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HYBRID CONTROL SYSTEMS AND VISCOSITY SOLUTIONS*

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Abstract. We investigate a model of hybrid control system in which both discrete and continuous controls are involved. In this general model, discrete controls act on the system at a given set interface. The state of the system is changed discontinuously when the trajectory hits predefined sets, namely, an autonomous jump set A or a controlled jump set C where the controller can choose to jump or not. At each jump, the trajectory can move to a different Euclidean space. We prove the continuity of the associated value function V with respect to the initial point. Using the dynamic programming principle satisfied by V, we derive a quasi-variational inequality satisfied by V in the viscosity sense. We characterize the value function V as the unique viscosity solution of the quasi-variational inequality by the comparison principle method.

Key words. dynamic programming principle, viscosity solution, quasi-variational inequality, hybrid control

AMS subject classifications. 34H05, 34K35, 49L20, 49L25

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1. Introduction.

Many complicated control systems, like flight control and transportation, perform computer coded checks and issue logical as well as continuous control commands. The interaction of these different types of dynamics and information leads to hybrid control problems. Thus hybrid control systems are those having continuous and discrete dynamics and continuous and discrete controls. Many control systems, which involve both logical decision making and continuous evolution, are of this type. Typical examples of such systems are constrained robotic systems [1] and automated highway systems [8]. See [5], [6], and the references therein for more examples of such systems.

In [5], Branicky, Borkar, and Mitter presented a model for the most general hybrid control system in which continuous controls are present and, in addition, discrete controls act at a given set interface, which corresponds to the logical decision making process as in the above examples. The state of the system is changed discontinuously when the trajectory hits these predefined sets, namely, an autonomous jump set A or a controlled jump set C where the controller can choose to jump or not. They prove right continuity of the value function corresponding to this hybrid control problem. Using the dynamic programming principle they arrive at the partial differential equation satisfied by the value function, which turns out to be the quasi-variational inequality, referred hereafter as QVI.

In [4], Bensoussan and Menaldi study a similar system and prove that the value function u is close to a certain u_ε which they mention to be continuous indicating the use of the basic ordinary differential equation estimate for continuous trajectories and the continuity of the first hitting time (see [4, Theorem 2.5 and Remark 3.5]). They

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prove its uniqueness as a viscosity solution of the QVI in a certain special case where the autonomous jump set is empty and the controlled jump set is the whole space.

In our work, we study this problem in a more general case in which the autonomous jump set is nonempty and the controlled jump set can be arbitrary. Our model is based on that of [5]. Our main aim is to prove uniqueness in the most general case when the sets A and C are nonempty and also to obtain precise estimates to improve the earlier continuity results. Our motivation comes from the fact that in all the real-life models mentioned above, logical decision making is always involved as well as the continuous control. This will correspond to a nonempty autonomous jump set A.

Here we prove the local Hölder continuity of the value function under a transversality condition, the same as the one assumed in [5] and [4] (see (2.36) in [4]). For this we need to follow the trajectories starting from two neighboring points, through their continuous evolution, and through their discrete jumps since the autonomous jump set is nonempty. This involves careful estimation of the distance between the trajectories in various time intervals and summing up these terms to show that the distance remains small for initial points sufficiently close enough. Although the basic estimates used are similar to those available in the literature (e.g., [3], [4]), the crucial point in our proof is the convergence of the above summation. This also allows us to get the precise Hölder exponent for the continuity of the value function.

As in [5] and [4], using the dynamic programming principle, we arrive at the QVI satisfied by the value function. Then we show that the value function is the unique viscosity solution of the QVI. Our proof is very different from [4]. Their approach using a fixed point method does not seem to be suitable, as it is for the general case of a nonempty autonomous jump set. Our approach is based on the comparison principle in the class of bounded continuous functions. It is inspired by earlier work on impulse and switching control and game theoretic problems in the literature, namely, [2], [7], [9], particularly the idea of defining a sequence of new auxiliary functions. But the presence of the autonomous and controlled jump sets leads to different equations on these sets, and hence some new ideas are needed to arrive at the conclusion.

2. Notation and assumptions. In a hybrid control system, as in [5], the state vector during continuous evolution is given by the solution of the following problem:

$$\dot{X}(t) = f(X(t), u(t)), \tag{2.1}$$

$$X(0) = x, \tag{2.2}$$

where $X(t) \in \Omega := \bigcup_i \Omega_i \times \{i\}$, with each Ω_i a closed connected subset of $\mathbb{R}^{d_i}$, i, $d_i \in Z_+$; $x \in \Omega$; and $f : \Omega \times \mathcal{U} \to \Omega$. Actually, $f = f_i$ with the understanding that $\dot{X}(t) = f_i(X(t), u(t))$ whenever $x \in \Omega_i$. $\mathcal{U}$ is the continuous control set

$$\mathcal{U} = \{u : [\,0, \infty) \to U \mid u \text{ measurable, } U \text{ compact metric space}\}.$$

The trajectory also undergoes discrete jumps when it hits predefined sets A, the autonomous jump set, and C, the controlled jump set. A predefined set D is the destination set for both autonomous jumps as well as controlled jumps:

$$\begin{aligned} A &= \bigcup_i A_i \times \{i\}, \quad A_i \subseteq \Omega_i \subseteq \mathbb{R}^{d_i}, \\ C &= \bigcup_i C_i \times \{i\}, \quad C_i \subseteq \Omega_i \subseteq \mathbb{R}^{d_i}, \\ D &= \bigcup_i D_i \times \{i\}, \quad D_i \subseteq \Omega_i \subseteq \mathbb{R}^{d_i}\ . \end{aligned}$$

The trajectory starting from $x \in \Omega_i$, on hitting A, that is the respective $A_i \subseteq \Omega_i$, jumps to the destination set D according to the given transition map g. g uses discrete controls from the discrete control set V_1 and can move the trajectory from A_i to $D_j \subseteq \Omega_j \subseteq \mathbb{R}^{d_j}$. The trajectory then will continue its evolution under f_j till it again hits A or C, in particular A_j or C_j. On hitting C the controller can choose either to jump or not to jump. If the controller chooses to jump, then the trajectory is moved to a new point in D. In this case the controller can also move from Ω_i to any of the Ω_j.

This gives rise to a sequence of hitting times of A, which we denote by σ_i, and a sequence of hitting times of C, where the controller chooses to make a jump which is denoted by ξ_i. Thus σ_i and ξ_i are the times when continuous and discrete dynamics interact. Hence the trajectory of this problem is composed of continuous evolution given by (2.1) between two hitting times and discrete jumps at the hitting times. We denote $(X(\sigma_i^-), u(\cdot))$ by x_i and $g(X(\sigma_i^-), v)$ by x_i' and the destination of $X(\xi_i^+, u(\cdot))$ by $X(\xi_i)'$. In general we take the trajectory to be left continuous so that $X_x(\sigma_i)$ means $X_x(\sigma_i^-)$ and $X_x(\xi_i)$ means $X_x(\xi_i^-)$, whereas $X_x(\sigma_i^+)$ will be denoted by x_i' and $X_x(\xi_i^+)$ will be denoted by $X_x(\xi_i)'$.

We give the inductive limit topology on Ω, namely,

$$(x_n, i_n) \in \Omega \text{ converges to } (x, i) \in \Omega \quad \text{if for some } N \text{ large and } \forall n \geq N,$$

$$i_n = i, \quad x, x_n \in \Omega_i, \quad \Omega_i \subseteq \mathbb{R}^{d_i} \quad \text{for some } i, \quad \text{and } \|x_n - x\|_{\mathbb{R}^{d_i}} < \varepsilon.$$

With the understanding of the above topology we suppress the second variable i from Ω. We follow the same for A, C, and D. We make the following basic assumptions on the sets A, C, D, and on functions f and g.

(A1): Each Ω_i is the closure of a connected, open subset of $\mathbb{R}^{d_i}$.

(A2): A_i, C_i, D_i are closed, ∂A_i,∂C_i are C^2. For all i and for all $x \in D_i$, $|x| < R$, and $\partial A_i \supseteq \partial \Omega_i$ for all i.

