The role of the critical layer in the stability of viscous shear flow

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The critical layer affects the stability of a variety of shear flows at the lowest order. This article studies the influence of the critical layer in boundary layers and in the parallel flow of miscible fluids, and concludes that the convective as well as the viscous terms of the Navier-Stokes equation can contribute towards the dominance of the critical layer. The result may be stabilizing or destabilizing depending on the balance between different terms in the stability equation.

1. Background

It is well known[1–5] that the inviscid equations for stability of shear flows display singularities at the critical layer, where the mean velocity is close to the phase speed of the disturbance, and at the wall, and that viscous effects must be included in these layers to smooth out these singularities. This implies that some viscous terms must be large in each of these layers. It is also known that uniformly valid solutions may be obtained only if the lowest-order equation as well as the next higher order equation in the critical layer are considered. The first is for eliminating the singularities while the second is essential for achieving an asymptotic matching between the solutions in the critical and bulk layers. Based on these ideas, Govindarajan and Narasimha[6–8] have derived the lowest order ('minimal composite') stability equations for spatially growing incompressible two-dimensional boundary layers. The rationale for this approach and the resulting equations have been summarized in a paper by Narasimha and Govindarajan[9] elsewhere in this issue. The derivation of minimal composite equations is based on the fact that the critical layer, the wall layer and the bulk of the boundary layer form three distinguished limits. Stability equations are formulated separately in each of these limits and used to construct a composite equation containing all effects up to a desired order of accuracy and valid across the entire boundary layer.

Ongoing work at the Nehru Centre, Bangalore has extended this approach to study three-dimensional boundary layers over swept wings[10], as well as compressible two-dimensional boundary layers[11]. Apart from these, the stability of channel flow of two miscible fluids of equal densities but different viscosities is being studied under the parallel flow approximation[12]. It is noticed that a common feature in all these flows is the dominance of the critical layer in the stability process: in boundary layer flows, all non-parallel effects at the low orders come from the critical layer alone, while in two-fluid flow, stability characteristics when the mixed layer overlaps the critical layer are very different from those when the two layers are well-separated.

The present paper is aimed at understanding how the critical layer affects stability in the flow situations mentioned above. It is shown that the viscous as well as the convective terms in the Navier-Stokes equations can contribute towards large effects within the critical layer. The result may be stabilizing or destabilizing as will be discussed below, but the point of interest is that the critical layer has an important role to play.

In §2 the stability equations for boundary layers are discussed, with emphasis on the role of the critical layer. The stability of the symmetric flow of two miscible fluids through a channel is discussed in §3.

2. The critical layer in boundary layer stability

Consider a flow quantity, q, where q could be component of velocity, for example. In a normal mode analysis, the stability equation is formulated in terms of the disturbance amplitude ϕ, given by

$$q(x,y,t)=\overline{q}(x,y)+\phi(x,y)\exp(\mathrm{i}[\textstyle\int\alpha\,\mathrm{d}x-\omega t]), \qquad (1)$$

where x and y are the streamwise and normal coordinates respectively, t is time, the overbar indicates a mean quantity, α and ω respectively are the streamwise wave number and the frequency of the disturbance.

In boundary layer flows, the disturbance amplitude ϕ may be expanded in terms of a small parameter ε[8,10,11] as

$$\phi=\sum_k \varepsilon^k\chi_k+\sum_m \varepsilon^m(\log\varepsilon)\lambda_m, \qquad (2)$$

$$k=0,1,2,\ldots, \qquad m=1,2,\ldots$$

In three-dimensional as well as in compressible boundary layers, an appropriate choice for ε is

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Prof. Satish Dhawan's work (with Prof. Narasimha) on boundary layer transition was my introduction to research and to fluid dynamics – it is a special pleasure to dedicate this article to him on his 80th birthday.

$$\varepsilon \equiv (\alpha R)^{-1/6}, \tag{3}$$

where R is the boundary layer Reynolds number. In all these cases, the critical layer thickness $\varepsilon_c \propto (\alpha R)^{-1/3}$, the wall layer thickness $\varepsilon_w \propto (\alpha R)^{-1/2}$, while the characteristic thickness of the bulk layer is of course $\varepsilon_b \propto R^{-1}$. Eq. (2) may be written as[8]

$$\phi = \chi_0 + \varepsilon^2\chi_{c1} + \varepsilon^2 \log(\varepsilon)\lambda_{c1} + \varepsilon^3\chi_{w1} + \varepsilon^4\chi_{c2} + \varepsilon^4 \log(\varepsilon)\lambda_{c2} + O(R^{-1}), \tag{4}$$

where the subscripts c and w denote critical and wall quantities respectively.

