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SYMMETRY OF POSITIVE SOLUTIONS OF SOME NONLINEAR EQUATIONS

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1. Introduction

In recent years, a lot of interest has been shown in the study of symmetry properties of solutions of nonlinear elliptic equations, reflecting the symmetry of the domain. In a famous paper, Gidas, Ni and Nirenberg [4] showed that if Ω is smooth, convex and symmetric in one direction, say, that of x_1, then any positive classical solution of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a locally Lipschitz continuous function, must be also symmetric with respect to x_1. The proof of this result is based on the moving plane method and the maximum principle.

In a recent paper, Berestycki and Nirenberg [2] have substantially simplified the moving plane method obtaining, among other results, the symmetry of the positive solutions of (1.1) without assuming any smoothness on Ω.

When the dimension of the space is two, Lions [9] suggested a method of proving the radial symmetry of positive solutions in a ball when f is positive, without assuming anything on the smoothness of f. While previous results were proved using variants of the moving plane method, this result can be proved using

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a combination of an isoperimetric inequality and Pohozaev's identity. It does not work, however, if $N > 2$.

No results are known, to the best of the authors' knowledge, about symmetry of solutions when we drop the hypothesis of Lipschitz continuity or positivity of f. Thus one would like to know if f is just continuous, but nonnegative, whether positive solutions reflect the symmetry of the domain as before.

Very little is known, even assuming f to be smooth, if we replace the Laplace operator in (1.1) by a closely related nonlinear operator, viz the p-laplacian (for $p \neq 2$). More precisely, we look at positive solutions of the equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

and pose analogous question as before. If Ω were a ball, Badiale and Nabana [1], using the same method as Berestycki and Nirenberg [2] and again taking f Lipschitz continuous, prove the symmetry of the positive solutions under the crucial assumption that we know a priori that the gradient of the solution vanishes only at the origin. Another result, again for the ball, is due to Kesavan and Pacella [7] which shows the radial symmetry of positive solutions with the assumption that $p = N$, the dimension of the space. Their method is a completion and generalization of the idea of Lions [9] and thus uses isoperimetric inequalities, assuming that the nonlinearity f is only continuous but positive. The method does not give any result for $p \neq N$.

In the present paper, we wish to study problems (1.1) and (1.2) when f is nonnegative and only continuous. However, by virtue of a result due to Kichenassamy and Smoller [8], we cannot hope that all nonnegative solutions in a ball are radial if f changes sign.

By suitable approximation procedures, we show that isolated solutions with non-vanishing index (w.r.t. a canonical formulation of these problems as operator equations) are limits of symmetric functions and hence are symmetric themselves. Note that with our method we only get the symmetry of the solution of (1.1) or (1.2) but not the strict monotonicity in the x_1 direction. However we cannot expect anything better, even if Ω is the ball and $p = 2$, since it is possible to construct for any $p > 1$ an example of a nonnegative nonlinearity $f(u)$ for which there exists a symmetric but not strictly radially decreasing positive solution of (1.2) (see Example 5.1).

The paper is organized as follows. Section 2 gives an abstract approximation theorem which will permit us to obtain the above mentioned solution as limits of solutions of problems with greater regularity. In Section 3 we will set up the problem (1.1) in the abstract framework and deduce some symmetry results.

Sections 4 and 5 contain the case of the p-laplacian and the example mentioned above.

2. An abstract result

Let X be a Banach space and let $T : [0, \infty] \times X \to X$ be continuous and such that for each $t \geq 0$, the map $T(t, \cdot\,) : X \to X$ is compact. We set

$$\phi_t = I - T(t, \cdot\,). \tag{2.1}$$

If x is an isolated zero of ϕ_t, then the Leray–Schauder degree $\deg(\phi_t, B_\varepsilon(x), 0)$ where $B_\varepsilon(x) = \{y \in X \text{ such that } ||y - x|| < \varepsilon\}$, is well-defined and independent of ε for small values of ε. Thus the index of x, denoted $i(\phi_t, x, 0)$ is well-defined and given by

$$i(\phi_t, x, 0) = \lim_{\varepsilon \to 0} \deg(\phi_t, B_\varepsilon(x), 0). \tag{2.2}$$