(A3): $g : A \times V_1 \to D$ is a bounded, uniformly Lipschitz continuous map, with Lipschitz constant G with the understanding that $g = \{g_i\}$ and $g_i : A_i \times V \to D_j$.

(A4): Vector field f is Lipschitz continuous with Lipschitz constant L in the state variable x and uniformly continuous in control variable u. Also,

$$|f(x, u)| \leq F \quad \forall x \in \Omega \quad \text{and} \quad \forall u \in U. \tag{2.3}$$

(A5): We assume ∂A_i is compact for all i, and for some $\xi_0 > 0$, following transversality condition holds

$$f(x_0, u) \cdot \eta(x_0) \leq -2\xi_0 \;\; \forall x_0 \;\in\; \partial A_i \;\; \forall u \in U, \tag{2.4}$$

where $\eta(x_0)$ is the unit outward normal to ∂A_i at x_0. We assume a similar transversality condition on ∂C_i.

(A6):

$$\inf_i d(A_i, C_i) \geq \beta \quad \text{and} \quad \inf_i \; d(A_i, D_i) \geq \beta > 0, \tag{2.5}$$

where d is the appropriate Euclidean distance. Note that the above rules out infinitely many jumps in finite time.

(A7): We assume the control sets U and V_1 to be compact metric spaces.

Now $(u(\cdot), v, \xi_i, X(\xi_i)')$ is the control, and the total discounted cost is given by

$$(2.6)\quad J(x,u(\cdot),v,\xi_i,X(\xi_i)') = \int_0^\infty K(X_x(t),u(t))e^{-\lambda t}dt + \sum_{i=0}^{\infty} C_a(X(\sigma_i),v)e^{-\lambda\sigma_i} + \sum C_c(X(\xi_i),X(\xi_i)')e^{-\lambda\xi_i},$$

where λ is the discount factor, $K : \Omega \times \mathcal{U} \to \mathbb{R}_+$ is the running cost, $C_a : A \times V_1 \to \mathbb{R}_+$ is the autonomous jump cost, and $C_c : C \times D \to \mathbb{R}_+$ is the controlled jump cost. The value function V is then defined as

$$(2.7)\quad V(x) = \inf_{\theta \in (\mathcal{U}\times V_1 \times [0,\infty)\times D)} J(x,u(\cdot),v,\xi_i,X(\xi_i)').$$

We assume the following conditions on the cost functionals.

(C1): K is Lipschitz continuous in the x variable with Lipschitz constant K_1 and is uniformly continuous in the u variable. Moreover, K is bounded by K_0.

(C2): C_a and C_c are uniformly continuous in both variables and bounded below by $C' > 0$. Moreover, C_a is Lipschitz continuous in the x variable with Lipschitz constant C_1 and is bounded above by C_0. Also we assume

$$C_c(x,y) < C_c(x,z) + C_c(z,y) \quad \forall x \in C_i, z \in D \cap C_j, y \in D.$$

We now give two simple examples of hybrid control systems. For more examples, see [5].

Example 2.1 (collisions). Consider the ball of mass m which is moving in vertical and horizontal directions in a room under gravity with gravitational constant g. The dynamics can be given as

$$\dot{x} = v_x, \quad \dot{v_x} = 0,$$
$$\dot{y} = v_y, \quad \dot{v_y} = -mg.$$

On hitting the boundaries of the room $A_1 = \{(x,y)|y = 0, \text{ or } y = R_1\}$ we instantly set v_y to $-\rho v_y$ for some $\rho \in [0,1]$, the coefficient of restitution. Similarly we reset v_x to $-\rho v_x$ on hitting the boundary $A_2\{(x,y)|x = 0 \text{ or } x = R_2\}$. Thus in this case the sets A_1 and A_2 are autonomous jump sets. We can generalize the above system by allowing dynamics to occur in different $\mathbb{R}^d$ after hitting.

The next example illustrates the importance of the transversality condition, in the absence of which the optimal trajectory and hence the optimal control may fail to exist.

Example 2.2. Consider the dynamical system in $\mathbb{R}^2$ given by

$$\dot{x_1}(t) = 1, \quad x_1(0) = 0,$$
$$\dot{x_2}(t) = u, \quad x_2(0) = 0,$$

where $u \in [0,1]$, and when the trajectory hits the set A given by $A = \{(x_1,x_2)|(x_1 - 1)^2 + (x_2+1)^2 = 1\}$ it jumps to $(10^{10}, 10^{10})$. The cost is given by $\int_0^\infty e^{-t}\min\{|x_1(t) + x_2(t)|, 210^{10}\}$.

Here the vector field $(u,1)$ is not transversal to the boundary at $(1,0)$ for $u = 0$. Hence optimal trajectory does not exist and, moreover, the value function is discontinuous at $(1,0)$.

In the following sections we are interested in exploring the value function of the hybrid control problem defined in (2.7). In section 2 we show that the value function is bounded and locally Hölder continuous with respect to the initial point. In section 3, we use viscosity solution techniques and the dynamic programming principle to derive a partial differential equation satisfied by V in the viscosity sense, which turns out to be the Hamilton–Jacobi–Bellman QVI. Section 4 deals with uniqueness of the solution of the QVI. We give a comparison principle proof characterizing the value function as unique viscosity solution of the QVI.

3. Continuity of the value function. Let the trajectory given by the solution of (2.1) and starting from the point x be denoted by $X_x(t, u(\cdot))$. Since $x \in \Omega$, it belongs in particular to some Ω_i. Then we have from the theory of ordinary differential equations

$$|X_x(t, u(\cdot)) - X_z(t, u(\cdot))| \leq e^{Lt}|x - z|, \tag{3.1}$$

$$|X_x(t, u(\cdot)) - X_x(\bar{t}, u(\cdot))| \leq F|t - \bar{t}|, \tag{3.2}$$

where F and L are as in (A4).

Define the first hitting time of the trajectory as

$$T(x) = \inf_u \{t > 0 \mid \ X_x(t, u) \ \in \ A\}.$$

Notice that this $T(x)$ is in particular with respect to A_i as $x \in \Omega_i$. By assuming a suitable transversality condition on ∂A_i and ∂C_i we prove the continuity of T in the topology of $\mathbb{R}^{d_i}$. This is equivalent to proving the continuity of T on Ω with respect to the inductive limit topology on Ω. Hereafter by convention we assume the topology to be of that Ω_i, in which the respective points belong.

THEOREM 3.1. *Assume* (A1)–(A7). *Let $X(t)$ be the trajectory given by the solution of* (2.1)*. Let the first hitting time $T(x)$ be finite. Then it is locally Lipschitz continuous, i.e., there exists a $\delta_1 > 0$ depending on f, ξ_0, and the distance function from ∂A_i such that for all y, $\bar{y}$ in $B(x, \delta_1)$, a δ_1 neighborhood of x in Ω*

$$|T(y) - T(\bar{y})| \ < C|y - \bar{y}|, \quad \textit{where } C \textit{ depends on } \ \xi_0.$$

Proof. Step 1. *Estimates for points near ∂A.* First we show that there exist $\delta > 0$ and $C > 0$ such that

$$T(x) < C \ d(x) \quad \forall x \ \in B(A_i, \delta) \setminus \overset{\circ}{A},$$

where $B(A_i, \delta)$ is a δ neighborhood of A_i and $d(x)$ is a signed distance of x from ∂A_i given by

$$d(x) = \begin{cases} -\text{dist}(x, \partial A_i) & \text{if } x \in \overset{\circ}{A}_i, \\ 0 & \text{if } x \in \partial A_i, \\ \text{dist}(x, \partial A_i) & \text{if } x \in \bar{A}_i^c. \end{cases}$$

For simplicity of notation we drop the suffix i from now on, remembering that the distances are in $\mathbb{R}^{d_i}$. It is possible to choose $R \ > 0$ such that in a small neighborhood of ∂A, say $B(\partial A, R)$, the above signed distance function d is C^1, thanks to our assumption (A2).