The primary motivation for studying the critical layer and its equations in greater detail is the observation that in incompressible two or three-dimensional boundary layers (over swept wings) the expression for χ_{w1} is contained in that for χ_{c1}. This immediately implies that *at any order below* R^{-1} *there is no term in the stability equations which is significant in the wall layer alone.* (The wall layer does, however, contribute to some existing terms in the equations.) In particular, the low-order effects of boundary layer growth are all due to the critical layer alone. In the case of compressible boundary layers, the wall layer does contribute at the lowest order, especially to the energy equation. However, here too, all non-parallel effects at any order lower than $O(R^{-1})$ are due to the critical layer alone. We discuss below the lowest-order composite stability equations, i.e. those which are correct up to but not including $O(R^{-2/3})$.

Lowest-order boundary layer stability equation

An ordinary differential equation governs boundary layer stability at the lowest order[7]:

$$(c-\Phi')(\phi''-\alpha^2\phi)+\Phi'''\phi = \frac{\mathrm{i}}{\alpha R}(\phi^{\mathrm{iv}} + p\Phi\phi'''), \tag{5}$$

where c is the phase speed of the disturbance and the primes stand for differentiation with respect to y. All quantities have been non-dimensionalized by the momentum thickness, θ and the local freestream velocity, U_∞; Φ' is the basic velocity (correct to the lowest order) in the streamwise direction, and p is defined by

$$\frac{\mathrm{d}U_\infty\theta}{\mathrm{d}x_d} \equiv \frac{U_\infty p}{R}. \tag{6}$$

The boundary conditions are given by

$$\phi = \phi' = 0 \text{ at } y = 0 \text{ and } \phi, \phi' \to 0 \text{ for } y \to \infty. \tag{7}$$

The term $p\Phi\phi'''$ on the right-hand side of the stability eq. (5) is large only in the critical layer. It arises out of the convective term $V\phi'''$ on the left-hand side of the conservation equation for the disturbance vorticity, where V is the normal component of the basic velocity.

In swept wing boundary layers, the stability may be described at the lowest order by a single ordinary differential equation which is a generalization of (5) (refs 10, 13). In compressible boundary layers, the lowest-order stability equations are a set of four ordinary differential equations in the disturbance variables $\tilde{u}$ (streamwise velocity), $\tilde{v}$ (normal velocity), $\tilde{T}$ (temperature) and $\tilde{p}$ (pressure)[11,14]. In each of these equations, there is one term which is a consequence of boundary layer growth. As was the case in incompressible swept-wing boundary layers, the non-parallel terms at the lowest order all appear due to the critical layer and emanate from the convective part of the conservation equations: in this case too, they are due to the advection of the respective disturbance quantity by the normal component of the basic flow.

Discussion

The critical layer equations for incompressible two-dimensional flows is examined below and general conclusions are drawn for other boundary layer flows as well. The lowest order equations in the critical layer are given by[8]

$$\chi_{c0}^{(\mathrm{iv})} - \mathrm{i}\eta_c\Phi_c''\chi_{c0}'' = 0, \tag{8}$$

$$\chi_{c1}^{(\mathrm{iv})} - \mathrm{i}\eta_c\Phi_c''\chi_{c1}'' = \mathrm{i}\Phi_c'''\left(\frac{1}{2}\eta_c^2\chi_{c0}'' - \chi_{c0}\right) - p\Phi_c\chi_{c0}''', \tag{9}$$

and

$$\lambda_{c1}^{(\mathrm{iv})} - \mathrm{i}\eta_c\Phi_c''\lambda_{c1}'' = 0. \tag{10}$$

Here,

$$\eta_c \equiv \frac{y - y_c}{\varepsilon_c}. \tag{11}$$

In the expansion (4), terms at least up to χ_{c1} and λ_{c1} must compulsorily be retained to achieve an asymptotic matching between the critical layer solutions and the outer inviscid solutions. Thus, the lowest order equation that can be constructed for critical layer is[8]

$$(\Phi' - c)\chi'' - \Phi'''\chi = -\frac{\mathrm{i}}{\alpha R}[\chi^{\mathrm{iv}} + p\Phi\chi''']. \tag{12}$$