From the very definition of index the following theorem is deduced

Theorem 2.1. *Let x_0 be an isolated zero of ϕ_0 such that*

$$i(\phi_0, x_0, 0) \neq 0. \tag{2.3}$$

Then there exist sequences $\{\varepsilon_n\}, \varepsilon_n \to 0$ and $\{x_n\},\ x_n \to x_0$ in X, such that

$$x_n - T(\varepsilon_n, x_n) = 0. \tag{2.4}$$

Proof. By virtue of (2.3), there exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, the degree $\deg(\phi_0, B_\varepsilon(x_0), 0)$ is independent of ε and is non-zero. Let us consider the map $H(\theta, x) : [0, 1] \times X \to X$ where $H(\theta, x) = x - T(\theta\varepsilon_o, x)$. We have one of the two alternatives:

(a) $I - T(\theta\varepsilon_o, \cdot\,)$ does not vanish on $\partial B_{\varepsilon_o}(x_0)$ for all $\theta \in [0, 1]$
(b) there exists $0 < \theta \leq 1$ such that $I - T(\theta\varepsilon_o, \cdot\,)$ vanishes on $\partial B_{\varepsilon_o}(x_0)$.

If case (a) holds, then the degree $\deg(H(\theta, \cdot\,)B_{\varepsilon_o}(x_0), 0)$ is well defined and independent of θ by homotopy invariance. Thus

$$\deg(I - T(\varepsilon_0, \cdot\,), B_{\varepsilon_0}(x_0), 0) = \deg(I - T(0, \cdot\,), B_{\varepsilon_0}(x_0), 0) \neq 0$$

and so there exists $x_1 \in B_{\varepsilon_0}(x_0)$ such that $x_1 - T(\varepsilon_0, x_1) = 0$. Set $\varepsilon_1 = \varepsilon_0$ and $\eta_1 = ||x_1 - x_0||$. If case (b) holds, then let x_1 be such that $||x_1 - x_0|| = \varepsilon_0$ and $x_1 - T(\theta\varepsilon_0, x_1) = 0$. Now set $\varepsilon_1 = \theta\varepsilon_0$ and $\eta_1 = \varepsilon_0$. Thus in either case we have $\varepsilon_1 \leq \varepsilon_0$, $\eta_1 \leq \varepsilon_0$, $||x_1 - x_0|| = \eta_1$ and $x_1 - T(\varepsilon_1, x_1) = 0$. Now set $\varepsilon_1' = \min\{\varepsilon_1/2, \eta_1/2\}$ and repeat the above argument with ε_1' replacing ε_0. Proceeding thus, we get sequences $\{x_n\}$ in X, ε_n and η_n such that

$$\varepsilon_{n+1} \leq \varepsilon_n/2, \quad \eta_{n+1} \leq \eta_n/2 \quad ||x_n - x_0|| = \eta_n \quad x_n - T(\varepsilon_n, x_n) = 0,$$

which proves the result. □

We describe below some situations when condition (2.3) will be verified.

Theorem 2.2. *Assume that there exists a constant C such that,*

$$x_\sigma = \sigma T(0, x_\sigma) \tag{2.5}$$

implies that $||x_\sigma|| \leq C$ for all $\sigma \in [0,1]$. Then, if ϕ_0 has a unique solution x_0, condition (2.3) *holds. If ϕ_0 has only finitely many solutions then condition* (2.3) *holds for at least one of them.*

Proof. The hypothesis implies that the degree $\deg(I - \sigma T(0, \cdot\,), B_{C+1}(0), 0)$ is well defined and independent of $\sigma \in [0,1]$. Hence

$$\deg(\phi_0, B_{C+1}(0), 0) = \deg(I, B_{C+1}(0), 0) = 1.$$

If x_0 is the unique zero of ϕ_0, then by excision, for sufficiently small ε,

$$\deg(\phi_0, B_\varepsilon(x_0), 0) = \deg(\phi_0, B_{C+1}(0), 0) = 1$$

and the result follows.

