}^m \to F$ such that $H^0(\mathcal{O}_{X_0}^m) \to H^0(F)$ is an isomorphism; moreover let R^s be the $PGL(m)$ stable open subset of R such that F is stable (which is of course torsion free). Then we see that $(*)$ gives a point of R^s. Let $U(n,d)_s$ be the moduli space of stable torsion free sheaves of rank n and degree d on X_0. Recall that (cf. [7], [9], [10])

$$R^s \bmod PGL(m) \simeq U(n,d)_s.$$

In fact R^s is a principal $PGL(m)$ bundle over $U(n,d)_s$.

We shall now give the main steps in giving a canonical structure of a quasi-projective scheme on $G(n,d)$.

I The subset $Y^s = Y(n,d)^s \subset \mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$, $Y^s = \{\mathrm{Im}\,\psi_E\}$ (E representing elements of $G(n,d)$) is $PGL(m)$ stable and has a natural structure of an (irreducible) variety whose singularities are (analytic) normal crossings.

II The map $\theta : Y^s \to R^s$ defined by y (represented by ψ_E or $\mathrm{Im}\,\psi_E$) $\mapsto$ the element of R^s represented by $(*)$ above, is a $PGL(m)$ equivariant morphism.

III The morphism θ is proper.

We shall now see how admitting I, II and III, we would get a canonical structure of a quasi-projective variety on $G(n,d)$ with a proper birational morphism onto $U(n,d)_s$, the moduli space of stable torsion free sheaves on X_0.

Let R_v^s denote the $PGL(m)$ stable open subset of R^s such that the torsion free sheaves on X_0 represented by its points are locally free i.e. vector bundles.

Then a point of $\theta^{-1}(R_v^s)$ is represented by ψ_E such that the equivalence class of E is in $G(n,d)_0$, i.e. a closed immersion

$$\psi_E : X_0 \hookrightarrow X_0 \times \mathrm{Gr}(m,n).$$

Then it is rather easy to see that the morphism,

$$\theta : \theta^{-1}(R_v^s) \to R_v^s$$

is an isomorphism. Hence it follows that θ is a proper birational morphism. Since

$$\theta : Y^s \to R^s$$

is a $PGL(m)$ morphism and $R^s \to U(n,d)_s$ is a principal $PGL(m)$ bundle, it is seen easily that the quotient Y mod $PGL(m)$ exists and in fact that $Y^s \to Y^s \bmod PGL(m)$ is a principal $PGL(m)$ bundle (since $R^s \to U(n,d)_s$ is locally isotrivial, choose a $PGL(m)$ stable open subset W in R^s such that W has an isotrivial section s over $W \bmod PGL(m)$, then $\theta^{-1}(s)$ provides an isotrivial section of $\theta^{-1}(W)$ over $\theta^{-1}(W)$ mod $PGL(m)$ etc.). As we saw in Remark 4, $G(n,d) = Y^s$ mod $PGL(m)$, set theoretically. Thus we get a canonical scheme theoretic structure on $G(n,d)$ which is a variety. Further, since $Y^s \to Y^s$ mod $PGL(m)$, is a principal fibre space and Y has normal crossings as singularities, we see that the singularities of $G(n,d)$ are normal crossings. Besides θ gives rise to a proper birational morphism.

$$\begin{array}{ccc} \delta : G(n,d) & \to & U(n,d)_s \\ \| & & \| \\ Y^s \operatorname{mod} PGL(m) & & R^s \operatorname{mod} PGL(m). \end{array}$$

To prove that $G(n,d)$ is quasi-projective, we make use of GIT (cf. [7], [13]). This will also achieve, at the same time, giving the scheme theoretic structure on $G(n,d)$, which we did above.

Recall the basic fact in the construction of the moduli spaces of vector bundles (or torsion free sheaves), namely that there is a projective variety W with an action of $PGL(m)$ which lifts to an action of an ample line bundle $\mathcal{O}_W(1)$ on W, such that if W^{ss} (resp. W^s) represents the open subscheme of semi-stable (resp. stable or more precisely properly stable) points of W for this action, we have

(a) $W^s = R^s$
(b) $W^{ss} = R^{ss}$

where R^{ss} denotes the open subscheme of R represented by semi-stable torsion free sheaves F. These can be found in (cf. [12]). In the recent work of Simpson (cf. [14]), it is in fact proved that W can be taken to be the closure of R^{ss} in the Quot scheme $Q(m, P_2)$.

We claim now that we can find a $PGL(m)$ equivariant factorisation

$$\begin{array}{ccc} Y^s & \hookrightarrow & Z \\ \theta \downarrow & & \downarrow \lambda, \\ R^s & \hookrightarrow & W \end{array}$$

where Z is a projective variety with an action of $PGL(m)$ lifting to an ample line bundle $\mathcal{O}_Z(1)$. To see this take Z_1 to be the closure of Y^s in $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. Then we get a rational map $Z_1 \to W$ and we take Z to be the graph of this rational map. Note that Z is an (irreducible) variety since Y^s is a variety. The choice of $\mathcal{O}_Z(1)$ is obvious.

Consider now the polarisation $L = \lambda^*(\mathcal{O}_W(a)) \otimes \mathcal{O}_Z(1)$ on Z. Then with the usual notations one knows that if a is sufficiently large, we have ([7], [13])

(i) $\lambda^{-1}(R^s)(= \lambda^{-1}(W^s)) \subset Z(L)^s$
(ii) λ maps $Z(L)^{ss}$ onto R^{ss}.

It follows then that $Z(L)^s$ mod $PGL(m)$ exists as a quasi-projective scheme and $Z(L)^s \to Z(L)^s$ mod $PGL(m)$ is a principal fibre space. Note that we have an open immersion $Y^s \hookrightarrow \lambda^{-1}(R^s)$ and hence the following commutative diagram

$$\begin{array}{ccc} Y^s & \overset{i}{\hookrightarrow} & \lambda^{-1}(R^s) \\ \theta \searrow & & \swarrow \lambda \\ & R^s & \end{array}$$

where θ, λ are proper (birational). This implies that i is proper. But since i is also an open immersion and Z is a variety, it follows that i is an isomorphism. Hence $Y^s = \lambda^{-1}(R^s)$. Thus we conclude Y^s mod $PGL(m) = G(n,d)$ is quasi-projective. If moreover $(n,d)=1$, then $U(n,d) = U(n,d)_s$ is projective, so that in this case $G(n,d)$ is also projective.

We shall now indicate how I and II are proved. The assertion III will be proved in the next section.

DEFINITION 5

Let $\mathcal{G} = \mathcal{G}(n,d)$ be the functor (called the Gieseker functor) defined as follows:

$$\mathcal{G} : (k\text{-schemes}) \to (\text{sets})$$

$\mathcal{G}(T) =$ set of closed subschemes $\Delta \hookrightarrow X_0 \times T \times \mathrm{Gr}(m,n)$ such that

(i) the induced projection map $p_{23} : \Delta \to T \times \mathrm{Gr}(m,n)$ is a closed immersion. We denote by E the rank n vector bundle on Δ which is the pull-back of the tautological rank n quotient bundle on $\mathrm{Gr}(m,n)$,

(ii) the projection $p_2 : \Delta \to T$ is a flat family of curves $\Delta_t (t \in T)$ such that Δ_t is a curve of the form X_k. Besides, the canonical map $\Delta_t \to X_0$ is the map $\pi : X_k (= \Delta_t) \to X_0$ that we have been considering,

(iii) the vector bundle E_t on Δ_t $(E_t = E|_{\Delta_t})$ is of degree d (and rank n) with $d = m + n(g-1)$.

(iv) By the definition of E, we get a quotient representation

$$\mathcal{O}^m_{\Delta_t} \to E_t$$

and we assume that this induces an isomorphism

$$H^0(\mathcal{O}^m_{\Delta_t}) \xrightarrow{\sim} H^0(E_t).$$

In particular, dim $H^0(E_t) = m$. It follows that

$$H^1(E_t) = 0.$$

PROPOSITION 6

The Gieseker functor $\mathcal{G}$ is represented by a $PGL(m)$ stable open subscheme Y of $\mathrm{Hilb}^{P_1}(X_0 \times \mathrm{Gr}(m,n))$. ($P_1$ being the Hilbert polynomial of the closed subscheme Δ_t of $X_0 \times \mathrm{Gr}(m,n)$, choosing of course a polarisation). Further Y is an (irreducible) variety with singularities as (analytic) normal crossings.

Proof. See Proposition 8 (of this paper) where a more general result is stated.

PROPOSITION 7

Let Δ be the universal object representing the Gieseker functor $\mathcal{G}$ above. Consider the "universal" closed immersion

$$\Delta \hookrightarrow X_0 \times Y \times \mathrm{Gr}(m,n)$$

defined by $\mathcal{G}$. This defines a flat family of curves $\Delta \to Y$. We have also a vector bundle E on Δ obtained as the pull-back of the tautological quotient bundle of rank n on $\mathrm{Gr}(m,n)$. If Δ_y denotes the fibre of $\Delta \to Y$ over $y \in Y$, E defines a family $\{E_y\}$ of vector bundles on $\{\Delta_y\}, y \in Y$. We have the map $\Delta_y \to X_0$, defined by the first projection p_1, which we *denote by π_y to be consistent with our earlier notation. We observe that $(\pi_y)_*(E_y)$ comes with a quotient representation*

$$\mathcal{O}_{X_0}^m \to (\pi_y)_*(E_y) \text{ with } H^0(\mathcal{O}_{X_0}^m) \xrightarrow{\sim} H^0((\pi_y)_*(E_y)). \tag{$*$}$$

Hence (*) defines a point of the open subscheme R of $Q(m, p_2)$ (see Remark 4). Then

$$y \mapsto \text{the point of } R \text{ defined by } (*)$$

defines a morphism $\theta : Y \to R$.