Now for $x_0 \in \partial A$ choose u_0 in $\mathcal{U}$ such that $u_0(t) = u_0$ for all t and $r_0 < R$ such that

$$f(x, u_0) \cdot Dd(x) < -\xi_0 \quad \forall x \in B(x_0, r_0). \tag{3.3}$$

Observe that we can choose r_0 independent of x_0 by using compactness of ∂A. Now consider the trajectory starting from x, given by

$$\begin{aligned} \dot{X}(t) &= f(X(t), u_0), \\ X(0) &= x, \end{aligned}$$

where $x \in B(x_0, r_0)$. Then

$$\begin{aligned} d(X(s)) - d(x) = &\int_0^s Dd(x) \cdot f(x, u_0)\ d\tau + \int_0^s \big(Dd(X(\tau)) - Dd(x)\big) \cdot f(X(\tau), u_0)\ d\tau \\ &+ \int_0^s Dd(x) \cdot (f(X(\tau), u_0) - f(x, u_0))\ d\tau. \end{aligned}$$

By using (3.3) and (2.3),

$$\begin{aligned} d(X(s)) - d(x) \le &\int_0^s -\xi_0\ d\tau + F \int_0^s (Dd(X(\tau)) - Dd(x))\ d\tau \\ &+ \int_0^s Dd(x) \cdot (f(X(\tau), u_0) - f(x, u_0))\ d\tau. \end{aligned}$$

Let c be the bound on Dd on $B(\partial A, r_0)$. Restricting s to be small so that $X(\tau)$ is in the r_0 neighborhood of ∂A, we are assured that Dd is continuous. So is f. Thus

$$\begin{aligned} d(X(s)) - d(x) &\le -\xi_0 s + o(Fs)\ + o(cLs) \\ &< -\frac{1}{2}\xi_0 s \quad \text{for} \quad 0 < s < \bar{s} \end{aligned}$$

for some $\bar{s}$ dependent only on modulus of continuity of f and Dd and independent of x. Choose $\delta = \min\{r_0, \frac{\bar{s}\xi_0}{2}\}$. If x is in the δ ball around x_0, then $d(x) < \frac{\bar{s}\xi_0}{2}$ and, choosing $s_x = 2\frac{d(x)}{\xi_0}$, will imply

$$s_x < \bar{s} \quad \text{and hence} \quad d\left(X(s_x)\right) < 0.$$

Thus by our definition of d, $X(s_x) \in \overset{\circ}{A}$, which implies

$$T(x) < s_x = 2\frac{d(x)}{\xi_0}.$$

Then for $C = \frac{2}{\xi_0}$ we have

$$T(x) < Cd(x) \quad \forall x \in B(x_0, \delta) \setminus \overset{\circ}{A}\ .$$

Step 2. *Estimate for any two points in* Ω. In this step we estimate $|T(x) - T(\bar{x})|$ for any x, $\bar{x} \in \Omega$. Define

$$t(\bar{x}, \bar{u}) = \inf\{t > 0 \mid X(t) \in A, \dot{X}(t) = f(X(t), \bar{u}), X(0) = \bar{x}\}.$$

For given $0 < \epsilon < 1$, and $\bar{x} \in \Omega$ by the definition of $T(\bar{x})$, we can choose $\bar{u} \in \mathcal{U}$ such that

$$\bar{t} = t(\bar{x}, \bar{u}) < T(\bar{x}) + \epsilon. \tag{3.4}$$

Using estimate (3.1),

$$|X_{\bar{x}}(\bar{t}, \bar{u}) - X_x(\bar{t}, \bar{u})| \leq |\bar{x} - x| e^{L\bar{t}} \leq |\bar{x} - x| e^{L(T(\bar{x})+\varepsilon)}. \tag{3.5}$$

Define $\delta_1 = \delta e^{-L(T(\bar{x})+1)}$, where δ is as in Step 1. Let us choose x such that $|x - \bar{x}| < \delta_1$. Then

$$|X_{\bar{x}}(\bar{t}, \bar{u}) - X_x(\bar{t}, \bar{u})| \leq |\bar{x} - x| e^{L\bar{t}} < |\bar{x} - x| e^{L(T(\bar{x})+1)} < \delta.$$

Also we have $X_{\bar{x}}(\bar{t}, \bar{u}) \in \partial A$. Hence, $X_x(\bar{t}, \bar{u}) \in B(\partial A, \delta) \setminus \mathring{A}$. Therefore, by Step 1,

$$T(X_x(\bar{t}, \bar{u})) < C d(X_x(\bar{t}, \bar{u})). \tag{3.6}$$

We claim that

$$T(x) \leq \bar{t} + T(X_x(\bar{t}, \bar{u})). \tag{3.7}$$

For given $\varepsilon_1 > 0$, choose $u_1 \in \mathcal{U}$ such that

$$T(X_x(\bar{t}, \bar{u})) \geq t(X_x(\bar{t}, \bar{u}), u_1) - \varepsilon_1.$$

Define a new control u_2 by

$$u_2(s) = \begin{cases} \bar{u}(s) & \text{if } s \leq \bar{t}, \\ u_1(s - \bar{t}) & \text{if } s > \bar{t}. \end{cases}$$

Then

$$T(x) \leq t(x, u_2) \leq \bar{t} + t\left(X_x(\bar{t}, \bar{u}), u_1\right) \leq \bar{t} + T\left(X_x(\bar{t}, \bar{u})\right) + \varepsilon_1.$$

Since ε_1 is arbitrary, this proves (3.7). Using (3.4) and (3.7) for $x \in B(\bar{x}, \delta_1)$ we get

$$\begin{aligned} T(x) &\leq T(\bar{x}) + T(X_x(\bar{t}, \bar{u})) + \epsilon \\ &\leq T(\bar{x}) + C\ d(X_x(\bar{t}, \bar{u})) + \epsilon \quad \text{by (3.6)}. \end{aligned}$$

Notice that $d(X_x(\bar{t}, \bar{u})) \leq |X_x(\bar{t}, \bar{u}) - X_{\bar{x}}(\bar{t}, \bar{u})|$. So by (3.5)

$$T(x) \leq T(\bar{x}) + C\ |x - \bar{x}|\ e^{L(T(\bar{x}) + \varepsilon)} + \epsilon.$$

Interchanging the roles of x and $\bar{x}$ we get

$$|T(x) - T(\bar{x})| \leq C\ |x - \bar{x}|\ e^{L(T(\bar{x}) \vee T(x))} \tag{3.8}$$

as ϵ tends to 0, where $T(\bar{x}) \vee T(x) = \max\{T(\bar{x}),\ T(x)\}$. Also observe that

$$\begin{aligned} T(x) &\leq T(\bar{x}) + C\ |x - \bar{x}|\ e^{L(T(\bar{x})+\varepsilon)} + \epsilon \\ &\leq T(\bar{x}) + C\delta + \varepsilon \leq T(\bar{x}) + C\delta + 1 \\ &\leq T(\bar{x}) + 2. \end{aligned}$$

Hence for all x belonging to $B(\bar{x}, \delta_1)$, T is bounded. Let this bound be T_0. Then we have

$$|T(x) - T(\bar{x})| < C|x - \bar{x}|e^{LT_0}.$$

Hence we conclude that the first hitting time of trajectory is locally Lipschitz continuous with respect to the initial point. ∎

Now we take up the issue of continuity of the value function. For this proof we need some estimates on hitting times of trajectories starting from two nearby points. We prove these estimates in the following lemmas. We fix the controls $\bar{u}$ and $\bar{v}$ and suppress them in the following calculations.