If ϕ_0 has only finitely many solutions, they are all isolated and again by excision and additivity properties of the degree,

$$\deg(\phi_0, B_{C+1}(0), 0) = \sum_{i=1}^{k} i(\phi_0, x_i, 0) \tag{2.6}$$

where $\{x_i : 1 \leq i \leq k\}$ is the solution set. Since the left hand side of (2.6) is equal to unity, at least one of the right-hand terms must be non-zero. □

3. The case of the laplacian

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set which is sufficiently smooth and let $f : \mathbb{R} \to \mathbb{R}$ be a function which is non-negative and continuous. We are interested in solutions $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ of the problem

$$\begin{cases} -\Delta u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Note that since $f(u) \in L^\infty(\Omega)$, by standard regularity theorems, $u \in W^{2,q}(\Omega) \cap C(\overline{\Omega})$ for every $q > 1$. Moreover, since $f \geq 0$, we automatically have $u > 0$ in Ω by the strong maximum principle.

We now set up (3.1) in a framework which will enable us to use the results of the previous section. The procedure is the obvious one.

For $u \in C(\overline{\Omega})$ we define $T_f(u) = v$ as the weak solution of

$$\begin{cases} -\Delta v = f(u) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Since f is continuous, again by regularity theory, we have that

$$T_f : C(\overline{\Omega}) \to C^1(\overline{\Omega}) \subset C(\overline{\Omega}).$$

Proposition 3.1. *Let $f_n \to f$ uniformly on compact sets of $\mathbb{R}$ and let $u_n \to u$ in $C(\overline{\Omega})$. Then*

$$T_{f_n}(u_n) \to T_f(u) \quad \text{in } C(\overline{\Omega}). \tag{3.3}$$

Proof. Set $v_n = T_{f_n}(u_n)$, $v = T_f(u)$. Then

$$-\Delta(v_n - v) = f_n(u_n) - f(u). \tag{3.4}$$

By the convergence of u_n to u in $C(\overline{\Omega})$, we may assume that

$$|u_n|, |u| \le M \quad \text{ in } \overline{\Omega}. \tag{3.5}$$

Since f is uniformly continuous on compact sets of $\mathbb{R}$, by (3.5) we have for any $\varepsilon > 0$,

$$|f_n(u_n) - f(u)| \le |f_n(u_n) - f(u_n)| + |f(u_n) - f(u)| < \varepsilon$$

if n is sufficiently large. Thus $f_n(u_n) \to f(u)$ in $C(\overline{\Omega})$ and by the usual estimates for the equation (3.2) we deduce that $v_n \to v$ in $C^1(\overline{\Omega})$ and hence in $C(\overline{\Omega})$. □

Proposition 3.2. *The map $T_f : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact.*

Proof. We just saw that T_f is continuous. Further if A is a uniformly bounded set in $C(\overline{\Omega})$, so is $f(A) = \{f(u) : u \in A\}$. Then, once again by standard estimates, $T_f(A) = \{T_f(u) : u \in A\}$ is bounded in $C^1(\overline{\Omega})$ and hence is a compact subset of $C(\overline{\Omega})$. □

Now if f is continuous, we set

$$f_\varepsilon = \rho_\varepsilon \star f \tag{3.6}$$

where ρ_ε, for $\varepsilon > 0$, are the usual mollifiers. We then know that $f_\varepsilon \to f$ uniformly on compact sets of $\mathbb{R}$. Further $f_\varepsilon \in C^\infty(\mathbb{R})$ and so is locally Lipschitz continuous. We now define $T : [0, \infty[\times C(\overline{\Omega}) \to C(\overline{\Omega})$ as

$$\begin{cases} T(\varepsilon, u) = T_{f_\varepsilon}(u), & \varepsilon > 0, \\ T(0, u) = T_f(u). \end{cases} \tag{3.7}$$

Then we are in the situation described in Section 2. We can now prove

Theorem 3.1. *Let Ω be a sufficiently smooth bounded open set in $\mathbb{R}^N$, convex in the x_1-direction and symmetric w.r.t. the plane $x_1 = 0$. Let $u \in H_0^1(\Omega) \cap C(\overline{\Omega})$ be a solution of* (3.1), *with f nonnegative and continuous on $\mathbb{R}$, which is isolated and such that $i(I - T_f, u, 0) \neq 0$ in $C(\overline{\Omega})$. Then u is symmetric in x_1 and non decreasing in the x_1-direction for $x_1 < 0$.*