Proof. We have a commutative diagram.

$$\begin{array}{ccc} \Delta & \xrightarrow{\pi} & X_0 \times Y \\ p \searrow & & \swarrow q \\ & Y & \end{array}$$

where π is the projection p_{12}, p = projection p_2, and q = canonical projection onto Y. We observe that

$$\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{X_0 \times Y}. \tag{a}$$

To see this since the fibres of π are connected (either a point or a chain of projective lines) and π is proper, we have $\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_Z$, where $Z \to X_0 \times Y$ is a proper bijective morphism. Note also that $Z \to X_0 \times Y$ is birational since π is birational (if Y_v denotes the subset defined by $y \in Y$ such that $\Delta_y \simeq X_0$, then π is an isomorphism over $X_0 \times Y_v$). Since X_0 and Y have normal crossing singularities, the proper bijective map $Z \to X_0 \times Y$ becomes an isomorphism (since all the "analytic branches" of $X_0 \times Y$ are again normal). This proves (a).

Now to prove the proposition, we claim that it suffices to prove

$$\begin{cases} \pi_*(E)|_{q^{-1}(y)} \simeq (\pi_y)_*(E_y) \\ q^{-1}(y) \simeq X_0 \times y \simeq X_0. \end{cases} \tag{b}$$

To prove this claim suppose then that (b) holds.

We have the quotient representation

$$\mathcal{O}_\Delta^m \to E \text{ on } \Delta. \tag{c}$$

Using (a), we get applying $(\pi)_*$

$$\mathcal{O}_{X_0 \times Y}^m \to \pi_*(E). \tag{d}$$

We have then the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{X_0}^m & \to & (\pi_*(E))|_{q^{-1}(y)} \\ \| & & \downarrow \\ \mathcal{O}_{X_0}^m & \to & (\pi_y)_*(E_y), \end{array} \tag{e}$$

where the first horizontal row is the restriction of (d) to $q^{-1}(y) \simeq X_0$ and the second horizontal row is obtained by applying $(\pi_y)_*$ to the quotient

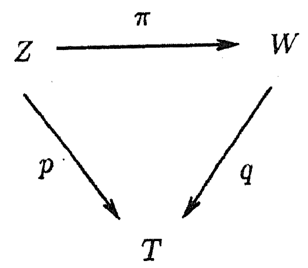
$$\mathcal{O}_{q^{-1}(y)}^m \to E_y.$$

One knows that the second horizontal map of (e) is surjective by (ii) (a) of Remark 3. By (b) this implies that the first horizontal map of (e) is also surjective (for all $y \in Y$).

This implies that (d) is surjective i.e. $\pi_*(E)$ is a quotient of $\mathcal{O}^m_{X_0 \times Y}$. Besides by (b) again the Hilbert polynomial of $\pi_*(E)|_{q^{-1}(y)}$ is P_2 and since Y is reduced, we see that $\pi_*(E)$ is flat over Y. This shows that (d) defines a morphism of Y into $Q(m, P_2)$ and the above claim is proved.

Thus to prove the proposition, it suffices to prove the following:

Lemma 4. *Suppose that we have a commutative diagram*



such that p and q are projective morphisms (which implies π is proper), $\pi_*(\mathcal{O}_Z) = \mathcal{O}_W$ and p is flat. Let E be a vector bundle on Z such that

$$R^1(\pi_t)_*(E|_{Z_t}) = 0, \quad i \geq 1.$$

Then $\pi_*(E)$ behaves well for restriction to fibres over T i.e.

$$\pi_*(E)|_{W_t} = (\pi_t)_*(E|_{Z_t}) \quad \textit{for all } t \in T.$$

Further $H^0(Z_t, E|_{Z_t}) \simeq H^0(W_t, (\pi_t)_*(E|_{Y_t}))$.

Proof. The required property is local with respect to T so that we can suppose that $T = \operatorname{Spec} A$. Let $\mathcal{O}_W(1)$ be a relatively ample sheaf with respect to q. Then by Serre's theorem, the coherent sheaf $\pi_*(E)$ on W is the sheaf associated to the graded module

$$\bigoplus_{n \geq 0} H^0(q_*(\pi_*(E)(n))). \tag{i}$$

If we denote by $E[n]$ the sheaf $E \otimes \pi^*(\mathcal{O}_W(n))$, we have

$$\pi_*(E)(n) = \pi_*(E[n]), \quad \text{since } \pi_*(\mathcal{O}_Z) = \mathcal{O}_W.$$

Further, since $q_*(\pi_*(E[n])) = p_*(E[n])$, we see that the graded module in (i) can be identified with

$$\bigoplus_{n \geq 0} H^0(p_*(E[n])) = \bigoplus_{n \geq 0} H^0(Z, E[n]). \tag{ii}$$

Similarly the sheaf $(\pi_t)_*(E|_{Z_t})$ on Z_t is the one associated to the graded k vector space ($k = k(t)$ residue at the closed point $t \in T$)

$$\bigoplus_{n \geq 0} (p_t)_*((E|_{Z_t})[n]) = \bigoplus_{n \geq 0} H^0(Z_t, (E|_{Z_t})[n]). \tag{iii}$$

Note that our hypothesis also implies that

$$R^i(\pi_t)_*((E|_{Y_t})[n]) = 0, \quad i \geq 1.$$

This implies the last assertion of the lemma. Further, we have

$$H^1(Z_t, (E|_{Z_t})[n]) = H^1(W_t, ((\pi_t)_*(E|_{Z_t}))(n)).$$

But the RHS is zero for $n \geq n_0$ (for some n_0 and for all $t \in T$) since $\mathcal{O}_W(1)$ is relatively ample with respect to q. Thus we see that

$$H^1(Z_t, (E|_{Z_t})[n]) = 0 \quad \text{for } n \geq n_0.$$

Since p is flat, we deduce that

$$\begin{cases} (p_*(E[n]) \otimes k(t) = (p_t)_*((E|_{Z_t})[n]) \\ \text{for } n \geq n_0 \ (k(t) = k \text{ residue field at } t). \end{cases}$$

Hence the graded module in (ii) tensored by $k(t)$ coincides with that in (iii) if we neglect terms of degree $\leq n_0$. But for determining the corresponding sheaves this suffices. We see that this proves the lemma. Consequently Proposition 7 also follows.

COROLLARY 1

(Proof of I and II). The subset $Y^s = Y(n,d)^s$ is open in Y and represents the subfunctor $\mathcal{G}^s$ of $\mathcal{G}$, for which in the definition of $\mathcal{G}$ (cf. Def. 4), we add moreover the condition

$$E_t \text{ is stable on } \Delta_t \text{ i.e. its equivalence class is in } G(n,d). \tag{v}$$

Besides, $\theta : Y^s \longrightarrow R^s$ is a morphism.

Proof. Since R^s is open in R (being torsion free and stable give open conditions) and $\theta : Y \longrightarrow R$ is a morphism, the corollary follows immediately.

Thus admiting the properness property III, we have shown the following (a more general version will be given in the next section).

Theorem 1. *There exists a canonical structure of a quasi-projective variety on $G(n,d)$ and a canonical proper birational morphism*

$$\pi_* : G(n,d) \longrightarrow U(n,d)_s$$

onto the moduli space of stable torsion free sheaves on X_0. The singularities of $G(n,d)$ are (analytic) normal crossings. If $(n,d) = 1$, $G(n,d)$ is projective, since $U(n,d)_s = U(n,d)$ is projective.

Remark 6 (semi-stable moduli). Consider the morphism $\theta : Y \longrightarrow R$ of Proposition 7. Set $Y^0 = \theta^{-1}(R^{ss})$. As we shall see in the next section, $\theta : Y^0 \longrightarrow R^{ss}$ is also proper. As we did in the discussions before Def. 4 for constructing a quasi-projective scheme structure on $G(n,d) = Y^s \bmod PGL(m)$, we can find a $PGL(m)$ equivariant factorisation

$$\begin{array}{ccc} Y^0 & \hookrightarrow & Z \\ \theta\downarrow & & \downarrow\lambda \\ R^{ss} & \hookrightarrow & W \end{array}$$

where Z and W are projective varieties with the actions of $PGL(m)$ lifting to ample line bundles etc. As we saw, $W^{ss} = R^{ss}$. Then by the same results and arguments which we used there, we see λ maps $Z(L)^{ss}$ onto W^{ss} and we have

$$Y^s \subset Z(L)^{ss} \subset Y^0.$$

However, $Z(L)^{ss}$ may not be equal to Y^0. Now the GIT quotient $Z(L)^{ss}//PGL(m)$ exists as a projective variety and we have a morphism of this GIT quotient onto $U(n,d) = R^{ss}//PGL(m)$, the moduli space of semi-stable torsion free sheaves of rank n and degree d on X_0. We could call $Z(L)^{ss}//PGL(m)$ the *generalized Gieseker semi-stable moduli space* of rank n and degree d on X_0. One has to show that it is intrinsically defined (there has been some choice of polarisations).

4. Properness and specialisation

To prove the specialisation property of $G(n,d)$, we have to take a family, say of smooth projective curves specialising to X_0 and show that the corresponding moduli spaces vary nicely. For simplicity, we work in the following context:

DEFINITION-NOTATION 6

Notation 6. Let $S = \operatorname{Spec} A$, where A is a discrete valuation ring with residue field the ground field k (which has been assumed algebraically closed). Let $\mathcal{X} \longrightarrow S$ be a flat family of projective curves such that the closed fibre $\mathcal{X}_{s_0} \simeq X_0$ and the generic fibre $\mathcal{X}_\xi$ (s_0-closed point of S and ξ the generic point of S) is smooth of genus g (recall g = arithmetic genus of X_0 with $g \geq 2$). We assume that as a scheme over k, $\mathcal{X}$ is regular. Let $\mathcal{O}_{\mathcal{X}}(1)$ be an S-ample line bundle on $\mathcal{X}$ (we could assume that it is of degree one on the fibres over S).

One can formulate the definition of the Gieseker functor (cf. Def. 4) over the base S and Prop. 6, Prop. 7 go through easily in this generality. Further the construction of $G(n,d)$ over the base S also goes through, since we are taking quotients for free actions of the projective group and also the fact that *GIT* works over an arbitrary base (cf. [11]). Of course one has to add that all these go through, provided the required property of properness holds.