LEMMA 3.2. *Let σ_1 and Σ_1 be the first hitting times of trajectories evolving with fixed controls $\bar{u}$ and $\bar{v}$ according to* (2.1) *starting from x and z, respectively. Let x_1 and z_1 be points where these trajectories hit A for the first time:*

$$x_1 = X_x(\sigma_1), \quad z_1 = X_z(\Sigma_1), \quad x_1, z_1 \in \partial A.$$

If $|x - z| < \delta_1$, where δ_1 is as in Theorem 3.1*, then*

$$|x_1 - z_1| \leq (1 + FC)e^{L(\Sigma_1 \vee \sigma_1)}|x - z|. \tag{3.9}$$

Proof. Note here that by Theorem 3.1 we have the estimate on $|\sigma_1 - \Sigma_1|$ given by (3.8),

$$|\sigma_1 - \Sigma_1| < Ce^{L(\Sigma_1 \vee \sigma_1)}\ |x - z|. \tag{3.10}$$

Using this we estimate $|x_1 - z_1|$. Without loss of generality we assume that $\Sigma_1 > \sigma_1$,

$$\begin{aligned} |x_1 - z_1| &= |X_x(\sigma_1) - X_z(\Sigma_1)| \\ &\leq |X_x(\sigma_1) - X_z(\sigma_1)| + |X_z(\sigma_1) - X_z(\Sigma_1)|. \end{aligned}$$

Using (3.1) we get

$$|X_x(\sigma_1) - X_z(\sigma_1)| < e^{L\sigma_1}|x - z|,$$

while (3.2) and (3.10) lead to

$$|X_z(\sigma_1) - X_z(\Sigma_1)| \leq F|\sigma_1 - \Sigma_1| \leq FCe^{L\Sigma_1}|x - z|.$$

Combining these estimates, we get

$$|x_1 - z_1| \leq e^{L\Sigma_1}|x - z|(1 + FC) \quad \text{for} \quad z \in B(x, \delta_1). \qquad \square$$

Observe that the destination points of x_1 and z_1, which are denoted by ${x_1}' = g(x_1, \bar{v})$ and ${z_1}' = g(z_1, \bar{v})$, may belong to $\Omega_j \subseteq \mathbb{R}^{d_j}$. Without loss of generality we assume that ${x_1}', {z_1}' \in \Omega_2 \subseteq \mathbb{R}^{d_2}$, and the evolution of trajectories takes place in Ω_2 till the next hitting time. Let σ_2 and Σ_2 be the next hitting times of the trajectories when they hit A once again. The next lemma deals with the estimate of $|\sigma_2 - \Sigma_2|$.

LEMMA 3.3. *Let the first hitting time of trajectories starting from x and z, and evolving with fixed control $\bar{u}$, be σ_1 and Σ_1, and the second hitting times are σ_2 and Σ_2. Then there exists δ_2 such that for $|x - z| < \delta_2$,*

$$|\sigma_2 - \Sigma_2| \leq Ce^{(\Sigma_2 \vee \sigma_2)}(FC + G(FC + 1))|x - z| \tag{3.11}$$

and if we denote

$$x_2 = X_{x_1'}(\sigma_2 - \sigma_1), \quad x_2' = g(x_2),$$
$$z_2 = X_{z_1'}(\Sigma_2 - \Sigma_1), \quad z_2' = g(z_2),$$

then

$$|x_2 - z_2| \le (FC+1)e^{L(\Sigma_2 \vee \sigma_2)}(FC + G(FC+1))|x - z|. \tag{3.12}$$

Proof. Without loss of generality let $\sigma_1 < \Sigma_1$. Observe that σ_2 and Σ_2 are the first hitting times of trajectories starting from points $X_{x_1'}(\Sigma_1 - \sigma_1)$ and z_1' at time $t = \Sigma_1$. Then

$$T(z_1') = (\Sigma_2 - \Sigma_1) \quad \text{and} \quad T(X_{x_1'}(\Sigma_1 - \sigma_1)) = \sigma_2 - \Sigma_1.$$

Hence by (3.8)

$$|\sigma_2 - \Sigma_2| \le Ce^{L(\Sigma_2 - \Sigma_1)}|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'|$$

whenever $|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'| \le \delta_1$. Now

$$|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'| \le |X_{x_1{}'}(\Sigma_1 - \sigma_1) - x_1'| + |x_1' - z_1'|.$$

Hence by using estimate (3.2) and (3.10) for the first term we have

$$|X_{x_1'}(\Sigma_1 - \sigma_1) - x_1'| \le F|\Sigma_1 - \sigma_1| \le FCe^{L\Sigma_1}|x - z|,$$

whereas using Lipschitz continuity of g and (3.9) for the second term we get

$$|x_1' - z_1'| \le G|x_1 - z_1| \le G(FC+1)e^{L\Sigma_1}|x - z| \quad \text{for} \quad z \in B(x, \delta_1).$$

Combining the above two estimates we have

$$|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'| \le e^{L\Sigma_1}(FC + G(FC+1))|x - z| \tag{3.13}$$

and by our choice of $\delta_2 = \min\{\delta_1, \frac{\delta_1 e^{-L\Sigma_1}}{FC+G(FC+1)}\}$, $|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'| < \delta_1$. Using (3.13) in the estimate of $|\sigma_2 - \Sigma_2|$ for $z \in B(x, \delta_2)$ we have

$$|\sigma_2 - \Sigma_2| \le Ce^{L\Sigma_2}(FC + G(FC+1))|x - z|. \tag{3.14}$$

Now we estimate $|x_2 - z_2|$:

$$\begin{aligned}|x_2 - z_2| &= |X_{x_1'}(\sigma_2 - \sigma_1) - X_{z_1'}(\Sigma_2 - \Sigma_1)| \\ &\le |X_{x_1'}(\sigma_2 - \sigma_1) - X_{z_1'}(\sigma_2 - \Sigma_1)| + |X_{z_1'}(\sigma_2 - \Sigma_1) - X_{z_1'}(\Sigma_2 - \Sigma_1)|.\end{aligned}$$

Observe that by the semigroup property

$$X_{x_1'}(\sigma_2 - \sigma_1) = X_{X_{x_1'}(\Sigma_1 - \sigma_1)}(\sigma_2 - \Sigma_1).$$

Hence

$$|X_{x_1'}(\sigma_2 - \sigma_1) - X_{z_1'}(\sigma_2 - \Sigma_1)| = |X_{X_{x_1'}(\Sigma_1 - \sigma_1)}(\sigma_2 - \Sigma_1) - X_{z_1'}(\sigma_2 - \Sigma_1)|$$

and by (3.1)

$$|X_{x_1'}(\sigma_2 - \sigma_1) - X_{z_1'}(\sigma_2 - \Sigma_1)| \le e^{L(\sigma_2 - \Sigma_1)}|X_{x_1'}(\Sigma_1 - \sigma_1) - z_1'|. \tag{3.15}$$

From (3.2) and (3.14) we get

$$\begin{aligned} |X_{z_1'}(\sigma_2 - \Sigma_1) - X_{z_1'}(\Sigma_2 - \Sigma_1)| &\le F|\sigma_2 - \Sigma_1 - (\Sigma_2 - \Sigma_1)| \\ &\le FCe^{L\Sigma_2}(FC + G(FC+1))|x - z|. \end{aligned} \tag{3.16}$$

Together these estimates yield, for $z \in B(x, \delta_2)$,

$$|x_2 - z_2| \le e^{L\Sigma_2}(FC+1)(FC + G(FC+1))|x - z|. \qquad \square$$

Let σ_i and Σ_i be the ith hitting times of trajectories starting from x and z, respectively. With the above notation we assume that ${x_i}', {z_i}' \in \Omega_{i+1} \subseteq \mathbb{R}^{d_{i+1}}$. We apply Theorem 3.1 and the above lemmas recursively to find estimates on successive hitting times and points where trajectories hit A. We generalize the above estimates for the ith hitting times of trajectories when they hit A. For simplicity of calculations we denote $FC + G(FC+1)$ by P hereafter.

REMARK 3.4. *Let the control $\bar{u}$ be fixed. Let σ_i and Σ_i be the ith consecutive hitting time of the trajectory starting from x and z, respectively, when they hit A, and let x_i, z_i be the points on ∂A where trajectories hit A. Then proceeding along lines similar to those of Lemmas* 3.2 *and* 3.3 *we get the estimates for* $|\sigma_i - \Sigma_i|$ *and* $|x_i - z_i|$ *which are given by*

$$\begin{aligned} |\sigma_i - \Sigma_i| &\le Ce^{L\Sigma_i}P^{i-1}|x - z|, \\ |x_i - z_i| &\le e^{L\Sigma_i}(FC+1)P^{i-1}|x - z| \end{aligned}$$

whenever $|x - z| < \delta_i$, *where* $\delta_i := \min\{\delta_1, \delta_2, \ldots, \frac{\delta_1 e^{-L\Sigma_i}}{P^{i-1}}\}$.