Proof. By Theorem 2.1, we have that there exists $u_\varepsilon \in C(\overline{\Omega})$ such that $u_\varepsilon \to u$ in $C(\overline{\Omega})$ and $u_\varepsilon = T_{f_\varepsilon}(u_\varepsilon)$. Hence u_ε solves (weakly) the problem

$$\begin{cases} -\Delta u_\varepsilon = f_\varepsilon(u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

Since $f \geq 0$, we also have $f_\varepsilon \geq 0$ and so $u_\varepsilon > 0$ in Ω. By the regularity of f_ε we deduce that $u_\varepsilon \in C^2(\overline{\Omega})$ and hence the result of Berestycki and Nirenberg [2] yields that u_ε is symmetric and strictly increasing in the x_1-direction for $x_1 < 0$. Thus the result follows. □

Corollary 3.1. *Under the assumption of the previous theorem, if Ω is a ball, then u is radial and radially decreasing.*

Proof. If Ω is a ball, then the u_ε are all radial and so u is radial. Hence it verifies the equation

$$-(r^{N-1}u'(r))' = f(u(r)).$$

Integrating this from 0 to r gives $u'(r) < 0$, since f is nonnegative. □

Corollary 3.2. *Let $f \geq 0$ Hölder continuous of order $\alpha \in]0,1[$ on $\mathbb{R}$. Assume that* (3.1) *possesses a finite number of solutions. Then one of them is symmetric* (*in the sense of Theorem* 3.1).

Proof. The result would follow directly from Theorem 2.2, provided we show that there exists a constant $C > 0$ such that if u_σ is a solution of

$$\begin{cases} -\Delta u_\sigma = \sigma f(u_\sigma) & \text{in } \Omega, \\ u_\sigma = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.9}$$

for some $\sigma \in [0,1]$, then $||u_\sigma||_\infty \leq C$. Now,

$$|f(u_\sigma)| \leq |f(u_\sigma) - f(0)| + |f(0)| \leq C_1|u_\sigma|^\alpha + |f(0)|.$$

Thus

$$||f(u_\sigma)||_\infty \leq C_1||u_\sigma||_\infty^\alpha + C_2.$$

Hence, from standard estimates for the solution of (3.2), it follows that

$$||u_\sigma||_\infty \leq C_3||u_\sigma||_\infty^\alpha + C_4.$$

If $||u_\sigma||_\infty$ were not bounded we get $||u_\sigma||_\infty^{1-\alpha} \leq C_3 + C_4||u_\sigma||_\infty^{-\alpha}$ which would give a contradiction. Hence the result. □

4. The case of the p-laplacian

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth open set and let $p > 1$. Let $f : \mathbb{R} \to \mathbb{R}$ be a given function such that

$$f \in C^1(\mathbb{R}) \quad \text{and} \quad f \geq 0. \tag{4.1}$$

Let $0 \leq \varepsilon \leq 1$ and $u \in C(\overline{\Omega})$. We set $w = T(\varepsilon, u)$ to be the weak solution in $W_0^{1,p}(\Omega)$ of the problem

$$\begin{cases} -\operatorname{div}((|\nabla w|^2 + \varepsilon)^{(p-2)/2} \nabla w) = f(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

If $u \in C(\overline{\Omega})$, then $f(u)$ is continuous on $\overline{\Omega}$ and hence belongs to $L^\infty(\Omega)$. Thus the solution of (4.2) exists uniquely and, by regularity results (see [3], [13], [5] and [10]), we have that $T(\varepsilon, u) \in C^{1,\alpha}(\overline{\Omega})$. Hence $T(\varepsilon, u)$ maps $C(\overline{\Omega})$ into itself.

We now proceed to show that the map T verifies the hypothesis of the abstract result of Section 2.