We shall now go through the generalisations rapidly and then take up properness.

Let $\operatorname{Gr}(m,n)$ or rather $\operatorname{Gr}(m,n)_S$ be the Grassmannian over S of n dimensional quotient spaces of the standard m dimensional space (we denote it by the same letter as we did for the case when the base field is k. Our object is simply the base change by S by the one we considered over k).

DEFINITION 7

Let $\mathcal{G}_S = \mathcal{G}(n,d)_S$ be the functor (called the Gieseker functor) defined as follows:

$$\mathcal{G}_S : (S\text{-schemes}) \longrightarrow (\text{sets}).$$

$\mathcal{G}_S(T)$ = set of closed subschemes $\Delta \hookrightarrow \mathcal{X} \times_S T \times_S \operatorname{Gr}(m,n)$ such that

(i) the induced projection map $p_{23} : \Delta \longrightarrow T \times_S \operatorname{Gr}(m,n)$ is a closed immersion. We denote by E the rank n vector bundle on Δ which is induced by the tautological rank n quotient bundle on $\operatorname{Gr}(m,n)$.

(ii) the projection map $p_1 : \Delta \longrightarrow T$ is a flat family of curves Δ_t ($t \in T$) such that Δ_t is a curve of the form X_k if t is over s_0 and is the curve $\mathcal{X}_\xi$ if t is over ξ.

Further consider the canonical commutative diagram

$$\begin{array}{ccc} \Delta & \longrightarrow & (\mathcal{X} \times_S T) \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

This induces a canonical morphism $\Delta_t \longrightarrow (\mathcal{X} \times_S T)_t$. This morphism should be an isomorphism when $(\mathcal{X} \times_S T)_t$ is smooth (i.e. when t does not map to the closed point s_0 of S); further when t maps to s_0, in which case $(\mathcal{X} \times_S T)_t \simeq X_0$, this morphism should reduce to the canonical morphism $\Delta_t (\simeq X_k) \longrightarrow X_0$. We observe that Δ_t is a closed subscheme of the $k(t)$ scheme $(\mathcal{X} \times_S \mathrm{Gr}(m,n)) \times_S \operatorname{Spec} k(t)$ ($k(t)$ residue field at $t \in T$) and its Hilbert polynomial is P_1.

(iii) the vector bundle E_t on Δ_t ($E_t = E|_{\Delta_t}$) is of degree d (and rank n) with $d = m + n(g-1)$.

(iv) By the definition of E, we get a quotient representation

$$\mathcal{O}^m_{\Delta_t} \longrightarrow E_t$$

and we assume that this induces an isomorphism

$$H^0(\mathcal{O}^m_{\Delta_t}) \xrightarrow{\sim} H^0(E_t).$$

In particular, $\dim H^0(E_t) = m$.

PROPOSITION 8

The functor $\mathcal{G}_S$ is represented by a $PGL(m)$ stable open subscheme $\mathcal{Y}$ of the S-scheme $\mathrm{Hilb}^{P_1}(\mathcal{X} \times_S \mathrm{Gr}(m,n))$. *Further the S-scheme $\mathcal{Y}$ has the following properties:*

(i) *the closed fibre $\mathcal{Y}_{s_0}$ (of $\mathcal{Y}$ over S) is the variety Y with normal crossings (in Proposition* 6)

(ii) *the generic fibre $\mathcal{Y}_\xi$ is smooth,*

(iii) *as a scheme over k, $\mathcal{Y}$ is regular.*

Proof. The proof is essentially to be found in Gieseker's work (cf. Prop. 4.1, [5]). In the appendix to this paper we give a brief outline of the proof.

Let V be a vector bundle on $\mathcal{X}$ of rank n and degree d on the fibres over S. Let P_2 be the Hilbert polynomial of V. Let $Q_S(m, P_2)$ be the Quot scheme of quotients with Hilbert polynomial P_2 of the trivial vector bundle of rank m on $\mathcal{X}$. Then recall that $Q_S(m, P_2)$ is projective over S. Recall that an element of $Q_S(m, P_2)(T)$ (T being an S-scheme) is the following:

$$\left\{ \begin{array}{l} \text{A quotient } \mathcal{O}^m_{\mathcal{X}\times_S T} \xrightarrow{j} F, \text{ where } F \text{ is coherent on} \\ \mathcal{X} \times_S T, \text{ flat over } T \text{ with Hilbert polynomial } P_2. \end{array} \right.$$

In particular, we can take $T = Q_S(m, P_2)$ and we get the universal quotient:

$$\mathcal{O}^m_{\mathcal{X}\times_S Q_S(m,P_2)} \longrightarrow \mathcal{F}.$$

Now if $q \in Q_S(m, P_2)$, we denote by $\bar{q}$ the image of q in S ($Q_S(m, P_2)$ is an S-scheme). Let $\mathcal{X}_{\bar{q}}$ denote the fibre of $\mathcal{X} \longrightarrow S$ over q. With this notation, for $q \in Q(m, P_2)$, $\mathcal{F}_q$ is canonically a quotient of $\mathcal{O}^m_{\mathcal{X}_{\bar{q}}}$.

Let $\mathcal{R}$ be the $PGL(m)$ stable open subscheme of $Q_S(m, P_2)$ formed of $q \in Q_S(m, P_2)$ such that the canonical map $H^0(\mathcal{O}_{\mathcal{X}_{\bar{q}}}) \longrightarrow H^0(\mathcal{F}_q)$ is an isomorphism. Then we see that $H^1(\mathcal{F}_q) = 0$. Let $\mathcal{R}^s$ be the $PGL(m)$ stable open subscheme of $\mathcal{R}$ formed of q such that $\mathcal{F}_q$ is a stable (torsion free) sheaf on $\mathcal{X}_{\bar{q}}$. Let $\mathcal{U}(n, d)$ be the moduli space of semi-stable torsion free sheaves along the fibres of $\mathcal{X} \to S$. This is an S-projective scheme whose construction is based on the usual considerations when the base is a field and the fact that GIT works over an arbitrary base (cf. [7], [11]). Let $\mathcal{U}(n, d)_s$ be the open subscheme of $\mathcal{U}(n, d)$ corresponding to stable torsion free sheaves. Then we have $\mathcal{R}^s$ mod $PGL(m) \simeq \mathcal{U}(n, d)_s$. In fact $\mathcal{R}^s$ is a principal $PGL(m)$ fibre bundle over $\mathcal{U}(n, d)_s$.

PROPOSITION 9

Let Δ be the universal object defining $\mathcal{G}_S$ so that we have the universal closed immersion

$$\Delta \hookrightarrow \mathcal{X} \times_S \mathcal{Y} \times_S \mathrm{Gr}(m, n).$$

This gives a commutative diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\ \pi\ } & \mathcal{X} \times_S \mathcal{Y} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & \mathcal{Y} & \end{array}$$

where π = projection p_{12}, p = projection p_2, q = canonical projection. We observe that for $y \in \mathcal{Y}$, $(\pi_y)_*(E_y)$ is a $k(y)$ valued point of $\mathcal{R}$. Then

$$y \mapsto (\pi_y)_*(E_y)$$

defines a morphism θ of $\mathcal{Y}$ into $\mathcal{R}$ (to be very formal we have to work with T valued points of $\mathcal{R}$, rather than $k(y)$ valued points).

Proof. The proof is on the same lines as that of Proposition 7. There is a mild difference in proving

$$\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{\mathcal{X} \times_S \mathcal{Y}}.$$

As in the case of Prop. 7, again π is birational as we note that the base change of the S-morphism π by $k(\xi)$ (ξ generic point of S) is an isomorphism (easy consequence of the fact that if $y \in \mathcal{Y}$ maps to ξ by the structure morphism $\mathcal{Y} \longrightarrow S$, then $\Delta_y \longrightarrow (\mathcal{X} \times_S \mathcal{Y}) \times_S k(y)$ is an isomorphism. Note we have assumed $\mathcal{X}_\xi$ is smooth). We claim that $\mathcal{X} \times_S \mathcal{Y}$ is normal. This can be seen as follows. We see that the closed subscheme $\mathcal{X} \times_S \mathcal{Y}_{s_0}$ ($\mathcal{Y}_{s_0}$-closed fibre of $\mathcal{Y} \longrightarrow S$) is of codimension one and defined by one equation (since the closed point s_0 of S is defined by one equation). We see that

$$\mathcal{X} \times_S \mathcal{Y}_{s_0} \simeq X_0 \times \mathcal{Y}_{s_0}$$

is of course Cohen–Macaulay (since X_0 and $\mathcal{Y}_{s_0}$ have normal crossing singularities). It follows that $\mathcal{X} \times_S \mathcal{Y}$ is Cohen–Macaulay. Further, it is easily seen that its singularity is of codimension ≥ 2. It follows that $\mathcal{X} \times_S \mathcal{Y}$ is normal. Then since π is proper birational and $\mathcal{X} \times_S \mathcal{Y}$ is normal, we see that $\pi_*(\mathcal{O}_\Delta) = \mathcal{O}_{\mathcal{X} \times_S \mathcal{Y}}$.

COROLLARY 2

Let $\mathcal{Y}^s = \Theta^{-1}(\mathcal{R}^s)$. Then $\mathcal{Y}^s$ is the open subset of $\mathcal{Y}$ such that $(\pi_y)_*(E_y)$ is stable, further it represents the subfunctor $\mathcal{G}_S^s$ defined in a way similar to that of $\mathcal{G}^s$ (cf. Corollary 1). Let $\mathcal{R}^f$ denote the open subset of $\mathcal{R}$ defined by $q \in Q_S(m, P_2)$ such that $\mathcal{F}_q$ is torsion free. We set $\mathcal{Y}^f = \Theta^{-1}(\mathcal{R}^f)$ and $\mathcal{Y}^0 = \Theta^{-1}(\mathcal{R}^{ss})$. Then we have open immersions

$$\mathcal{Y}^s \subset \mathcal{Y}^0 \subset \mathcal{Y}^{ss}$$

and Θ induces morphisms

$$\mathcal{Y}^s \longrightarrow \mathcal{R}^s, \quad \mathcal{Y}^0 \longrightarrow \mathcal{R}^{ss}, \quad \mathcal{Y}^f \longrightarrow \mathcal{R}^f.$$

Remark 7. Consider the above S-morphisms Θ (e.g. $\Theta : \mathcal{Y}^s \longrightarrow \mathcal{R}^s$). They are all isomorphisms over $S - \{s_0\}$ or equivalently Θ induces an isomorphism of their generic fibres over S (e.g. $\Theta_\xi : \mathcal{Y}^s_\xi \xrightarrow{\sim} \mathcal{R}^s_\xi$). In fact Θ is an isomorphism over the bigger open subset $\mathcal{R}_v$ defined by $q \in Q_s(m, P_2)$ such that $\mathcal{F}_q$ is locally free.