Close to neutral stability, a balance must be achieved between the left hand side of eq. (12) operating on the real part of the eigenfunction χ_r and the right hand side operating on the imaginary part χ_i, i.e.

$$(\Phi' - c_r)\chi_r'' - \Phi'''\chi_r = \frac{1}{\alpha_r R_{\text{neutral}}}[\chi_i^{\text{iv}} + p\Phi\chi_i''']. \quad (13)$$

In the parallel flow approximation, only the first term on the right hand side of eq. (13) would be present. What then is the effect of the addition of the lowest-order non-parallel term on the neutral Reynolds number? We assume that the left-hand side is not very different for the parallel and non-parallel solutions – this is to be expected and can be shown numerically to be the case. In particular, the eigenfunction is practically the same[15] for the Orr-Sommerfeld and the full non-parallel equations. It is therefore plausible that a higher value of the neutral Reynolds number would be required to satisfy eq. (13) if the third and the fourth derivatives of the imaginary part of χ were to be of the same sign. (The factor $p\Phi_c$ is almost always positive in the boundary layer.) Conversely, if χ_i^{iv} and χ_i''' were of opposite signs, the non-parallel term would decrease the neutral Reynolds number.

Several computations of χ_i^{iv} and χ_i''' under different conditions show that in general the two quantities are of the same sign close to the location $y_{\max}$ of the maximum in the disturbance eigenfunction $\tilde{u}$, and of opposite signs at intermediate heights above this location. The critical layer is located just below $y_{\max}$ for most of the lower limb of the stability boundary. Therefore, including the non-parallel term increases R_{neutral} on the lower limb. On the upper limb, the critical layer is located at a greater distance from the wall – it is often in the region where the signs of χ_i^{iv} and χ_i''' are different. Again, the non-parallel term would have a stabilizing influence, this time by decreasing the neutral Reynolds number. We may conclude that boundary layer growth *at the lowest order* is expected (as a first guess) to exert a *stabilizing* influence.

The location of the critical layer y_c as a fraction of the boundary layer thickness is shown for flat-plate flow ($m = 0$, where m is the Falkner-Skan pressure gradient parameter) in Figure 1. In order to get a better estimate of the critical layer height in terms of the relevant stability parameters, we plot y_c along the stability boundary as a function of $y_{\max}$ in Figure 2. The qualitative difference in the critical layer height between the lower and upper limbs is immediately apparent. The two heights y_c and $y_{\max}$ are plotted for an adverse pressure gradient flow with $m = -0.06$ in Figure 3. The trend there too is the same. Thus, we do not expect any qualitative differences in the effect on stability with changing pressure gradients.

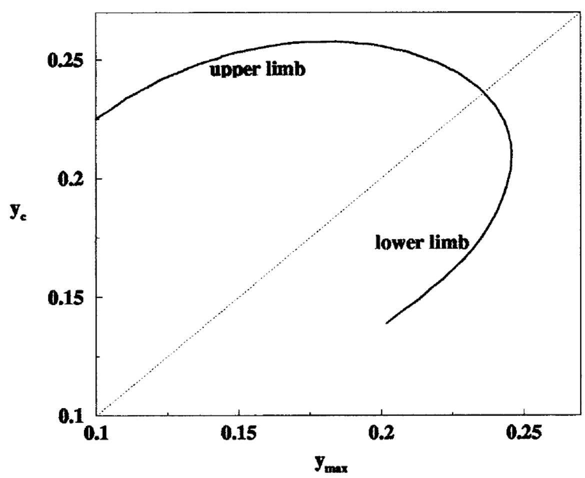

Figure 2. Location of the critical layer as a function of the height of the maximum in $\tilde{u}$. $m = 0$. The dashed line indicates where the two would be equal.