Proposition 4.1. *Let $p > 1$ and let $T : [0,1] \times C(\overline{\Omega}) \to C(\overline{\Omega})$ be defined as above. Then T is continuous and for each $\varepsilon \geq 0$, $T(\varepsilon, \cdot\,) : C(\overline{\Omega}) \to C(\overline{\Omega})$ is compact.*

Proof. Let $||u||_\infty \leq M$. Then $||f(u)||_\infty \leq M' = \sup_{[-M,M]} |f|$. By the regularity estimates (see [3], [13], [5], [10] for example) we have

$$||T(\varepsilon, u)||_{C^{1,\alpha}(\overline{\Omega})} \leq C||f(u)||_\infty \tag{4.3}$$

with a constant C which does not depend on u and ε. From (4.3) we deduce that T is compact. Moreover let $\varepsilon_n \to \varepsilon_0 \geq 0$ and $u_n \to u$ in $C(\overline{\Omega})$. Set $w_n = T(\varepsilon_n, u_n)$ and $w = T(\varepsilon_0, u)$. Then we can apply (4.3) to w_n obtaining the existence of a subsequence w_{n_k} converging to w in $C(\overline{\Omega})$. This proves the continuity of T. $\square$

Since T satisfies the hypothesis of Theorem 2.1, an isolated solution u of (1.2) with non-zero index can be realized as the limit in $C(\overline{\Omega})$ of u_ε where

$$u_\varepsilon = T(\varepsilon, u_\varepsilon), \tag{4.4}$$

as $\varepsilon \to 0$. Note that (4.4) says that $u_\varepsilon \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$ is a weak solution of

$$\begin{cases} -\operatorname{div}((|\nabla u_\varepsilon|^2 + \varepsilon)^{(p-2)/2} \nabla u_\varepsilon) = f(u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.5}$$

Moreover, since $f(u_\varepsilon) \in L^\infty(\Omega)$, we have that u_ε belongs to L^∞, and hence by the regularity theory for quasilinear equations (see [3], [5], [13], [10]) we get that $u_\varepsilon \in C^2(\overline{\Omega})$, for $\varepsilon > 0$ (i.e. when the equation is nondegenerate).

If $u > 0$ we also have that $u_\varepsilon > 0$ by the strong maximum principle (see [14]). We now just have to prove symmetry results for positive solutions of (4.5) to deduce those for solutions of (1.2). This we proceed to do in the next section.

It is not difficult to see that we may define $T(\varepsilon, u) = w$ as the solution of

$$
\begin{cases} -\operatorname{div}((|\nabla w|^2 + \varepsilon)^{(p-2)/2} \nabla w) = f_\varepsilon(u) & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.6}
$$

where $f_\varepsilon \to f$ uniformly on compact sets of $\mathbb{R}$ and again prove that T verifies all the hypothesis of Section 2. Thus using $f_\varepsilon = f \star \rho_\varepsilon$ as in the case of the Laplacian, we can still assume that f is just continuous.

5. Symmetry results for the p-laplacian

In this section we will deduce some symmetry results for positive solutions of the problem (1.2). To use the approximation procedure outlined earlier, we need first to prove symmetry results for solution of the perturbated p-laplacian. We now do this, closely following the approach of Berestycki and Nirenberg [2].

Theorem 5.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain which is convex in the x_1-direction and symmetric w.r.t. the plane $x_1 = 0$. Let $f \in C^1(\mathbb{R})$ and $u \in C^2(\overline{\Omega})$ be a positive solution of the problem*

$$
\begin{cases} -\operatorname{div}((|\nabla u|^2 + \varepsilon)^{(p-2)/2} \nabla u) = f(u) & \textit{in } \Omega, \\ u = 0 & \textit{on } \partial\Omega, \end{cases} \tag{5.1}
$$

where $p > 1$. Then u is symmetric w.r.t. x_1 and $\partial u / \partial x_1 < 0$ for $x_1 > 0$ in Ω.

Remark 5.1.

(i) It is in fact enough to consider f locally Lipschitz continuous.
(ii) If Ω were a ball, then it follows that u is radially symmetric and decreasing.

Proof. Let $x = (x_1, y) \in \mathbb{R}^N$, with $x_1 \in \mathbb{R}$, $y = (x_2, \ldots, x_N) \in \mathbb{R}^{N-1}$. Let $-a = \inf_{x \in \Omega} x_1$, $a > 0$. We denote by T_λ, the plane $x_1 = \lambda$ and set

$$
\Sigma_\lambda = \{x \in \Omega \mid x_1 < \lambda\}. \tag{5.2}
$$

In Σ_λ, we define the functions v_λ and w_λ by

$$
\begin{cases} v_\lambda(x_1, y) = u(2\lambda - x_1, y), \\ w_\lambda(x) = v_\lambda(x) - u(x). \end{cases} \tag{5.3}
$$

Step 1. We start by proving that w_λ satisfies in Σ_λ an uniformly elliptic equation. To do this we follow the procedure of [11].