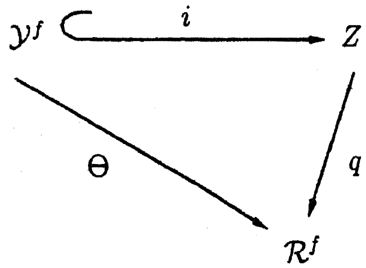
PROPOSITION 10

The morphism

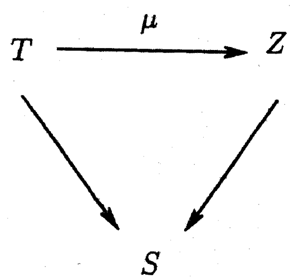
$$\Theta : \mathcal{Y}^s \longrightarrow \mathcal{R}^s \quad (\textit{resp. } \mathcal{Y}^0 \longrightarrow \mathcal{R}^{ss}, \quad \mathcal{Y}^f \longrightarrow \mathcal{R}^f)$$

is proper.

Proof. It suffices to prove that $\Theta : \mathcal{Y}^f \longrightarrow \mathcal{R}^f$ is proper, as $\mathcal{Y}^s = \Theta^{-1}(\mathcal{R}^s)$ and $\mathcal{Y}^0 = \Theta^{-1}(\mathcal{R}^{ss})$. We have a commutative diagram



where i is an open immersion and $q : Z \longrightarrow \mathcal{R}^f$ is a projective morphism (since $\mathcal{Y}^f \longrightarrow \mathcal{R}^f$ is a quasi-projective morphism). As we saw in Remark 6, $\Theta^{-1}(\mathcal{R}^f_\xi) = \mathcal{Y}^f_s \longrightarrow \mathcal{R}^f_\xi$ is an isomorphism, in particular it is a proper morphism (note that $\mathcal{R}^f_\xi$ is open in $\mathcal{R}^f$). From this it follows that we can assume without loss of generality that Z is the closure of $\mathcal{Y}^f_\xi$. Then to prove that Θ is proper, we have only to show that $Z = \mathcal{Y}^f$ or equivalently $\mathcal{Y}^f_{s_0} = Z_{s_0}$ (these represent the closed fibres, s_0 being the closed point of S). Suppose that $z_0 \in Z_{s_0}$. Then we can find an S-morphism $T \longrightarrow Z$



such that

(i) $T = \operatorname{Spec} B$, B a d.v.r with k as residue field.
(ii) μ (closed point of T) $= z_0$.
(iii) $T \longrightarrow S$ is surjective, which implies that μ (generic point of T) $\in \mathcal{Y}^f_\xi \simeq \mathcal{R}^f_\xi$.

Consider the morphism $q \circ \mu : T \longrightarrow \mathcal{R}^f$. Then this gives a coherent sheaf $\mathcal{F}$ on $\mathcal{X}_T = (\mathcal{X} \times_S T)$ which is flat over T and torsion free over the fibres of $\mathcal{X}_T \longrightarrow T$. Further $\mathcal{F}$ has a quotient representation $\mathcal{O}^m_{\mathcal{X}_T} \longrightarrow \mathcal{F}$. Note that the closed fibre of $\mathcal{X}_T \simeq$ closed fibre of $\mathcal{X} \simeq X_0$ and $\mathcal{X}_T$ is regular outside the singular point p of X_0. Further, if η is the generic point of T, we denote the generic fibre of $\mathcal{X}_T \longrightarrow T$ by $\mathcal{X}_\eta$, which is a base change of the generic fibre $\mathcal{X}_\xi$ of $\mathcal{X} \longrightarrow S$. Now $\mathcal{F}$ is locally free on $\mathcal{X}_T$ outside p and the quotient representation defines a T-morphism

$$g : \mathcal{X}_T \setminus \{p\} \longrightarrow \operatorname{Gr}(m,n) \text{ (rather } \operatorname{Gr}(m,n)_T).$$

We can assume that g is an immersion (by suitable tensorisation by a power of $\mathcal{O}_{\mathcal{X}}(1)$, a similar property can be supposed to hold for the defining torsion free sheaf on $\mathcal{X} \times_S Q$ (m, P_2), so that this property follows by base change). Let Γ_g be the graph of g, considered as a rational morphism of $\mathcal{X}_T$, so that we have a closed immersion of T-schemes

$$\Gamma_g \hookrightarrow \mathcal{X}_T \times_T \operatorname{Gr}(m,n). \tag{$*$}$$

Let π_g denote the canonical projection (a T-morphism)

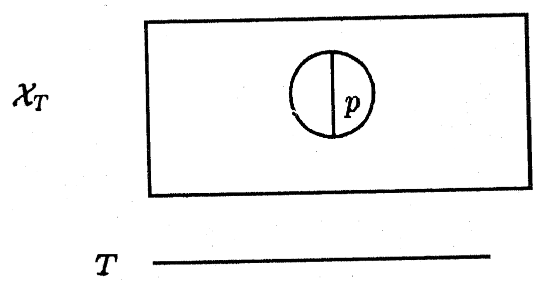
$$\pi_g : \Gamma_g \longrightarrow \mathcal{X}_T.$$

Obviously, π_g induces an isomorphism of the generic fibres over T, in fact it is an isomorphism over $\mathcal{X}_T \setminus \{p\}$. Let E be the vector bundle of rank n on Γ_g induced by the tautological quotient vector bundle of rank n on $\operatorname{Gr}(m,n)$ (through $(*)$). Consider the following:

$$\left\{\begin{array}{l}\text{(i) The closed fibre of } \Gamma_g \text{ over } T \text{ is a curve of the form } X_k\\ \text{(this implies that the morphism induced by } \pi_g \text{ on the closed}\\ \text{fibres is the canonical morphism } X_k \longrightarrow X_0)\\ \text{(ii) } (\pi_g)_*(E) = \mathcal{F}.\end{array}\right. \tag{A}$$

We claim that (A) $\Longrightarrow$ properness of $\Theta : \mathcal{Y}^f \longrightarrow \mathcal{R}^f$ i.e. $z_0 \in \mathcal{Y}^f_{s_0}$. To see this observe first that $\Gamma_g \longrightarrow T$ is a flat family of curves. To prove the claim, we have only to show that $(*)$ defines a T valued point of the Gieseker functor $\mathcal{G}_s$. We see that all the properties in the definition of the Gieseker functor are satisfied (Def. 6). This follows by Prop. 3 (especially the property (iii) of this proposition) and the fact that $(\pi_g)_*(E)$ behaves well for restriction to fibres over T (Lemma 4, to apply this Lemma we require the property $(\pi_g)_*(\mathcal{O}_{\Gamma_g}) = \mathcal{O}_{\mathcal{X}_T}$, which follows using the fact that $\mathcal{X}_T$ is normal and π_g is birational).

We see that to check the assertions in (A) it suffices to check them over a neighbourhood of the point p in $\mathcal{X}_T$. More precisely, let C be the local ring of $\mathcal{X}_T$ at p. Let C_0 be the local ring at p of the closed fibre ($\approx X_0$) of $\mathcal{X}_T \longrightarrow T$.



Ball $\leftrightarrow$ Spec C

Vertical line $\leftrightarrow$ Spec C_0

Let Γ_C (resp. Γ_{C_0}) be the base change of $\pi_g : \Gamma_g \longrightarrow \mathcal{X}_T$ by $\operatorname{Spec} C \longrightarrow \mathcal{X}_T$ (resp. $\operatorname{Spec} C_0 \longrightarrow \mathcal{X}_T$). Let $\mathcal{F}_C$ denote the stalk of $\mathcal{F}$ at p and π_C (resp. π_{C_0}) the canonical morphism

$$\Gamma_C \longrightarrow \operatorname{Spec} C \ (\text{resp. } \Gamma_{C_0} \longrightarrow \operatorname{Spec} C_0).$$

For a curve of the form X_k with its canonical morphism $X_k \longrightarrow X_0$ we denote by $(X_k)_{C_0}$, its base change by $\operatorname{Spec} C_0 \longrightarrow X_0$. We denote by E_C the restriction of E to Γ_C. Note that we have a commutative diagram

$$\begin{array}{ccc} \Gamma_C & \longrightarrow & \operatorname{Spec} C \\ & \searrow & \downarrow \\ & & \operatorname{Spec} B = T \end{array}$$

Then (A) is equivalent to

$$\left\{\begin{array}{l} \text{(i) The closed fibre of } \Gamma_C \longrightarrow T \text{ is of the form } (X_k)_{C_0} \text{ and} \\ \Gamma_{C_0} \longrightarrow \operatorname{Spec} C_0 \text{ identifies with the canonical morphism} \\ (X_k)_{C_0} \longrightarrow \operatorname{Spec} C_0 \\ \text{(ii) } (\pi_C)_*(E_C) = \mathcal{F}_C. \end{array}\right. \qquad (A_1)$$

We have a quotient representation of the C-module $\mathcal{F}_C$

$$C^m \longrightarrow \mathcal{F}_C$$

induced by the quotient representation $\mathcal{O}^m_{\mathcal{X}_T} \longrightarrow \mathcal{F}$. Then if m_1 denotes the minimal number of generators of the C-module $\mathcal{F}_C$, we have a commutative diagram

$$\begin{array}{ccc} C^m & \longrightarrow & \mathcal{F}_C \\ & \searrow \quad \nearrow & \\ & C^{m_1} & \end{array}$$

where $C^m \longrightarrow C^{m_1}$ is surjective. We have a canonical closed immersion $\operatorname{Gr}(m_1, n) \hookrightarrow \operatorname{Gr}(m, n)$ and the closed immersion

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times_T \operatorname{Gr}(m, n) \qquad (*)'$$

induced by $(*)$ factorises as follows:

$$\begin{array}{ccc} \Gamma_C & \hookrightarrow & \operatorname{Spec} C \times_T \operatorname{Gr}(m, n) \\ & \searrow & \uparrow \\ & & \operatorname{Spec} C \times_T \operatorname{Gr}(m_1, n) \end{array}$$

i.e. Γ_C is the graph of the rational map

$$\operatorname{Spec} C \longrightarrow \operatorname{Gr}(m_1, n)$$

defined by the canonical morphism

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \operatorname{Gr}(m_1, n)$$

which is defined by a minimal set of generators of the C-module $\mathcal{F}_C$ which is locally free on $\operatorname{Spec} C \setminus \{p\}$.