THEOREM 3.5 (continuity of the value function). *Under the assumptions of Theorem* 3.1*, value function V of hybrid control problem defined by* (2.7) *is bounded and locally Hölder continuous with respect to the initial point.*

Proof. First we show that the value function is bounded. For any $u \in \mathcal{U}$ and $v \in V_1$,

$$V(x) \le \int_0^\infty K(X_x(t), u(t))e^{-\lambda t}dt + \sum_{i=0}^{\infty} C_a(X(\sigma_i), v)e^{-\lambda\sigma_i}.$$

By our assumptions (C1) and (C2),

$$V(x) \le K_0 \int_0^{+\infty} e^{-\lambda t}\,dt \; + \sum_{i=1}^{+\infty} C_0 e^{\lambda\sigma_i} \le \frac{K_0}{\lambda} \; + C_0 \sum_{i=1}^{+\infty} e^{-\lambda\sigma_i}.$$

From (A5), recalling that $\beta = \inf\ d(A_i, D_i)$,

$$\sigma_{i+1} \;\ge\; \sigma_i + \frac{\beta}{\sup|f(x,u)|} \;\ge\; \sigma_i + \beta/F. \tag{3.17}$$

Hence we get

$$\sum_{i=1}^{\infty} e^{-\lambda\sigma_i} \le e^{-\lambda\sigma_1} \sum_{i=1}^{\infty} \left(e^{-\lambda\beta/F}\right)^i \le e^{-\lambda\sigma_1} \frac{1}{1 - e^{-\lambda\beta/F}}, \tag{3.18}$$

leading to

$$V(x) \le \frac{K}{\lambda} + C_0 e^{-\lambda\sigma_1} \frac{1}{1 - e^{-\lambda\beta/F}}.$$

This proves $V(x)$ is bounded.

We now show that V defined in (2.7) is locally Hölder continuous with respect to the initial point. Let $x, z \in \Omega$. Regarding $V(x)$ as in (2.7), we assume that the controller chooses not to make any controlled jumps. Note that the controller has this choice because in the interior of C he can always choose not to jump. On the boundary of C that is ∂C by the transversality condition, vector field is nonzero and hence he can continue the evolution without jumping. Thus in any case he can choose not to jump. Then given $\varepsilon > 0$, we can choose the controls $\overline{u}, \overline{v}$ depending on ε such that

$$V(z) \ge \int_0^\infty K(X_z(t), \overline{u}(t)) e^{-\lambda t} dt + \sum_{i=1}^\infty C_a(X_z(\Sigma_i), \overline{v}) e^{-\lambda \Sigma_i} - \varepsilon.$$

Also

$$V(x) \le \int_0^\infty K(X_x(t), \bar{u}(t)) e^{-\lambda t} dt + \sum_{i=1}^\infty C_a(X_x(\sigma_i), \overline{v}) e^{-\lambda \sigma_i}.$$

Hence

$$\begin{aligned} V(x) - V(z) \le & \int_0^\infty |K(X_x(t), \overline{u}(t)) - K(X_z(t), \overline{u}(t))| e^{-\lambda t} dt \\ & + \sum_{i=1}^\infty |C_a(X_x(\sigma_i), \overline{v}) - C_a\big(X_z(\Sigma_i), \overline{v}\big)| e^{-\lambda(\sigma_i \vee \Sigma_i)} + \varepsilon, \end{aligned}$$

where $\sigma_i \vee \Sigma_i = \max\{\sigma_i, \Sigma_i\}$. Now for T large to be chosen precisely later on we split the integral and summation as follows:

$$\begin{aligned} V(x) - V(z) \le & \int_0^T |K(X_x(t), \overline{u}(t)) - K(X_z(t), \overline{u}(t))| e^{-\lambda t} dt \\ & + \sum_{i=1}^N |C_a(X_x(\sigma_i), \overline{v}) - C_a(X_z(\Sigma_i), \overline{v})| e^{-\lambda(\sigma_i \vee \Sigma_i)} \\ & + \int_T^\infty |K(X_x(t), \overline{u}(t)) - K(X_z(t), \overline{u}(t))| e^{-\lambda t} dt \\ & + \sum_{i=N+1}^\infty |C_a(X_x(\sigma_i), \overline{v}) - C_a(X_z(\Sigma_i), \overline{v})| e^{-\lambda(\sigma_i \vee \Sigma_i)} + \varepsilon, \end{aligned} \tag{3.19}$$

where T will be chosen so that the tail end of the integral and summation become small and T is in between the Nth and $(N+1)$th hitting times of the trajectories. By using the bound K_0 on K given by (C1) we get

$$\int_T^\infty |K(X_x(t), \overline{u}(t)) - K(X_z(t), \overline{u}(t))| e^{-\lambda t} dt \le \frac{2K_0}{\lambda} e^{-\lambda T} \tag{3.20}$$

and by using bound C_0 on C_a given by (C2) and doing calculations along lines similar to those of (3.18) we get the estimate

(3.21)
$$\sum_{i=N+1}^{\infty} |C_a(X_x(\sigma_i),\overline{v}) - C_a(X_z(\Sigma_i),\overline{v})| e^{-\lambda(\sigma_i \vee \Sigma_i)} \leq 2C_0\big(e^{-\lambda\beta/F}\big)^N \frac{1}{1-e^{-\lambda\beta/F}}.$$

Now we calculate $\int_0^T |K(X_x(t),\overline{u}(t)) - K(X_z(t),\overline{u}(t))| e^{-\lambda t} dt$. We will show that there exists $\bar{\delta} > 0$ such that if $|x-z| < \bar{\delta}$, then the sequence of σ_i and Σ_i can be, for example,

$$0 \leq \sigma_1 \leq \Sigma_1 \leq \sigma_2 \leq \Sigma_2 \leq \cdots \leq \sigma_n \leq \Sigma_n \leq T \tag{3.22}$$

$$\text{or} \quad 0 \leq \Sigma_1 \leq \sigma_1 \leq \cdots \leq \Sigma_n \leq \sigma_n \leq T.$$

That is, every A hitting time of trajectory starting from x is followed by A hitting time of trajectory starting from z.

Without loss of generality let us assume $\sigma_1 < \Sigma_1$. If $\Sigma_1 < \sigma_1$, the following calculations go through with appropriate changes and hence we split this integral, assuming (3.22) as follows:

$$\begin{aligned} \int_0^T Ie^{-\lambda t}dt \leq \int_0^{\sigma_1} Ie^{-\lambda t}dt + \int_{\sigma_1}^{\Sigma_1} Ie^{-\lambda t}dt + \int_{\Sigma_1}^{\sigma_2} Ie^{-\lambda t}dt + \cdots \\ + \int_{\sigma_n}^{\Sigma_n} Ie^{-\lambda t}dt + \int_{\Sigma_n}^{\sigma_{n+1}} Ie^{-\lambda t}dt, \end{aligned} \tag{3.23}$$

where $I = |K(X_x(t),\overline{u}(t)) - K(X_z(t),\overline{u}(t))|$. In this there are two types of integrals:

1. $\int_{\sigma_i}^{\Sigma_i} Ie^{-\lambda t}dt$;
2. $\int_{\Sigma_i}^{\sigma_{i+1}} Ie^{-\lambda t}dt$.