Since $u \in C^2(\overline{\Omega})$ is a solution of (5.1), we can write (in non-divergence form, using the summation convention)

$$-(|\nabla u|^2+\varepsilon)^{(p-2)/2}\Delta u-(p-2)(|\nabla u|^2+\varepsilon)^{(p-4)/2}\partial_i u\partial_j u\partial_{ij}u=f(u) \tag{5.4}$$

and

$$-(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}\Delta v_\lambda-(p-2)(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\partial_i v_\lambda\partial_j v_\lambda\partial_{ij}v_\lambda=f(v_\lambda). \tag{5.5}$$

Multiplying these equations by 2 and subtracting we obtain

$$\begin{aligned}&[(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}+(|\nabla u|^2+\varepsilon)^{(p-2)/2}]\Delta w_\lambda\\&\quad+(p-2)[(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\partial_i v_\lambda\partial_j v_\lambda\\&\quad+(|\nabla u|^2+\varepsilon)^{(p-4)/2}\partial_i u\partial_j u]\partial_{ij}w_\lambda\\&\quad+[(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}-(|\nabla u|^2+\varepsilon)^{(p-2)/2}]\Delta(u+v_\lambda)\\&\quad+(p-2)[(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\partial_i v_\lambda\partial_j v_\lambda\\&\quad-(|\nabla u|^2+\varepsilon)^{(p-4)/2}\partial_i u\partial_j u]\partial_{ij}(u+v_\lambda)=2(f(u)-f(v_\lambda)).\end{aligned} \tag{5.6}$$

Applying the mean value theorem to the terms

$$\begin{aligned}&(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}-(|\nabla u|^2+\varepsilon)^{(p-2)/2},\\&(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\partial_i v_\lambda\partial_j v_\lambda-(|\nabla u|^2+\varepsilon)^{(p-4)/2}\partial_i u\partial_j u,\\&f(u)-f(v_\lambda),\end{aligned}$$

we get that w_λ satisfies the following equation in Σ_λ

$$a_{ij}^\lambda(x)\partial_{ij}w_\lambda+b_i^\lambda(x)\partial_i w_\lambda+c^\lambda(x)w_\lambda=0. \tag{5.7}$$

where the coefficients a_{ij}^λ, b_i^λ and c^λ are all bounded. In particular

$$\begin{aligned}a_{ij}^\lambda(x)=&[(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}+(|\nabla u|^2+\varepsilon)^{(p-2)/2}]\delta_{ij}\\&+(p-2)[(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\partial_i v_\lambda\partial_j v_\lambda\\&+(|\nabla u|^2+\varepsilon)^{(p-4)/2}\partial_i u\partial_j u].\end{aligned} \tag{5.8}$$

Now consider

$$\begin{aligned}\sum_{i,j}a_{ij}^\lambda\xi_i\xi_j=&[(|\nabla v_\lambda|^2+\varepsilon)^{(p-2)/2}+(|\nabla u|^2+\varepsilon)^{(p-2)/2}]|\xi|^2\\&+(p-2)\bigg[(|\nabla v_\lambda|^2+\varepsilon)^{(p-4)/2}\Big(\sum_i\partial_i v_\lambda\xi_i\Big)^2\\&+(|\nabla u|^2+\varepsilon)^{(p-4)/2}\Big(\sum_i\partial_i u\xi_i\Big)^2\bigg].\end{aligned} \tag{5.9}$$