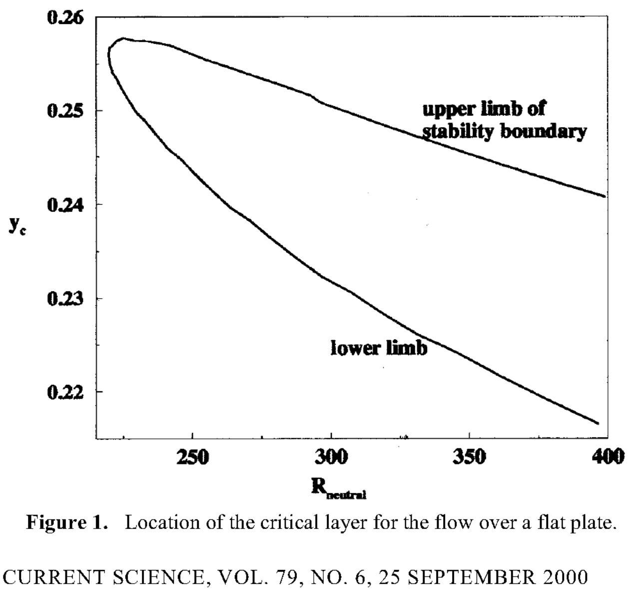

Figure 1. Location of the critical layer for the flow over a flat plate.

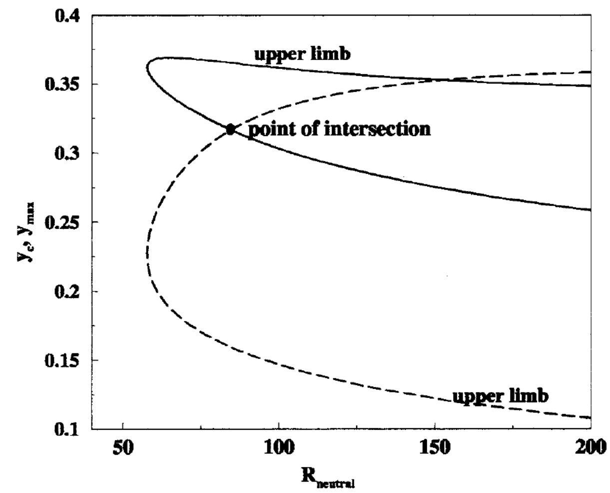

Figure 3. Location of the critical layer (———) and the height of the maximum in $\tilde{u}$ (----) for an adverse pressure gradient flow, $m = -0.06$.

In comparisons of stability boundaries predicted by the minimal composite equation with parallel-flow results for two-dimensional and three-dimensional flows, it was noticed that the lowest-order non-parallel equation always predicted a more stable flow. In the present paper, a direct estimate of the influence of the non-parallel term is obtained by solving the lowest order stability equation for a Falkner-Skan flow with and without the third derivative term: the equations solved are (5) and

$$(c-\Phi')(\phi''-\alpha^2\phi)+\Phi'''\phi=\frac{i}{\alpha R}\phi^{iv}. \qquad (14)$$

Eq. (14) is the lowest-order stability equation under the parallel-flow approximation.

In the flow over a flat plate (Figure 4) as well as in adverse pressure gradient flow (Figure 5) the non-parallel term stabilizes the flow, confirming expectations from the crude arguments above. In these figures, the quantity F is proportional to the dimensional disturbance frequency, and is given by $F \equiv \omega / R^{(1-3m)/(1+m)}$.

It has been mentioned above that for incompressible flow, the equation for the lowest order explicit wall term χ_{w1} in the expansion (4) is contained in the equation for χ_{c1}. An examination of (4) reveals that a stability equation of the next higher order can be derived by including the terms in χ_{c2} and λ_{c2}, whose equations are given respectively[8] by

$$\chi_{c2}^{(iv)}-i\eta_c\Phi_c''\chi_{c2}''=i\Phi_c'''\left(\frac{1}{2}\eta_c^2\chi_{c1}''-\chi_{c1}\right)$$

$$-p\Phi_c\chi_{c1}'''-i\eta_c[\Phi_c^{(iv)}+\alpha^2\Phi_c'']\chi_{c0}-p\eta_c\Phi_c'\chi_{c0}'''$$

$$+\left\{\frac{i\eta_c^3\Phi_c^{(iv)}}{6}+2\alpha^2-(2q-p)\Phi_c'\right\}\chi_{c0}''+R\Phi_c'\frac{\partial\chi_{c0}''}{\partial x}, \qquad (15)$$

where $q \equiv d\theta/dx$, and

$$\lambda_{c2}^{(iv)}-i\eta_c\Phi_c''\lambda_{c2}''=i\Phi_c'''\left(\frac{1}{2}\eta_c^2\lambda_{c1}''-\lambda_{c1}\right)-p\Phi_c\lambda_{c1}'''. \qquad (16)$$