If $|x-z| < \delta_N$, where $\delta_N = \min\{\delta_1, \delta_2, \ldots, \frac{\delta_1 e^{-L\Sigma_N}}{P^{N-1}}\}$, we can estimate the above integrals using Lemmas 3.2 and 3.3 and Remark 3.4. We use the bound on K to evaluate the first integral.

$$\int_{\sigma_i}^{\Sigma_i} Ie^{-\lambda t}dt \leq \frac{2K_0}{\lambda}\big(e^{-\lambda\sigma_i} - e^{-\lambda\Sigma_i}\big) \leq \frac{2K_0}{\lambda}\, \lambda|\sigma_i - \Sigma_i|.$$

Using Remark 3.4,

$$\int_{\sigma_i}^{\Sigma_i} Ie^{-\lambda t}dt \leq 2K_0CP^{i-1}e^{L\Sigma_i}. \tag{3.24}$$

To evaluate the second integral we use the Lipschitz continuity of K.

$$\begin{aligned} \int_{\Sigma_i}^{\sigma_{i+1}} Ie^{-\lambda t}dt &= \int_{\Sigma_i}^{\sigma_{i+1}} |K\big(X_{x_i'}(t-\sigma_i)\big) - K\big(X_{z_i'}(t-\Sigma_i)\big)| e^{-\lambda t}dt \\ &\leq K_1 \int_{\Sigma_i}^{\sigma_{i+1}} |X_{x_i'}(t-\sigma_i) - X_{z_i'}(t-\Sigma_i)| e^{-\lambda t}dt. \end{aligned} \tag{3.25}$$

By the semigroup property,

$$|X_{x_i'}(t-\sigma_i) - X_{z_i'}(t-\Sigma_i)| = |X_{X_{x_i'}(\Sigma_i-\sigma_i)}(t-\Sigma_i) - X_{z_i'}(t-\Sigma_i)|$$
$$\leq e^{L(t-\Sigma_i)}|X_{x_i'}(\Sigma_i-\sigma_i) - z_i'| \quad \text{by (3.1)}.$$

Now by generalizing the estimate in (3.13) we get

$$|X_{x_i'}(\Sigma_i-\sigma_i) - z_i'| \leq P^i e^{L\Sigma_i}|x-z|. \tag{3.26}$$

Hence substituting the above estimates in (3.25), we get

$$\int_{\Sigma_i}^{\sigma_{i+1}} Ie^{-\lambda t}dt \leq K_1 e^{-L\Sigma_i} P^i e^{L\Sigma_i}|x-z| \int_{\Sigma_i}^{\sigma_{i+1}} e^{(L-\lambda)t}dt.$$

For $L \neq \lambda$,

$$\int_{\Sigma_i}^{\sigma_{i+1}} Ie^{-\lambda t}dt \leq K_1 P^i\ |x-z|\, \frac{e^{(L-\lambda)(\sigma_{i+1})} - e^{(L-\lambda)\Sigma_i}}{L-\lambda} \tag{3.27}$$
$$\leq K_1 P^i |x-z|\, \frac{e^{(L-\lambda)T}-1}{L-\lambda}$$

and for $L = \lambda$,

$$\int_{\Sigma_i}^{\sigma_{i+1}} Ie^{-\lambda t}dt \leq K_1 e^{-L\Sigma_i} P^i e^{L\Sigma_i}|x-z| \int_{\Sigma_i}^{\sigma_{i+1}} dt \tag{3.28}$$
$$\leq K_1 P^i |x-z|\ |\sigma_{i+1}-\Sigma_i|$$
$$\leq K_1 P^i\ |x-z|\ \ 2T.$$

For $L \neq \lambda$, by using (3.24), (3.27), $\int_0^T Ie^{-\lambda t}dt$ becomes

$$\int_0^T Ie^{-\lambda t}dt \leq \sum_{i=1}^N 2K_0 C P^{i-1} e^{LT}|x-z| + \sum_{i=1}^N \frac{K_1}{L-\lambda} P^i \big(e^{(L-\lambda)T}-1\big)|x-z|.$$

Hence

$$\left.\begin{aligned} \int_0^T Ie^{-\lambda t}dt \leq 2K_0 C \left[\tfrac{P^{N-1}}{P-1}\right]|x-z| \\ + K_1 \left[\tfrac{P^N-1}{P-1}\right]\tfrac{e^{(L-\lambda)T}-1}{L-\lambda}|x-z| \end{aligned}\right\} \quad \text{for } L \neq \lambda \tag{3.29}$$

and for $L = \lambda$, using (3.24) and (3.28),

$$\int_0^T Ie^{-\lambda t}dt \leq \sum_{i=1}^N 2K_0|\sigma_i - \Sigma_i| + \sum_{i=1}^N K_1 T P^i |x-z|$$
$$\leq \sum_{i=1}^N 2K_0 C P^{i-1}|x-z| + \sum_{i=1}^N K_1 T P^i |x-z|.$$

Thus

$$\left.\begin{aligned} \int_0^T Ie^{-\lambda t}dt \leq 2K_0 C \left(\tfrac{P^N-1}{P-1}\right)|x-z| \\ +2K_1 T \left(\tfrac{P^N-1}{P-1}\right)|x-z| \end{aligned}\right\} \quad \text{for } L = \lambda. \tag{3.30}$$

Furthermore, by using (C2) and Remark 3.4 we get

$$\begin{aligned}\sum_{i=1}^{N} |C_a(x_i,\overline{v}) - C_a(z_i,\overline{v})| e^{-\lambda(\sigma_i \vee \Sigma_i)} &\leq \sum_{i=1}^{N} 2C_1 |x_i - z_i| e^{-\lambda(\sigma_i \vee \Sigma_i)} \\ &\leq 2C_1 \sum_{i=1}^{N} (FC+1) e^{LT} P^{i-1} |x-z|,\end{aligned}$$

(3.31)

$$\sum_{i=1}^{N} |C_a(x_i,\overline{v}) - C_a(z_i,\overline{v})| e^{-\lambda(\sigma_i \vee \Sigma_i)} \leq 2C_1(FC+1)e^{LT}|x-z| \left(\frac{P^{N-1}-1}{P-1}\right).$$

Since P is a constant, without loss of generality we can assume

$$\frac{P^N}{P-1} < 2P^N. \tag{3.32}$$

Also observe that $\sigma_i - \sigma_{i+1} \geq \beta/F$ implies that $T \geq \sigma_{N+1} - \sigma_1 \geq N\beta/F$ and hence

$$N < TF/\beta. \tag{3.33}$$

Using (3.20), (3.21), (3.29), (3.31), (3.32), (3.33) in (3.19) for $L \neq \lambda$ we have

$$\begin{aligned}V(x) - V(z) \leq\ & 4K_0 C e^{LT} P^{TF/\beta}\ |x-z| + 2K_1 P^{TF/\beta}\ \frac{e^{(L-\lambda)T}-1}{L-\lambda}|x-z| \\ & + \frac{2K_0}{\lambda} e^{-\lambda T} + 2C_1 e^{LT} P^{TF/\beta}\ |x-z| \\ & + 2C_0 \left(e^{-\lambda\beta/F}\right)^{TF/\beta}\ \frac{1}{1-e^{-\lambda\beta/F}}.\end{aligned}$$

Now we further restrict $|x-z| < (\delta_1)^{\frac{1}{1-\theta}}$ for some θ such that $0 < \theta < 1$. Then choose T such that

$$P^{TF/\beta} e^{LT} = |x-z|^{-\theta}.$$

This gives

$$T = \frac{-\theta \log|x-z|}{\lambda + F\log P/\beta}. \tag{3.34}$$

This together with the choice of $|x-z|$ implies

$$\delta_N = \frac{\delta_1}{e^{L\Sigma_N} P^{N-1}} > \frac{\delta_1}{e^{LT} P^{TF/\beta}}\ =\ \delta_1 |x-z|^{\theta}\ >\ |x-z|. \tag{3.35}$$

Thus $|x-z| < \delta_N$ and hence the above estimate holds true for our choice of T. Then substituting the value of T in the above estimate, for $L \neq \lambda$, we get

$$\begin{aligned}V(x) - V(z) \leq\ & 4K_0 C|x-z|^{1-\theta} + \frac{K_1}{L-\lambda}|x-z|^{1-\theta} + C_1|x-z|^{1-\theta} \\ & + \frac{2K_0}{\lambda}|x-z|^{\frac{\lambda\theta}{(F\log P/\beta)+L}} + 2C_0|x-z|^{\frac{\lambda\theta}{(F\log P/\beta)+L}}.\end{aligned}$$