If $p \geq 2$, we may ignore the second term of the right-hand side. If $p < 2$, then

$$\bigg(\sum_i \partial_i u \xi_i\bigg)^2 \leq |\nabla u|^2 |\xi|^2 \leq (|\nabla u|^2 + \varepsilon)|\xi|^2,$$

$$\bigg(\sum_i \partial_i v_\lambda \xi_i\bigg)^2 \leq |\nabla v_\lambda|^2 |\xi|^2 \leq (|\nabla v_\lambda|^2 + \varepsilon)|\xi|^2,$$

and (5.9) yields

$$\begin{aligned}\sum_{i,j} a_{ij}^\lambda \xi_i \xi_j \geq{}& [(|\nabla v_\lambda|^2 + \varepsilon)^{(p-2)/2} + (|\nabla u|^2 + \varepsilon)^{(p-2)/2}]|\xi|^2 \\ &+ (p-2)[(|\nabla v_\lambda|^2 + \varepsilon)^{(p-2)/2} + (|\nabla u|^2 + \varepsilon)^{(p-2)/2}]|\xi|^2 \\ ={}& (p-1)[(|\nabla v_\lambda|^2 + \varepsilon)^{(p-2)/2} + (|\nabla u|^2 + \varepsilon)^{(p-2)/2}]|\xi|^2.\end{aligned}$$

Thus, in either case, we have

$$\sum_{i,j} a_{ij}^\lambda \xi_i \xi_j \geq k[(|\nabla v_\lambda|^2 + \varepsilon)^{(p-2)/2} + (|\nabla u|^2 + \varepsilon)^{(p-2)/2}]|\xi|^2 \tag{5.10}$$

(with $k = 1$ if $p \geq 2$ and $k = p - 1$ if $1 < p < 2$), which proves the uniform ellipticity of the equation (5.7).

Step 2. We are now in a position to argue exactly as in Berestycki and Nirenberg ([2]). Let

$$-a = \inf\{x_1 \mid (x_1, y) \in \Omega\}.$$

Observe that for $0 < \lambda + a$ small, the domain Σ_λ is narrow in the x_1-direction. Hence, by a version of the maximum principle (see [2]), since $w_\lambda \geq 0$ on $\partial\Sigma_\lambda$ but $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$ (as $u = 0$ while $v_\lambda > 0$ on $\partial\Sigma_\lambda \cap \partial\Omega$),

$$w_\lambda > 0 \quad \text{in } \Sigma_\lambda. \tag{5.11}$$

Let $(-a, \mu)$ be the largest interval of values k such that (5.11) holds. We claim that $\mu = 0$. If not, we have $\mu < 0$. By continuity, we know that $w_\mu \geq 0$ on Σ_μ and if $\mu < 0$, we have $w_\lambda \not\equiv 0$ on $\partial\Sigma_\lambda$. Hence, by the maximum principle, we again have $w_\mu > 0$ in Σ_μ. Fix a $\theta > 0$ arbitrarily small and consider a compact set $K \subset \Sigma_\mu$ such that

$$|\Sigma_\mu \setminus K| < \theta/2.$$

By compactness, we have $w_\mu \geq \eta > 0$ in K and so, by continuity, it follows that for small $\delta, |\Sigma_{\mu+\delta} \setminus K| < \theta$ and $w_{\mu+\delta} > 0$ on K.

In the remaining portion $\Sigma_{\mu+\delta} \setminus K = \widetilde{\Sigma}$, $w_{\mu+\delta}$ verifies equation (5.7), with λ obviously replaced by $\mu + \delta$; further, $w_{\mu+\delta} \geq 0$ on $\partial\Sigma_{\mu+\delta}$ and is not identically zero there. Then by the Proposition 1.1 of [2] and the strong maximum principle, it follows that $w_{\mu+\delta} > 0$ in $\Sigma_{\mu+\delta}$ which contradicts the maximality of μ.

Thus $\mu = 0$. Now, applying the Hopf lemma to w_λ on the plane $T_\lambda \subset \partial\Sigma_\lambda$, we get $\frac{\partial w}{\partial x_1}(x) < 0$, i.e. $\frac{\partial u}{\partial x_1}(x) > 0$ for $x_1 = \lambda < 0$ since $\frac{\partial w}{\partial x_1} = -2\frac{\partial u}{\partial x_1}$.

We can apply the same procedure starting from $\lambda = a$ to get the symmetry result since it will prove that $w_\mu \equiv 0$ at $\mu = 0$. $\square$

Remark 5.2. In the proof of the above theorem, the hypothesis $u \in C^2(\overline{\Omega})$ is used to derive the equation (5.7) and to ensure that the coefficients of the differential operator are bounded.