We observe that in the foregoing discussion about the local nature of the assertion (A) over a neighbourhood of p, we could have supposed that A ($S = \operatorname{Spec} A$) and B are complete and C is the completion of the local ring of $\mathcal{X}_T$ at p. We assume this in the sequel.

We shall now give a concrete determination of $\mathcal{F}_C$, which would facilitate the checking of (A_1). For this we need the claim

$$\mathcal{F}^{**} = \mathcal{F}(\Longleftrightarrow \mathcal{F}_C^{**} = \mathcal{F}_C),$$

where $\mathcal{F}^{**}$ denotes the double dual of $\mathcal{F}$. To prove this claim note that we have a canonical inclusion $\mathcal{F} \hookrightarrow \mathcal{F}^{**}$ and $\mathcal{F}^{**}$ is flat over T. Consider the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{**} \longrightarrow N \longrightarrow 0.$$

Now the support of N is at p, in particular it is a torsion C-module. Let us use the notation, say $\mathcal{F}_t$ for the restriction $\mathcal{F}$ to the fibre of $\mathcal{X}_T \longrightarrow T$ over t. Let t_0 denote the closed point of T.

Since $\mathcal{F}_{t_0}$ is torsion free, the above exact sequence restricted to the fibre over t_0, remains exact i.e. we have the exact sequence

$$0 \longrightarrow \mathcal{F}_{t_0} \longrightarrow \mathcal{F}_{t_0}^{**} \longrightarrow N_{t_0} \longrightarrow 0.$$

Since $\mathcal{F}^{**}$ is flat over T and $\mathcal{F}_\eta = \mathcal{F}_\eta^{**}$ (η generic point of T), we conclude that Hilbert polynomial of $\mathcal{F}_{t_0}$ = Hilbert polynomial of $\mathcal{F}_{t_0}^{**}$. Since N_{t_0} is of finite length, it follows that $N_{t_0} = 0$, which in turn implies $N = 0$. This shows that $\mathcal{F} = \mathcal{F}^{**}$.

For simplicity we shall assume that the base field k is of characteristic zero, say $k = \mathbb{C}$.

Let $\widehat{\mathcal{O}}_p$ be the completion of the local ring of $\mathcal{X}$ at p. Then since $\mathcal{X}$ has been supposed to be regular, $\widehat{\mathcal{O}}_p = k[[x, y]]$. Besides, it is not difficult to see that x, y can be chosen so that the canonical homomorphism $A \longrightarrow \widehat{\mathcal{O}}_p$ is given by $t \mapsto xy$ ($A \simeq k[[t]]$). The canonical homomorphism $A \longrightarrow B$ ($B \simeq k[[t]]$) is defined by $t \mapsto t^r$ (r a positive integer, for a choice of the uniformising parameter t). Then we see easily that the completion C of the local ring of $\mathcal{X}_T$ at p is of the form

$$C \simeq k[[x, y, t]]/(xy = t^r) \tag{**}$$

and the canonical homomorphism $B \longrightarrow C$ is defined by $t \mapsto$ the image of t in C. Let $D = k[[u, v]]$. Consider the action of the cyclic group Γ_r operating on D by the action

$$\begin{cases} (u, v) \mapsto (\zeta u, \bar{\zeta} v), \ \zeta \in \Gamma_r \text{ represented by an} \\ r\text{th root of unity, } \bar{\zeta} \text{ being the complex conjugate of } \zeta. \end{cases}$$

Then we have

$$C = D^{\Gamma_r} \ (\Gamma_r \text{ invariants in } D), \text{ taking } x = u^r,\ y = v^r\ t = uv.$$

Now C is normal with an isolated singularity at p and the representation (**) means that this singularity is an ordinary double point of type A (cf. [1]). Let f denote the canonical morphism

$$f : \operatorname{Spec} D \longrightarrow \operatorname{Spec} C \ (C \hookrightarrow D)$$

which induces an etale covering:

$$f_0 : \operatorname{Spec} D \setminus (0) \longrightarrow \operatorname{Spec} C \setminus \{p\}.$$

Consider $f_0^*(\mathcal{F}_C)$ (here $\mathcal{F}_C$ denotes the restriction of $\mathcal{F}_C$ to $\operatorname{Spec} C \setminus \{p\}$). Then it is locally free and extends to a locally free coherent sheaf $\mathcal{F}'$ on $\operatorname{Spec} D$ i.e. $\mathcal{F}'$ is represented by a free D-module of rank n, which we denote again by $\mathcal{F}'$. In fact we have an action of Γ_r on $\mathcal{F}'$ consistent with its action on D (we call $\mathcal{F}'$ a $D - \Gamma_r$ module). It is easy to see that $(\mathcal{F}')^{\Gamma_r}$ is a reflexive C-module and that the restriction of $(\mathcal{F}')^{\Gamma_r}$ to $\operatorname{Spec} C \setminus \{p\}$ can be canonically identified with the restriction of $\mathcal{F}_C$ to $\operatorname{Spec} C \setminus \{p\}$. Now since $\mathcal{F}_C$ is reflexive, it follows that

$$\mathcal{F}_C \simeq (\mathcal{F}')^{\Gamma_r}.$$

It is known (cf. [6]) that a free $(D - \Gamma_r)$ module is associated to a representation of Γ_r i.e. the space of sections over $\operatorname{Spec} D$ of the trivial vector bundle $\operatorname{Spec} D \times V$, where V is a finite dimensional representation of Γ_r. We take the diagonal representation of Γ_r on $\operatorname{Spec} D \times V$ and through this action, the above space of sections acquires a canonical $(D - \Gamma_r)$ module structure. Now a finite dimensional Γ_r representation is a direct sum of 1-dimensional representations. Then if $\dim V = 1$ so that $V \simeq \mathbb{C}$, we see that a Γ_r action on $(\operatorname{Spec} D \times \mathbb{C}) = L$ is given as follows:

$$\zeta \cdot \{(u, v) \times \mathbb{C}\} = (\zeta u, \bar{\zeta} v, \zeta^s \theta), \quad \theta \in \mathbb{C}$$

where we take $\operatorname{Spec} D$ as a 2-dimensional disc with the origin as centre and Γ_r is identified with rth roots of unity. A Γ_r-invariant section of this line bundle L is easily identified with a function F on the disc satisfying the following condition:

$$F(\zeta u, \bar{\zeta} v) = \zeta^s F(u, v).$$

We see easily that the Γ_r-invariant sections of L are generated by u^s and v^{r-s} as a C-module. We have

$$u^{r-s}(u^s, v^{r-s}) = (u^r, (uv)^{r-s}) = (x, t^{r-s})$$

i.e. the C-module $(L)^{\Gamma_r}$ is isomorphic to an ideal in C of the form (x, t^{r-s}). Thus the C-module $\mathcal{F}_C$ is of the following form:

$$\mathcal{F}_C \simeq \bigoplus_{i=1}^{n} (t^{a_i}, x), \quad 0 \le a_1 \le a_2 \le \cdots \le a_n.$$

We see that (t^{a_i}, x) is principal if and only if $a_i = 0$. Hence we can write

$$\begin{array}{ccc} \mathcal{F}_C = (\mathcal{O}_C^b) & \displaystyle\bigoplus_{i=1}^{n-b} (t^{a_i}, x), & 0 < a_1 \le a_2 \le \cdots \le a_n \\ \| & \| & \\ = \mathcal{F}_1 \quad \oplus & \mathcal{F}_2. & \end{array}$$

Now $\mathcal{F}_1$ is free of rank b. We saw above that Γ_C is the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \operatorname{Gr}(m_1, n)$$

efined by a minimal set of generators of the C-module $\mathcal{F}_C$. Choosing $2(n-b)$ generators of $\mathcal{F}_2$ (and adding the canonical generators of $\mathcal{F}_1$), we see easily that the above map factorises as follows:

$$\begin{array}{ccc} \operatorname{Spec} C \setminus \{b\} & \longrightarrow & \operatorname{Gr}(m_1, n) \\ & \searrow & \uparrow \\ & & \operatorname{Gr}(2(n-b), (n-b)) \end{array}$$

Thus for our purpose (studying Γ_C and checking (A_1) above), we can assume without loss of generality that $b = 0$ or equivalently $\mathcal{F}_1 = 0$ i.e.

$$\mathcal{F}_C = \bigoplus_{i=1}^{n} (t^{a_i}, x), \quad 0 < a_1 \leq a_2 \leq \cdots \leq a_n.$$

Then we have $m_1 = 2n$ (minimal number of generators for the C-module $\mathcal{F}_C$) and Γ_C is the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow \operatorname{Gr}(2n, n)$$

defined by a minimal number of generators of $\mathcal{F}_C$. Now

$$\mathcal{F}_C = \bigoplus_{k=1}^{n} I_k, \quad I_k = (t^{a_k}, x).$$

Picking up the two generators t^{a_k} and x from each I_k, we see easily that the above rational map factorises as follows:

$$\begin{array}{ccc} \operatorname{Spec} C - \{p\} & \longrightarrow & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 = (\mathbb{P}^1)^n \\ & \searrow & \downarrow \\ & & \operatorname{Gr}(2n, n) \end{array}$$