Here we have used the fact that $e^{(L-\lambda)T} - 1 < e^{LT}$. Thus we have proved that in the $\delta_1^{\frac{1}{1-\theta}}$ ball around x,

$$V(x) - V(z) < C_1|x-z|^{\theta_1} \quad \text{for some constant } C_1,$$

where

$$\theta_1 = \min\left\{1-\theta,\ \frac{\lambda\,\theta}{(F\log P/\beta)+L}\right\} \quad \text{for } 0<\theta<1.$$

For $L = \lambda$, using (3.20), (3.21), (3.30), (3.31), (3.32), and (3.34) in (3.19), we have

$$\begin{aligned}V(x) - V(z) \le 4K_0C|x-z|^{1-\theta} &+ 2\frac{K_1}{(F\log P/\beta)+L}\log(|x-z|)|x-z|^{1-\theta}\\ &+ 2C_1(FC+1)|x-z|^{1-\theta} + \frac{2K_0}{L}|x-z|^{\frac{L\theta}{(F\log P/\beta)+L}}\\ &+ 2C_0|x-z|^{\frac{L\theta}{(F\log P/\beta)+L}}.\end{aligned}$$

Since $|x-z|^{1-\theta}$ goes to 0 faster than $\log(|x-z|)$ goes to $-\infty$ as $|x-z|\to 0$, all terms on the right-hand side (RHS) go to 0. The modulus of continuity of V is the same as that of $\log(r)r^{1-\theta}$. This suggests that in the $\delta_1^{\frac{1}{1-\theta}}$ ball around x,

$$V(x) - V(z) < C_1|x-z|^{\theta_1} \quad \text{for some constant } C_1$$

and for all θ_1 such that

$$\theta_1 < \min\left\{1-\theta,\ \frac{L\theta}{(F\log P/\beta)+L}\right\} \quad \text{for } \ 0<\theta<1.$$

Thus in any case we have shown that (for θ_1 chosen depending on $L \ne \lambda$ or $L=\lambda$)

$$V(x) - V(z) \le C_1|x-z|^{\theta_1} \quad \text{for some constant } C_1.$$

Interchanging the roles of x and z we will get

$$V(z) - V(x) \le C_2|x-z|^{\theta_1} \quad \text{for some constant } C_2.$$

Together these will give

$$|V(x) - V(z)| \le C|x-z|^{\theta_1} \quad \text{for some constant } C.$$

This proves the Hölder continuity of V.

Now we want to justify our claim in (3.22), i.e., if $\sigma_1 < \Sigma_1$, we can choose $|x-z|$ small enough such that (3.22) holds. If we restrict $|x-z|$ such that $|x-z| \le \min(\frac{\beta}{4FC}, (\frac{\beta}{4CF})^{\frac{1}{1-\theta}})$, then by Remark 3.4,

$$|\Sigma_i - \sigma_i| \le Ce^{LT}(FC + G(FC+1))^{TF/\beta}|x-z|.$$

By our choice of T,

$$|\Sigma_i - \sigma_i| \le C|x-z|^{1-\theta} \le \frac{1}{4}\frac{\beta}{F} < \frac{1}{2}|\sigma_i - \sigma_{i+1}|$$

and this together with the assumption $\sigma_1 < \Sigma_1$ implies $\sigma_i < \Sigma_i < \sigma_{i+1}$ for all i. So our claim is justified. $\square$

4. Dynamic programming principle and the QVI. Under our assumptions (A1)–(A7), an optimal trajectory exists for any initial condition as shown in [5, Theorem 6.4]. The following dynamic programming principle and derivation of the QVI is also found in the literature [5], [4]. For the sake of completeness we prove it in detail here.

THEOREM 4.1 (dynamic programming principle). *Let V be the value function of the hybrid control problem as given in* (2.7). *If t_1 is the first hitting time of A, then*

$$V(x) = \inf_u \left\{ \int_0^{t_1} K(X(t), u(t))e^{-\lambda t} dt + e^{-\lambda t} MV(X_x(t_1)) \right\}, \tag{DPPA}$$

where

$$M\phi(x) = \inf_{v \in \mathcal{V}} \{\phi(g(x, v)) + C_a(x, v)\}$$

and if t_1 is the first hitting time of C, then

$$V(x) = \inf_u \left\{ \int_0^{t_1} K(X(t), u(t))e^{-\lambda t} dt + e^{-\lambda t} NV(X_x(t_1)) \right\}, \tag{DPPC}$$

where

$$N\phi(x) = \inf_{x' \in D} \{(\phi(x')) + C_c(x, x')\}.$$

For any $T > 0$,

$$\begin{aligned} V(x) = \inf_{u, v, \xi_i, X(\xi_i)'} \Bigg\{ &\int_0^T K(X_x(t), u(t))e^{-\lambda t} dt + \sum_{\sigma_i < T} e^{-\lambda \sigma_i} C_a(X(\sigma_i), v) \\ &+ \sum_{\xi_i < T} e^{-\lambda \xi_i} C_c\big(X(\xi_i), X(\xi_i)'\big) + e^{-\lambda T} V(X_x(T)) \Bigg\}. \end{aligned} \tag{DPP}$$

Proof. Let t_1 be the first hitting time of trajectory when it hits $A \cup C$. If t_1 is a first hitting time of A, we denote it by σ_1,

$$\begin{aligned} V(x) \le &\int_0^{\sigma_1} K(X(t), u(t))e^{-\lambda t} dt + C_a(X(\sigma_1), v)e^{-\lambda \sigma_1} \\ &+ \left[\int_{\sigma_1}^{\infty} K(X(t), u(t))e^{-\lambda t} dt + \sum_{i=2}^{\infty} C_a(X(\sigma_i), v)e^{-\lambda \sigma_i} \right. \\ &\left. + \sum_{i=1}^{\infty} C_c\big(X(\xi_i), X(\xi_i)'\big)e^{-\lambda \xi_i} \right]. \end{aligned}$$

We change the variable $t' = t - \sigma_1$ in the square bracket. Then taking the infimum in the square brackets over the control variables we get a value function of the trajectory starting from the point $g(X_x(\sigma_1), v)$. Hence,

$$\begin{aligned} V(x) \le &\int_0^{\sigma_1} K(X(t), u(t))e^{-\lambda t} dt + e^{-\lambda \sigma_1} C_a(X(\sigma_1), v) \\ &+ e^{-\lambda \sigma_1} V(g(X_x(\sigma_1), v). \end{aligned}$$

Now taking the infimum over discrete controls v belonging to $\mathcal{V}$ in the last two terms we get

$$V(x) \le \int_0^{\sigma_1} K(X(t), u(t))e^{-\lambda t}dt + MV(X_x(\sigma_1)).$$

Further taking the infimum over continuous controls u in $\mathcal{U}$ we have the one-way inequality in (DPPA). For the reverse inequality, let $\varepsilon > 0$ be given. Choose the control $\theta_\varepsilon = (u_\varepsilon, v_\varepsilon, \xi_{i\varepsilon}, X(\xi_i)'_\varepsilon)$ such that

$$\begin{aligned} V(x) + \varepsilon \ge &\int_0^{\sigma_1} K(X(t), u_\varepsilon(t))e^{-\lambda t}dt + C_a(X(\sigma_1), v_\varepsilon)e^{-\lambda\sigma_1} \\ &+ e^{-\lambda\sigma_1}\left[\int_{\sigma_1}^{\infty} K(X(t), u_\varepsilon(t))e^{-\lambda t}dt + \sum_{i=2}^{\infty} C_a(X(\sigma_i), v_\varepsilon)e^{-\lambda\sigma_i}\right. \\ &\left. + \sum_{i=1}^{\infty} C_c\big(X(\xi_{i\varepsilon}), X(\xi_i)'_\varepsilon\big)e^{-\lambda\xi_{i\varepsilon}}\right] \end{aligned}$$

with calculations similar to those earlier, we can conclude that

$$\begin{aligned} V(x) + \varepsilon &\ge \int_0^{\sigma_1} K(X(t), u(t))e^{-\lambda t}dt + MV(X_x(\sigma_1)) \\ &\ge \inf_u \int_0^{\sigma_1} K(X(t), u(t))e^{-\lambda t}dt + MV(X_x(\sigma_1)). \end{aligned}$$