Theorem 5.2. *Let Ω be a bounded smooth open set of $\mathbb{R}^N$ and f a nonnegative C^1-function. Let $u \in W_0^{1,p}(\Omega) \cap C(\overline{\Omega})$, be a positive and isolated solution of*

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.12}$$

such that $i(\phi, u, 0) \neq 0$ in $C(\overline{\Omega})$ (here $\phi = I - T(0, \cdot\,)$ is as defined in Section 1). Then u is symmetric in x_1. If Ω is a ball, then u is radially symmetric.

Proof. The proof follows immediately from the fact that $u = \lim_{\varepsilon\to 0} u_\varepsilon$ in $C(\overline{\Omega})$ where the $u_\varepsilon \in C^2(\overline{\Omega})$ satisfy (5.1) and hence are all symmetric. $\square$

Corollary 5.1. *Let f be Hölder continuous of order $\alpha \in \,]0,1[$ on $\mathbb{R}$. Then whenever (5.12) admits only a finite number of solutions, at least one of them is symmetric.*

Proof. By Theorem 2.2, it suffices to show that for $\sigma \in [0,1]$, solutions u_σ of

$$\begin{cases} -\operatorname{div}(|\nabla u_\sigma|^{p-2}\nabla u_\sigma) = \sigma f(u_\sigma) & \text{in } \Omega, \\ u_\sigma = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.13}$$

are uniformly bounded in $C(\overline{\Omega})$. But, since f is Hölder continuous and estimates analogous to (4.3) hold for solutions of (5.13) we have

$$||u_\sigma||_\infty \le C||f(u_\sigma)||_\infty \le C_1||u_\sigma||_\infty^\alpha + C_2, \quad \alpha < 1$$

which implies the uniform boundedness of the solution u_σ. $\square$

As mentioned in the introduction we now exhibit an example of a positive solution of (5.12) in a ball which is symmetric but not strictly radially decreasing when $p > 2$ and $N \ge 1$.

Example 5.1. Let us consider the equation (5.12) where $\Omega = B(0,2) \equiv B \subset \mathbb{R}^N$, $N \ge 1$. Let us consider the functions

$$u(x) = \begin{cases} 1 - (|x| - 1)^\alpha & \text{if } 1 \le |x| \le 2, \\ 1 & \text{if } |x| < 1, \end{cases}$$

and

$$\begin{aligned} f(t) &= \alpha^{p-1}(\alpha - 1)(p-1)(1-t)^{[(\alpha-1)(p-2)+\alpha-2]/\alpha} \\ &\quad + \alpha^{p-1}(N-1)\frac{(1-t)^{(\alpha-1)(p-1)/\alpha}}{1+(1-t)^{1/\alpha}}, \quad t \in [0,1] \end{aligned}$$

with $\alpha > p/(p-2)$ if $p > 2$ or $\alpha > p/(p-1)$ if $1 < p < 2$. So $f(t) \geq 0$ and f is C^1 when $p > 2$ while f is only hölder continuous when $1 < p < 2$.

Let us prove that u is a solution of (5.12) corresponding to the previous nonlinearity f. Let $\phi \in C_0^\infty(B)$ and

$$\begin{aligned}
\int_B |\nabla u|^{p-2}\nabla u \nabla\phi &= -\,\alpha^{p-1} \int_{1<|x|<2} \sum_{i=1}^{N} (|x|-1)^{(\alpha-1)(p-1)} \frac{x_i}{|x|}\frac{\partial\phi}{\partial x_i} \\
&= \alpha^{p-1} \int_{1<|x|<2} \phi \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\bigg[(|x|-1)^{(\alpha-1)(p-1)}\frac{x_i}{|x|}\bigg] \\
&= \alpha^{p-1} \int_{1<|x|<2} \phi\bigg((\alpha-1)(p-1)(|x|-1)^{(\alpha-1)(p-1)-1} \\
&\quad + (N-1)\frac{(|x|-1)^{(\alpha-1)(p-1)}}{|x|}\bigg) \\
&= \int_{1<|x|<2} f(u(x))\phi = \int_B f(u(x))\phi.
\end{aligned}$$

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