Let $\alpha_1, \ldots, \alpha_l$ be the distinct ones among the above $\{a_i\}$ occurring with multiplicity $\{\beta_i\}$, $1 \leq i \leq l$. Then the above rational map of $\operatorname{Spec} C \setminus \{p\}$ into $(\mathbb{P}^1)^n$ factorises as follows:

$$\begin{array}{ccc} \operatorname{Spec} C \setminus \{p\} & \longrightarrow & \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \ (l \text{ times}) \\ & & \Delta_1 \downarrow \quad \cdots \quad \downarrow \Delta_l \\ & \searrow & (\mathbb{P}^1)^{\beta_1} \times \cdots \times (\mathbb{P}^1)^{\beta_l} = (\mathbb{P}^1)^n \\ & & \downarrow \\ & & \operatorname{Gr}(2n, n) \end{array}$$

where $\{\Delta_i\}$ are the diagonal morphisms of $\mathbb{P}^1$ into $(\mathbb{P}^1)^{\beta_i}$, $1 \leq i \leq l$. Thus we have

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l \qquad (***)$$

and it is obtained as the graph of the rational map

$$\operatorname{Spec} C \setminus \{p\} \longrightarrow (\mathbb{P}^1)^l$$

choosing the generators t^{α_k}, x of $I_k = (t^{\alpha_k}, x)$ with $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_k$. We observe that the pull-back of the tautological quotient bundle on $\mathrm{Gr}(2n, n)$ to $(\mathbb{P}^1)^l$ (by the canonical map given above) identifies with

$$\begin{cases} \mathcal{O}(1)^{\beta_1} \oplus \cdots \oplus \mathcal{O}(1)^{\beta_l}, \ \mathcal{O}(1)^{\beta_i} \text{ denotes the line bundle} \\ \text{coming from } i\text{th } \mathbb{P}^1\text{-factor.} \end{cases} \tag{a}$$

We shall now determine the fibre of the morphism $\Gamma_C \longrightarrow \operatorname{Spec} C$ in $(***)$ over the closed point of $\operatorname{Spec} C$ (which is the point p of $\mathcal{X}_T$).

A curve in $\operatorname{Spec} C$ (passing through p) is given by a morphism $\operatorname{Spec} E \longrightarrow \operatorname{Spec} C$, where E is a d.v.r (with residue field the base field), such that if π is a uniformising parameter of E, we have

$$\begin{cases} x = \pi^\lambda u, \ y = \pi^\mu v, \ t = \pi^{(\lambda+\mu)/r}(uv)^{1/r} \\ u, v \text{ units in } E. \text{ Set } \delta = (\lambda + \mu)/r \text{ (positive integer).} \end{cases}$$

Then through the canonical map $\operatorname{Spec} C \setminus \{p\} \longrightarrow (\mathbb{P}^1)^l$ and the k-th projection of $(\mathbb{P}^1)^l$ onto $\mathbb{P}^1$, we get a map of $\operatorname{Spec} E$ into $\mathbb{P}^1$ given by

$$(\pi^{\delta\alpha_k} *, \pi^\lambda u).$$

If $\lambda < \delta\alpha_k$, the image is (1,0). If $\lambda = \delta\alpha_k$, varying the units i.e. the curves, we would get all the points of $\mathbb{P}^1$ in the image. From these considerations it follows easily that the fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$ identifies with the union of $\mathbb{P}^1$'s in $(\mathbb{P}^1)^l$ of the form:

$$\begin{aligned} &\{\mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \{(1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \\ &\cup \{(1,0) \times (1,0) \times \mathbb{P}^1 \times (0,1) \times \cdots \times (0,1)\} \cup \cdots \cup \{(1,0) \times \cdots \times (1,0) \times \mathbb{P}^1\}. \end{aligned} \tag{b}$$

Thus if we denote by F the fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of $\operatorname{Spec} C$, we see easily that

$$F_{\mathrm{red}} = \text{a chain } R \text{ of } \mathbb{P}^1\text{'s in } (\mathbb{P}^1)^l \text{ of length } l \text{ of the above form.}$$

Recall that Γ_{C_0} is the base change of $\Gamma_C \longrightarrow \operatorname{Spec} C$ by $\operatorname{Spec} C_0 \longrightarrow \operatorname{Spec} C$ (or the inverse of $\operatorname{Spec} C_0$), where (recall that) C_0 denotes the completion of the local ring at p of the closed fibre of $\mathcal{X}_T \longrightarrow T$. We have

$$\begin{aligned} &\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l \\ &\Gamma_{C_0} \hookrightarrow \operatorname{Spec} C_0 \times (\mathbb{P}^1)^l. \end{aligned}$$

Now the analytic local ring $C_0 \simeq k[[x,y]]/(xy)$ so that $\operatorname{Spec} C_0$ has two smooth components $\operatorname{Spec} E_1$ and $\operatorname{Spec} E_2$, where $\operatorname{Spec} E_1$ (resp. $\operatorname{Spec} E_2$) is defined by $y = 0$ (resp. $x = 0$). Of course as closed subschemes of $\operatorname{Spec} C$, we have $t = 0$ on $\operatorname{Spec} E_1$ and $\operatorname{Spec} E_2$. To find the image of the closed point of $\operatorname{Spec} E_1$ (resp. $\operatorname{Spec} E_2$) in Γ_{C_0}, we observe that

$$\begin{aligned} &(t^{\alpha_i}, x) = (0, x) \text{ on } \operatorname{Spec} E_1 \longrightarrow (0,1) \\ &(t^{\alpha_i}, x) = ((xy)^{\alpha_i/r}, x) \sim (y^{\alpha_i/r}, x^{(r-\alpha_i)/r}) \longrightarrow (1,0) \ (r - \alpha_i > 0). \end{aligned}$$

From these we conclude easily that the image $\operatorname{Spec} E_1$ in Γ_{C_0} is represented by

$$\operatorname{Spec} E_1 \times (0,1) \times \cdots \times (0,1)$$

and the image of Spec E_2 in Γ_{C_0} is represented by

$$\operatorname{Spec} E_2 \times (1,0) \times \cdots \times (1,0).$$

This shows that

$$(\Gamma_{C_0})_{\text{red}} \simeq (X_l)_{C_0} \tag{c}$$

and the canonical morphism $(\Gamma_{C_0})_{\text{red}} \longrightarrow \operatorname{Spec} C_0$ identifies with the canonical morphism $(X_l)_{C_0} \longrightarrow \operatorname{Spec} C_0$.

We now see easily that

$$(\Gamma_{C_0})_{\text{red}} = \Gamma_{C_0} \Longleftrightarrow F_{\text{red}} = F.$$

Then if $F = F_{\text{red}}$, by the arguments leading to (A_1), we see that if $\overline{\Gamma}_g$ represents the closed fibre of $\Gamma_g \longrightarrow T$, we have

$$\overline{\Gamma}_g = (\overline{\Gamma}_g)_{\text{red}} \simeq X_l. \tag{d}$$

We will now show that $F_{\text{red}} = F$, which would prove (d). If $l = 1$, this is rather immediate. Suppose then that $l \geq 2$.

Consider the closed immersion (see $(***)$ above)

$$\Gamma_C \hookrightarrow \operatorname{Spec} C \times (\mathbb{P}^1)^l.$$

Let us take the homogeneous coordinates

$$(X_1, Y_1), (X_2, Y_2), \ldots, (X_l, Y_l)$$

for $(\mathbb{P}^1)^l$, We see that the following equations hold on Γ_C:

$$\begin{aligned} Y_1 X_2 &= t^{\alpha_2 - \alpha_1} X_1 Y_2 \\ Y_2 X_3 &= t^{\alpha_3 - \alpha_2} X_2 Y_3 \\ &\vdots \\ Y_{l-1} X_l &= t^{\alpha_l - \alpha_{l-1}} X_{\alpha-1} Y_l. \end{aligned}$$

This implies that F (fibre of $\Gamma_C \longrightarrow \operatorname{Spec} C$ over the closed point of Spec C, $F \hookrightarrow (\mathbb{P}^1)^l$) satisfies the following equations:

$$Y_1 X_2 = 0, \quad Y_2 X_3 = 0, \ldots, Y_{l-1} X_l = 0.$$

It is easy to see that F is defined by these equations and that F is reduced. This proves (d).

Let E be the pull-back on Γ_g of the tautological quotient vector bundle of rank n on $\mathrm{Gr}(m,n)$. Then by (a) above, it follows that the restriction of E to $\overline{\Gamma}_g \simeq X_l$ is a standard vector bundle. Besides, by the representation (b) it follows easily that this restriction satisfies the properties of Lemma 2. Hence if $\overline{\pi}_g$ denotes the canonical morphism

$$\overline{\pi}_g : \overline{\Gamma}_g \simeq X_l \longrightarrow X_0 (\simeq \text{ closed fibre of } \mathcal{X}_T \longrightarrow T)$$

we see that $(\overline{\pi}_g)_*(E)$ is torsion free. Then by the arguments of Lemma 4, it follows that $(\pi_g)_*(E)$ is a family of torsion free sheaves on $\mathcal{X}_T$, parametrized by T. Then by an argument similar to proving that $\mathcal{F}$ is reflexive, we see that $(\pi_g)_*(E)$ is reflexive. But $(\pi_g)_*(E)$ and $\mathcal{F}$ coincide outside p. Hence it follows that

$$(\pi_g)_*(E) = \mathcal{F}.$$

Thus all the properties of (A_1) or (A) follow. This completes the proof of the required properness and Proposition 10 follows.