The composite equation thence derived turns out to be the lowest-order parabolic equation (LOP)[8] for incompressible boundary layer stability, correct up to but not including $O(R^{-1})$:

$$(c-\Phi')(\phi''-\alpha^2\phi)+\Phi'''\phi$$

$$=\frac{i}{\alpha R}\left\{\phi^{iv}+p\Phi\phi'''+\left(-2\alpha^2+\Phi'(2q-p-\frac{\partial}{\partial x}\right)\phi''\right\}. \qquad (17)$$

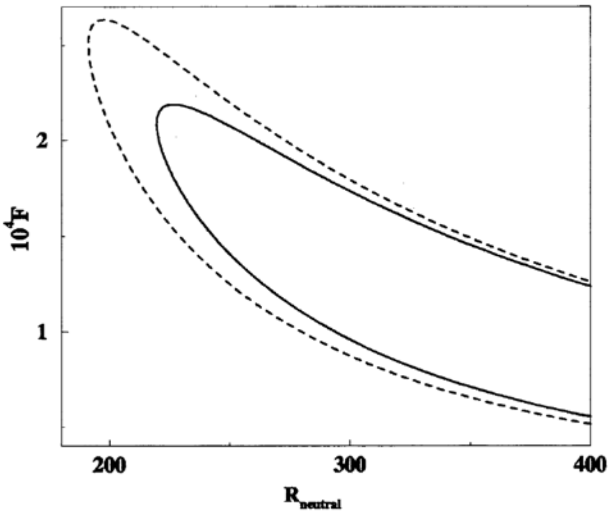

Figure 4. Stability boundary from lowest order theories. Non-parallel (eq. (5)):—— ; parallel (eq. (14)): ----. m = 0.0.

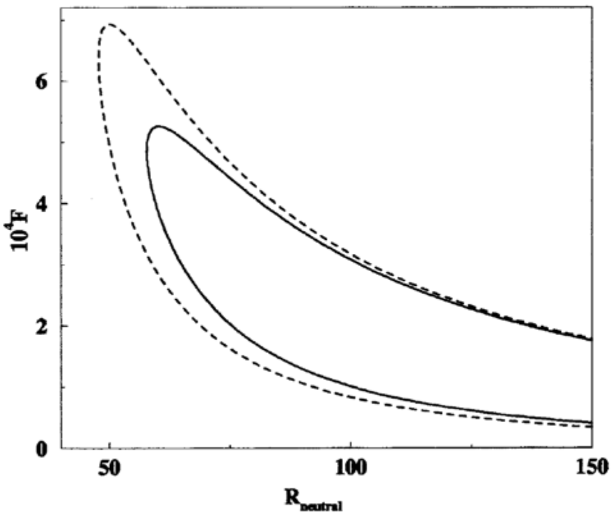

Figure 5. Stability boundary predicted by (eq. (5)) (——) compared to that from (eq. 14)) (----). m = –0.06.

Using computed results for the derivatives of the disturbance eigenfunction, it is possible to predict qualitatively what the effect of the $O(R^{-2/3})$ terms will be. The coefficient of ϕ'' on the right hand side of eq. (17) is usually positive in the neighbourhood of the critical layer. The sign of ϕ_i' is observed to be opposite to that of ϕ_i''' throughout the region of interest. Thus, in the neighbourhood of neutral stability, the effect of the second derivative term on the right hand side of eq. (17) may be expected to oppose the effect of the third derivative term. If this argument were to hold, the $O(R^{-2/3})$ effect of the critical layer would be to destabilize the flow. The neutral stability boundary obtained by solving the LOP eq. (17) is shown in Figure 6. It is seen that the effect of the critical layer at the higher order $[O(R^{-2/3})]$ is to cancel out its own stabilizing effect at the lower order $[O(R^{-2/3})]$.

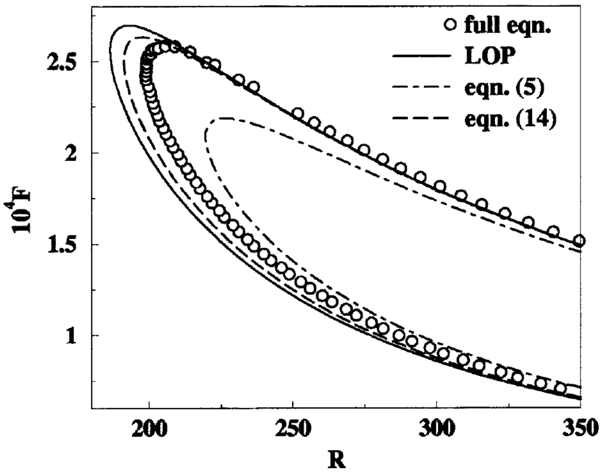

Figure 6. Effect of the critical layer at $O(R^{-2/3})$.