Remark 7. We have the well-known property that there is a desingularisation of $\mathcal{X}_T$ (whose singularity is an ordinary double point of type A) such that it is an isomorphism over $\mathcal{X}_T \setminus \{p\}$ and the fibre over p is a chain of $\mathbb{P}^1$'s (cf. [1]). We have not made use of this fact but it was a motivation for this proof. There is also the plausibility of another proof of properness along the following lines. One sees that to prove the properness of $\mathcal{G}_S$, it suffices to prove the properness of $\mathcal{G}$ i.e. in the case when $S = \operatorname{Spec} k$. For this let F be a family of torsion free sheaves on $X_0 \times T$ (T advr.) such that F is locally free of rank n outside $p \times t_0$ (t_0 closed point of T). It should be possible to write down a fairly explicit form of F in a neighbourhood of $(x_0 \times p)$, since one knows the versal deformation of the stalk of F at $(x_0 \times p)$, considered as a module over the local ring at $(x_0 \times p)$ (cf. [4], [10]). Then choosing a minimal set m_1 of generators of the stalk of F at $(x_0 \times p)$, we consider the canonical rational map of a neighbourhood of $(x_0 \times p)$ (which is a morphism outside $(x_0 \times p)$). Then the graph of this rational map should be proved to have the same properties as was done in the proof of the above proposition.

Now we have the main result of this paper.

Theorem 2. *Let $\mathcal{X} \longrightarrow S$ be a flat family of projective curves as in Definition–Notation 6. Then the Gieseker functor $\mathcal{G}(n,d)_S = \mathcal{G}_S$ (cf. Def. 7) is represented by a scheme $G(n,d)_S$ which is quasi-projective (and flat) over S. The closed fibre of $G(n,d)_S$ over S is a variety with analytic normal crossings. Besides, we have a canonical proper morphism*

$$\pi_* : G(n,d)_S \longrightarrow U(n,d)_S$$

where $U(n,d)_S$ is the relative moduli space of stable torsion free sheaves of $\mathcal{X} \longrightarrow S$ of rank n and deg *d. If $(n,d) = 1$, $G(n,d)_S$ is projective over S (of course it is known in this case that $U(n,d)_S$ is projective over S). Further (since we have assumed that $\mathcal{X}$ as a scheme is regular), $G(n,d)_S$ is regular as a scheme over k.*

Proof. Except the last assertion all the statements in the theorem have been proved above. The construction of the moduli space $G(n,d)_S$ follows on the same lines the proof given for the case $S = \operatorname{Spec} k$, given in the previous section before Def. 4. The proof of the last assertion follows on the same lines as in the work of Gieseker ([4]).

Remark 8. We have made simplifying hypotheses in the construction of the generalized moduli spaces. It should not be difficult to generalize it in the context of a general family $\mathcal{X} \longrightarrow S$ of stable curves and also work out the generalized Gieseker moduli in the semi-stable case (cf. Remark 5).

Remark 9. Let say $(n,d) = 1$. Consider the canonical morphism

$$\pi_* : G(n,d) \longrightarrow U(n,d).$$

Then it can be shown that if the torsion free sheaf $F \in U(n,d)$ on X_0 is of type r at p, then the fibre $(\pi_*)^{-1}(F)$ over F is isomorphic to the wonderful compactification of $PGL(r)$ (in the sense of De Concini and Procesi, cf. [2]). A crucial remark for determining this fibre is (1) of Remark 3 which states that for all vector bundles E on X_l such that $\pi_*(E) = F$ the restriction of E to X (normalisation of X_0) remains the same. Hence one has to investigate the patching data which extend $E|_X$ to X_l. These will be taken up in a later work.

Remark 10. It seems possible to construct $G(n,d)$ in a rather explicit manner, from the moduli space on X. Gieseker does this when $n=2$.

Appendix: Local theory

We shall now outline a proof of Proposition 8, essentially as in Gieseker (cf. [5]).

I. Let $\mathcal{G}'_S$ be the functor obtained from $\mathcal{G}_S$ by forgetting the imbeddings into Grassmannians, so that an element of $\mathcal{G}'_S(T)$ is represented by a family of curves $\Delta \longrightarrow T$ and a morphism $\Delta \longrightarrow \mathcal{X} \times_S T$ satisfying the condition (ii) of Def. 7. We have a canonical morphism

$$\mathcal{G}_S \longrightarrow \mathcal{G}'_S.$$

We claim that this functor is formally smooth. The proof of this is quite standard. Given an element θ of $\mathcal{G}'_S(T)$ where T is the spectrum of an artin local ring, such that if $\theta_0 \in \mathcal{G}'_S(T_0)$ (T_0 closed subscheme of T defined by an ideal of dimension one) obtained by restricting θ to T_0, can be lifted to an element of $\mathcal{G}_S(T_0)$, then we have to show that θ can be lifted to an element of $\mathcal{G}_S(T)$. Let θ be defined by $\Delta \longrightarrow T$ and $\Delta \longrightarrow \mathcal{X} \times_S T$. Then the lifting of θ_0 to an element of $\mathcal{G}_S(T_0)$ defines a vector bundle E_0 on Δ_0 (Δ_0 – the restriction of Δ to T_0), obtained as the pull-back of the tautological quotient bundle on $\mathrm{Gr}(m,n)$. It is easily seen that the problem is to extend E_0 to a vector bundle E on Δ and then the sections of E_0 to those of E. Let E_1 be the restriction of E_0 to the fibre of $\Delta \longrightarrow T$ over the closed point of T. The obstruction to extending E_0 lies in $H^2(\operatorname{End} E_1)$, which is zero. Extending sections is possible, since $H^1(E_1)=0$ (cf. (iv) of Def. 5).

II. We shall hereafter take $S=\operatorname{Spec} A$, $A=k[[t]]$. Let $W=\operatorname{Spec} k[[t_0,\dots,t_r]]$ endowed with an S-scheme structure $W \longrightarrow S$, defined by $t \mapsto t_0 \dots t_r$. A crucial point is the construction of an element θ of $\mathcal{G}'_S(W)$ (cf. Lemma 4.2 [5]) defined by a family of curves $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$, having the following properties:

(a) the closed fibre of $Z \longrightarrow W$ is X_r
(b) the closed subscheme of W corresponding to the singular fibres of $Z \longrightarrow W$ is the union of $t_i=0$ so that it has normal crossing singularities and is the inverse image of the closed point of S (by the morphism $W \longrightarrow S$).
(c) Let V be an affine open subset of the closed fibre of $Z \longrightarrow W$ containing its singular points (or we can take the semi-local ring at the singular points). Then $Z \longrightarrow W$ defines a deformation $Z_V \longrightarrow W$ of V. The property is that this is an (effective) miniversal deformation of V.
(d) $\psi^*(M)=L$, where L (resp. M) is the dualizing sheaf of Z (resp. $\mathcal{X} \times_S W$) relative to W.

Roughly speaking Z is obtained by taking the base change of $\mathcal{X}$ by $W \longrightarrow S$ and performing certain blow ups.

III. Let $T=\operatorname{Spec} B$, B an artin local ring ($k=$ algebra) such that T is also an S-scheme. We take elements of $\mathcal{G}'_S(T)$ represented by $Z' \longrightarrow T$ and $\psi' : Z' \longrightarrow \mathcal{X} \times_S T$ such that the fibre of $Z' \longrightarrow T$ over the closed point of T is the curve X_r. In this way we obtain a functor

$$\mathcal{G}'_r : (\text{Artin } S\text{-schemes}) \longrightarrow \text{sets}.$$

Through the element $\theta \in \mathcal{G}'_S(W)$ in II above, we get a canonical morphism

$$\lambda : W \longrightarrow \mathcal{G}'_r. \tag{i}$$

Gieseker shows that this morphism is formally smooth (cf. Proposition 4.5 and its proof, [5]). In other words suppose that we are given an element $\delta \in \mathcal{G}'_r(T)$ defined by $Z' \longrightarrow T$ and $\psi' : Z' \longrightarrow \mathcal{X} \times_S T$. We suppose further that there is a morphism $\phi_0 : T_0 \longrightarrow W$ such that the pull-backs by ϕ_0 of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ coincide with the restriction $\delta_0 \in \mathcal{G}'_r(T_0)$ of δ to T_0 (T_0 closed subscheme of T as above). Then it is shown that ϕ_0 can be extended to a morphism $\phi : T \longrightarrow W$ such that the pull-backs of $Z \longrightarrow W$ and $\psi : Z \longrightarrow \mathcal{X} \times_S W$ are isomorphic to $Z' \longrightarrow T$ and ψ'.

The proof can be sketched as follows. Given Z', ψ' and ϕ_0, we can find a morphism $\phi : T \longrightarrow W$ such that the pull-back (Z_1, ψ_1) of (Z, ψ) by ϕ is isomorphic to (Z', ψ') over T_0 (i.e. the restrictions to T_0 are isomorphic); besides (Z_1, ψ_1) and (Z', ψ') are locally isomorphic over T. This latter fact is a consequence of (c) of II. Given these local isomorphisms (whose restrictions to T_0 define the given isomorphism of $(Z_!, \psi_1)$ with (Z', ψ') over T_0), we find the obstruction to extending these local isomorphisms to a global one over T is an element μ

$$\mu \in H^1(X_r, \mathrm{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})),$$

where $\Omega^1_{X_r}$ denotes the sheaf of differentials and Hom denotes the "sheaf Hom". Similarly, the obstruction to extending

$$\psi'_0 : Z'_0 \longrightarrow \mathcal{X} \times_S T_0$$

to a morphism $Z' \longrightarrow \mathcal{X} \times_S T$ is an element μ'

$$\mu' \in H^1(X_r, \mathrm{Hom}(\pi * \Omega^1_{X_r}, \mathcal{O}_{X_r}))$$

and we see that μ maps to μ' under the canonical homomorphism

$$H^1(X_r, \mathrm{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow H^1(X_r, \mathrm{Hom}(\pi^* \Omega^1_{X_0}, \mathcal{O}_{X_r})), \tag{ii}$$

where π is the canonical morphism $X_r \longrightarrow X_0$. But since ψ' extends ψ'_0 we see that $\mu' = 0$. It is shown that (ii) is injective (Cor. 4.4, [5]) so that $\mu = 0$ and we get the required isomorphism of (Z', ψ') with (Z_1, ψ_1). Thus we see that λ is formally smooth.