The symbols shown in Figure 6 are the results from the 'full' non-parallel equation given by[6]

$$\Big\{(c-\Phi_0')(D^2-\alpha^2)+\Phi_0'''+\frac{1}{\mathrm{i}\alpha R}\Big(D^4+p\Phi_0 D^3$$
$$+[-2\alpha^2+(2q-p)\Phi_0'-\mathrm{i}\alpha\Phi_1']D^2+[2yq\alpha^2(c-\Phi_0')$$
$$-p\alpha^2\Phi_0+(2q-p)\Phi_0'']D+\alpha^4+(q-2p)\alpha\omega+p\Phi_0'''$$
$$+3(p-q)\alpha^2\Phi_0'+\mathrm{i}\alpha(\Phi_1'''+\alpha^2\Phi_1')+(-\omega+3\alpha\Phi_0')R\alpha'$$
$$+[\Phi_0'''+3\alpha^2\Phi_0'-2\alpha\omega-\Phi_0'D^2]R\frac{\partial}{\partial x}\Big)\Big\}\phi=0, \quad (18)$$

which includes all terms nominally of $O(R^{-1})$. Here D stands for differentiation with respect to y, and the mean flow is given by

$$\Phi=\Phi_0+\frac{1}{R}\Phi_1+\cdots. \quad (19)$$

For the case shown here, $m = 0$, Φ_0 obeys the Blasius equation and $\Phi_1 = 0$. An example of the effect of higher order mean velocity in a pressure gradient flow has been worked out in GN95. In that case too, results from eq. (17) agree well[8] with solutions of eq. (18).

It is significant to note that the wall layer does not introduce any terms up to the order $O(R^{-2/3})$ either. In this regard, however, compressible boundary layers differ from incompressible flows. In the former, several terms in the low-order equations are present in the composite equations by virtue of being large in the wall layer alone, such as the bulk viscosity terms and one term in the energy equation. The four compressible stability equations mentioned above may be reduced to three equations in three variables by eliminating the pressure, $\tilde{p}$. It is shown by Seshadri *et al.*[11] that the equations are considerably simplified in the process. In particular, several of the 'wall' terms, including all terms in the bulk viscosity vanish. However, the wall term in the energy equation stays. This term, which provides a means for energy transfer to the disturbances by the basic shear stress at the wall, is likely to be significant at high Mach numbers.

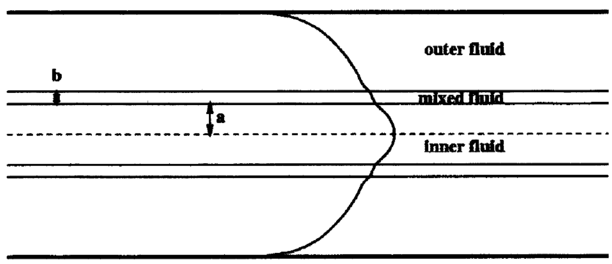

Figure 7. Schematic of flow of two miscible fluids through a channel.

3. Miscible flow of two fluids through a channel

In this section, we turn our attention to the channel flow of two fluids to demonstrate that the critical layer can come into play in determining stability characteristics in completely different situations and through different means. The effect of changes in viscosity normal to the flow has been studied extensively from two points of view: (i) continuous viscosity stratification in the normal direction, such as in ref. 16, and (ii) a viscosity jump at the interface of two immiscible fluids[17]. In the latter, the instability is driven by dynamics at the interface. In a real flow, the interface would be smeared out into a thin mixed layer. The stability of the resulting flow has been shown[12] to be qualitatively different from either of the two cases above. The primary reason for this difference is the behaviour in the critical layer.

The situation considered is the symmetric flow of two miscible fluids through a channel. The schematic of the flow is shown in Figure 7. The inner fluid occupies a width 2 a (scaled by the channel half-width H) on either side of the centre line while the outer fluid is in two layers, each occupying the width $1-(a+b)$. The two fluids are separated by a mixed layer of thickness b within which the viscosity varies from that of the inner fluid to that of the outer. Although in reality the thickness of the mixed layer would increase downstream, the parallel-flow approximation has been used here.