One can view $\mathcal{G}'_r$ as a functor

$$\mathcal{G}'_r : (\text{Artin } k\text{-schemes}) \longrightarrow \text{sets},$$

i.e. if T is an artin k-scheme, in the definition of an element of $\mathcal{G}'_r(T)$ we take also an S-scheme structure for T. Then we see that an element of $\mathcal{G}'_r(T)$ gives a deformation of the singularities of X_r; to be more precise we get a deformation of V (V as in (c) of II above). Thus we get a functor

$$\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V). \tag{iii}$$

Then by (c) of II above, we see that the dimension d of the k-linear space of first order deformations of $\mathcal{G}'_r$ is $\geq (r+1)$. On the other hand by the formal smoothness of λ in (i) above, we see that $d \leq (r+1)$. Hence $d = (r+1)$ and we conclude that the functor λ (as well as the functor in (iii) above) is an isomorphism. Note that we have

$$\left\{ \begin{array}{l} \text{the pull-back by } \psi' \text{ of the dualizing sheaf of } \mathcal{X} \times_S T \\ \text{is the dualizing sheaf of } Z' \text{ (all relative to } T). \end{array} \right. \tag{iv}$$

IV. Let h be a closed point of $H = \mathrm{Hilb}^{P_1}(\mathcal{X} \times_S \mathrm{Gr}(m,n))$ such that $h \in \mathcal{G}_S(k)$. We have a curve X_r associated to h. Let $T = \mathrm{Spec}\, B$, B an artin local ring (with residue field k) such that T is also an S-scheme. We take elements of $\mathcal{G}_S(T)$ (resp. $H(T)$), represented by $\Delta \hookrightarrow \mathcal{X} \times_S T \times_S \mathrm{Gr}(m,n)$ such that the canonical image $\mathcal{G}_S(k)$ in $\mathcal{G}_S(T)$ is h. Then the fibre of $\Delta \longrightarrow T$ over the closed point of T is the curve X_r. In this way we obtain functors

$$\mathcal{G}_h(\text{resp. } H_h) : (\text{Artin } S\text{-scheme}) \longrightarrow \text{sets}.$$

We observe also that

$$\mathcal{G}_h = H_h \; (\text{since } \mathcal{G}_h(T) = H_h(T)). \tag{i}$$

Then we get a canonical functor $\mathcal{G}_h \longrightarrow \mathcal{G}'_r$, which is formally smooth by I. By (III), we have

$$\mathcal{G}'_r \xrightarrow{\sim} \mathrm{Def}(V) \simeq W. \tag{ii}$$

If we assume now that $\mathcal{G}_S$ is represented by an open subscheme $\mathcal{Y}$ of H, then by all the above considerations, we see that $\mathcal{Y}$ is smooth over k and $\mathcal{Y}_{s_0}$ (fibre over the closed point s_0 of S) has normal crossing singularities.

We denote by

$$\begin{cases} p_H : \Delta_H \longrightarrow H, \text{ the universal family over } H, \text{ and } \psi_H \\ \text{the canonical morphism } \psi_H : \Delta_H \longrightarrow \mathcal{X} \times_S H = \mathcal{X}_H. \end{cases} \tag{iii}$$

Let $\mathcal{O}$ be the local ring of H at h. We write $C = \mathrm{Spec}\, \mathcal{O}$ and $C_n = \mathrm{Spec}\, \mathcal{O}/m^n$ (m maximal ideal of $\mathcal{O}$).

Let

$$\begin{cases} p_C : \Delta_C \longrightarrow C, \; \psi : \Delta_C \longrightarrow \mathcal{X}_C. \\ p_n : \Delta_{C_n} \longrightarrow C_n, \; \psi : \Delta_{C_n} \longrightarrow \mathcal{X}_{C_n} \end{cases} \tag{iv}$$

denote the base changes of (iii) by the canonical morphism $C \longrightarrow H$, $C_n \longrightarrow H$. Since the fibre of ψ_H over the closed point of C is X_r, by the deformation theory of ordinary double points, we see that the fibres of p_C have only ordinary double point singularities. Let L (resp. M) denote the dualizing sheaf (in our case a line bundle) of Δ_C (resp. $\mathcal{X}_C$) relative to C. We denote the base changes of L and M by the morphism $C_n \longrightarrow C$ by L_n and M_n respectively. Note that L_n (resp. M_n) is the dualizing sheaf of Δ_{C_n} (resp. $\mathcal{X}_{C_n}$) relative to C_n (cf. [3]). Then by (iv) of (III), we have

$$\psi_n^*(M_n) = L_n \quad \text{for all } n. \tag{v}$$

We claim that

$$\psi_C^*(M) \simeq L. \tag{vi}$$

To prove (vi), set $N = L^{-1} \otimes \psi_C^*(M)$ and $N_n = L_n^{-1} \otimes \psi_n^*(M_n)$. Now N_n are trivial line bundles. Then by Grothendeick's comparison theorems, we have

$$\widehat{(p_C)_*}(N) = \varprojlim_n (p_n)_*(N_n) \tag{vii}$$

where the LHS denotes the completion of the direct image $(p_C)_*(N)$ considered as an $\mathcal{O}$-module. Since N_n are trivial, we see that the RHS of (vii) is $\simeq \widehat{\mathcal{O}}$. Then the restriction of N to the generic fibre of p_C has a non-trivial section. Applying the semi-continuity theorem, the space of sections of N restricted to any fibre of p_C is 1-dimensional, which implies that "$(p_C)_*$ commutes with base change". From this one concludes easily that there is a section s of N which does not vanish identically on the closed fibre of p_C. Since the restriction of N to the closed fibre of p_C is trivial, we see that s is, in fact, everywhere non-zero on Δ_C. This shows that N is trivial and proves the claim (vi).

From the above considerations, we see easily that there is a neighbourhood U of h in H such that for the morphisms

$$p_U : \Delta_U \longrightarrow U, \psi_U : \Delta_U \longrightarrow \mathcal{X}_U \tag{viii}$$

obtained as base change of p_H and ψ_H by $U \longrightarrow H$, we have the following properties:

$$\begin{cases} \text{The fibres of } p_U \text{ have only ordinary double point singularities.} \\ \text{Besides } \psi_U^*(M_U) = L_U, \text{ where } L_U \text{ (resp.) is the dualizing} \\ \text{sheaf of } \Delta_U \text{ (resp. } \mathcal{X}_U) \text{ relative to } U. \end{cases} \tag{ix}$$

We claim that (ix) implies that the fibres of p_U are smooth or curves of the form X_n and ψ_U induces the canonical morphism on fibres over U i.e. the property (ii) of Def. 7 is satisfied. The claim is a consequence of the following:

Lemma. Let Y be a connected projective curve of arithmetic genus g with only ordinary double point singularities. Let $f : Y \longrightarrow D$ be a morphism, where either D is a smooth projective curve of genus g or $D \simeq X_0$. Suppose that the pull-back by f of the dualizing sheaf of D is isomorphic to the dualizing sheaf of Y. Then f is an isomorphism if D is smooth; otherwise Y is of the form X_n and f is the canonical morphism $X_n \longrightarrow X_0$.

The above lemma is an easy consequence of the characterization of the dualizing sheaf of Y by a sheaf of meromorphic differentials on the normalization of Y (cf. [3]).

Thus it follows that

$$\Delta_U \longrightarrow U \quad \text{and} \quad \psi_U : \Delta_U \longrightarrow \mathcal{X}_U \tag{x}$$

satisfy the property (ii) of Def. 7 define open subsets of H so that we can suppose that the morphisms in (x) satisfy all the conditions of Def. 7. Thus it follows that $\mathcal{G}_S$ is represented by an open subscheme $\mathcal{Y}$ of H.

Thus to conclude the proof of Proposition 8, it remains to prove that $\mathcal{Y}_{s_0}$ is irreducible. Since $\mathcal{Y}_{s_0}$ has normal crossing singularities, it follows that the open subset $\mathcal{Y}_1$ of smooth points of $\mathcal{Y}_{s_0}$ is dense in $\mathcal{Y}_{s_0}$. It suffices to prove that $\mathcal{Y}_1$ is irreducible. Then $\mathcal{Y}_1$ can be identified with an open subset of the open subscheme R of the quot scheme $Q(m, P_2)$. One knows that R is irreducible and the required irreducibility follows.

Remark. Let Def (X_r) denote the functor of deformations of X_r. Then we have morphisms

$$\mathcal{G}'_r \xrightarrow{i_1} \text{Def}\,(X_r) \xrightarrow{j_1} \text{Def}\,(V) \tag{i}$$

where i defines a subfunctor. Recall that the first order deformations of X_r can be identified with $\text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})$ and we have an exact sequence (cf. [3]).

$$\begin{aligned} 0 \longrightarrow H^1(X_r, \text{Hom}(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow \text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r}) \\ \xrightarrow{j_2} H^0(X_r, \text{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})) \longrightarrow 0. \end{aligned} \tag{ii}$$

Now j_2 can be identified with the canonical map of the first order deformations induced by j_1. The above considerations show that the first order deformations of $\mathcal{G}'_r$ can be identified with a supplement of $\ker j_2$.

One can arrange the above proof of Proposition 8 slightly differently as follows. The argument in IV above shows that if N denotes the subspace of $\mathrm{Ext}^1(\Omega^1_{X_r}, \mathcal{O}_{X_r})$ corresponding to the first order deformations of $\mathcal{G}'_r$, we have $N \cap \ker j_2 = (0)$. This shows that the linear map on first order deformations induced from the canonical morphism $\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V)$ is injective, in particular $\dim N \leq (r+1)$. On the other hand, as we saw before, by (c) of II above, $\dim N \geq (r+1)$. We then easily conclude that $\mathcal{G}'_r \longrightarrow \mathrm{Def}\,(V)$ is an isomorphism i.e. $\mathcal{G}'_r \longrightarrow W$ is an isomorphism, which would prove Proposition 8.

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