The stability of this flow is governed by the modified Orr-Sommerfeld equation for flow with varying viscosity[18]:

$$(c-\Phi')(\phi''-\alpha^2\phi)+\Phi'''\phi=\frac{\mathrm{i}}{\alpha R}[\mu\phi^{\mathrm{iv}}+2\mu'\phi'''+$$
$$(\mu''-2\alpha^2\mu)\phi''-2\alpha^2\mu'\phi'+(\alpha^4\mu+\alpha^2\mu'')\phi], \quad (20)$$

with the boundary conditions

$$\phi(\pm 1) = \phi'(\pm 1) = 0. \quad (21)$$

The Reynolds number here is defined with respect to the centreline velocity, the channel half-width and the viscosity of the inner fluid as $R \equiv \rho U_c H/\mu_i$; the viscosity μ has been scaled by μ_i.

An example of the results is shown in Figure 8, where the viscosity of the outer fluid is a mere 5% higher than that of the inner fluid. The critical layer is located in this case at about $a = 0.8$. When the critical layer and the mixed layer are well-separated, i.e. when $a < 0.65$, the critical Reynolds number (based on the inner fluid viscosity) is somewhat higher than that for a single fluid flow. However, when the mixed layer and the critical layer overlap, the flow is drastically destabilized. The 'lowest-order' reason for this effect lies in the third derivative term on the right hand side of eq. (20). The third derivative of the disturbance eigenfunction is largest within the critical layer while the coefficient of this term is non-zero only in the mixed layer: it is only when the two overlap that this term affects the stability of the flow. When the height a lies within the critical layer of the dominant disturbance mode, the effect is destabilizing when the outer fluid is more viscous, i.e. when $\mu' > 0$, and when the less viscous fluid is placed in the outer layer, the effect is stabilizing.

This result has implications for passive control of the flow. In order to trigger transition at a low Reynolds number, the viscosity of the fluid layer between the critical layer and the wall may be increased by a small amount by some means. In other words, the more viscous fluid must be placed in the outer layer, and the fraction of mass flux of the outer fluid must be so chosen as to locate the mixed layer at the height y_c of the critical layer of the primary disturbance mode. It is possible to use this idea to stabilize the flow as well, but this is not as straightforward, as discussed in ref. 12. When the more viscous fluid is placed in the core, more complicated phenomena emerge, second disturbance mode may become dominant at some Reynolds number. In this case, in order to achieve flow control, the viscosity ratio must be carefully adjusted as well.

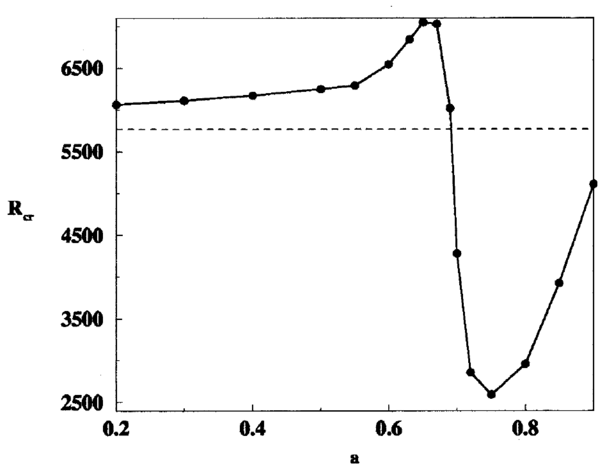

Figure 8. Dependence of critical Reynolds number on the location of the mixed layer; $\mu_0 = 1.05$, $b = 0.1$. The dashed line shows the critical Reynolds number for a single fluid flowing through a channel.

4. Conclusions

The critical layer often plays a significant role in stability. In boundary layers, the spatial development of the flow explicitly enters the stability equation at the order $R^{-1/3}$ due to the critical layer. Whether the boundary layer is two or three-dimensional, compressible or incompressible, this effect is due to the advection of disturbance quantities by the mean normal flow, i.e. it arises out of convective terms in the momentum conservation equations. The lowest order non-parallel term often has a stabilizing effect on the solution, which is neutralized by the destabilizing effect at the next higher order.

In the channel flow of two miscible fluids, the critical layer has a significant role to play if it lies in the vicinity of the mixed layer. The effect originates this time from the viscous part of the momentum equation. If the mixed layer is located correctly, the flow may be significantly destabilized or stabilized depending on the sign of the viscosity gradient.

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