Hence as $\varepsilon \to 0$ we have other way inequality. Thus (DPPA) is proved. Now we proceed to prove (DPPC). Let t_1 be the first hitting time of C where the controller chooses to jump. In this case we write $t_1 = \xi_1$. Then

$$\begin{aligned} V(x) \le &\int_0^{\xi_1} K(X(t), u(t))e^{-\lambda t}dt + C_c\big(X(\xi_1), X(\xi_1)'\big)e^{-\lambda\xi_1} \\ &+ \left[\int_{\xi_1}^{\infty} K(X(t), u(t))e^{-\lambda t}dt + \sum_{i=1}^{\infty} C_a(X(\sigma_i), v)e^{-\lambda\sigma_i}\right. \\ &\left. + \sum_{i=2}^{\infty} C_c\big(X(\xi_i), X(\xi_i)'\big)e^{-\lambda\xi_i}\right]. \end{aligned}$$

Doing the change of variables $t' = t - \xi_1$ in the square brackets and taking the infimum over the control variables, it is the value function of trajectory starting from $(X_x(\xi_1))'$. Hence,

$$V(x) \le \int_0^{\xi_1} K(X(t), u(t))e^{-\lambda t}dt + e^{-\lambda\xi_1}C_c(X(\xi_1), X(\xi_1)') + e^{-\lambda\xi_1}V(X_x(\xi_1)').$$

Now taking the infimum over $(X_x(\xi_1))' \in D$ in the last two terms we get

$$V(x) \le \int_0^{\xi_1} K(X(t), u(t))e^{-\lambda t}dt + NV(X_x(\xi_1)),$$

and taking the infimum over u in $\mathcal{U}$ on the RHS we will get the one-way inequality of (DPPC).

For the reverse inequality, given $\varepsilon > 0$ choose $\theta_\varepsilon = (u_\varepsilon, v_\varepsilon, \xi_{i_\varepsilon}, X(\xi_i)'_\varepsilon)$ such that

$$\begin{aligned} V(x) + \varepsilon &\geq \int_0^{\xi_{1_\varepsilon}} K(X(t), u_\varepsilon(t)) e^{-\lambda t} dt + NV(X_x(\xi_{1_\varepsilon})) \\ &\geq \inf_u \int_0^{\xi_{1_\varepsilon}} K(X(t), u(t)) e^{-\lambda t} dt + NV(X_x(\xi_{1_\varepsilon})). \end{aligned}$$

As $\varepsilon \to 0$ we will get

$$V(x) = \inf_u \left\{ \int_0^{\xi_1} K(X(t), u(t)) e^{-\lambda t} dt + NV(X_x(\xi_1)) \right\},$$

which proves (DPPC). The proof of (DPP) for any $T > 0$ follows similarly, which we skip here. □

Theorem 4.2 (quasi-variational inequality). *Under the assumptions* (A1)–(A7) *and* (C1), (C2)*, the value function V described in* (2.7) *satisfies the following the QVI in the viscosity sense:*

$$\text{(QVI)} \qquad V(x) = \begin{cases} MV(x) & \forall x \in A, \\ \min\{NV(x), -H(x, DV(x))\} & \forall x \in C, \\ -H(x, DV(x)) & \forall x \in \Omega \setminus A \cup C, \end{cases}$$

where H is the Hamiltonian given by

$$H(x, p) = \sup_{u \in U} \left\{ \frac{-K(x, u) - f(x, u) \cdot p}{\lambda} \right\}.$$

Proof. Let $x \in A$. In this case we have to show that $V(x) = MV(x)$. Since $x \in A$, the first hitting time of trajectory is $\sigma_1 = 0$. Hence, by (DPPA) we get $V(x) = MV(x)$.

Now we consider the case $x \in \Omega \setminus A \cup C$. In this case we want to show that V satisfies the Hamilton–Jacobi–Bellman (HJB) equation in the viscosity sense. For we need to show the following: for all $\phi \in C^1(\Omega)$ and x local maximum of $V - \phi$

$$V(x) + H(x, D\phi(x)) \leq 0$$

and for all $\phi \in C^1(\Omega)$ and x local minimum of $V - \phi$

$$V(x) + H(x, D\phi(x)) \geq 0.$$

Let $r = \min\{\mathbf{d}(x, \partial A), \mathbf{d}(x, \partial C)\}$. Choose $R < r$. Then in the ball $B(x, R)$ no impulses are applied. Now V is continuous at x, and assume that $V - \phi$ has local maximum at x. Choose τ small enough such that $X_x(\tau) \in B(x, R)$. By our choice of R and τ, τ is less than the first hitting time. Then, since x is the local maximum of $V - \phi$,

$$\begin{aligned} \phi(x) - \phi(X_x(\tau)) &\leq V(x) - V(X_x(\tau)) \\ &\leq \int_0^\tau K(X_x(t), u(t)) e^{-\lambda t} dt + (e^{-\lambda \tau} - 1) V(X_x(\tau)), \end{aligned}$$

where the second inequality follows by (DPP), since $\tau < \sigma_1$ and $\tau < \xi_1$. Dividing by τ and taking the limit as $\tau \to 0$ we get

$$-D\phi(x) \cdot f(x) \leq K(x, u(0)) - \lambda V(x),$$

which implies

$$V(x) + \frac{-K(x, u(0) - D\phi(x) \cdot f(x)}{\lambda} \leq 0.$$

Taking the supremum over all $u \in \mathcal{U}$ we will get

$$V(x) + H(x, D\phi(x)) \leq 0.$$

Hence V is a viscosity subsolution of HJB equation.

To show that V is a viscosity supersolution, let $V - \phi$ have local minimum at x. Then for τ such that $X_x(\tau) \in B(x, R)$,

$$\begin{aligned}\phi(X_x(\tau)) - \phi(x) &\leq V(X_x(\tau)) - V(x) \\ &\leq (1 - e^{-\lambda\tau})V(X_x(\tau)) - \int_0^\tau K(X_x(t), u(t))e^{-\lambda t}dt \quad \text{by (DPP).}\end{aligned}$$

Dividing by τ and taking the limit as $\tau \to 0$ we get

$$\lambda V(x) - K(x, u(0)) - D\phi(x) \cdot f(x) \geq 0,$$

$$V(x) + \frac{-K(x, u(0)) - D\phi(x) \cdot f(x)}{\lambda} \geq 0.$$

Taking the supremum over all u we will get

$$V(x) + H(x, D\phi(x)) \geq 0.$$

Hence V is a viscosity supersolution of the HJB equation. Thus we have shown that in the case $x \in \Omega \setminus A \cup C$, V satisfies the HJB equation in the viscosity sense.

Now consider the case $x \in C$. We observe that if $x \in C$, and the controller chooses to jump, then by (DPPC), V should satisfy $NV(x)$. Whereas if the controller decides not to jump, then the system undergoes some continuous evolution and we can analyze as before to conclude that V satisfies the HJB equation in the viscosity sense. In this case we have to show that V satisfies the following equation in the viscosity sense:

$$\min\{V(x) - NV(x), V(x) + H(x, DV(x))\} = 0.$$

For this we need to show that, for all $\phi \in C^1(\Omega), x$ local minimum of $V - \phi$

$$\min\{V(x) - NV(x), V(x) + H(x, DV(x))\} \geq 0,$$

and for all $\phi \in C^1(\Omega), x$ local maximum of $V - \phi$,

$$\min\{V(x) - NV(x), V(x) + H(x, DV(x))\} \leq 0.$$

Now if $V(x) = NV(x)$, there is nothing to prove.

Suppose $V(x) < NV(x)$; then we need to show that V satisfies the HJB equation in the viscosity sense. We show that whenever $V(x) < NV(x)$ there exists $r > 0$ and a ball $B(x, r)$ around x such that it is not optimal to apply any impulses on $B(x, r)$. Then we can do the analysis in this ball to conclude as in the case of $x \in \Omega \setminus A \cup C$. For we claim that there exists $\varepsilon > 0$ such that

$$V(x) = \inf_{u,v,\xi_i,X(\xi_i)'} \left\{ \int_0^{t_1} K(X_x(t), u(t))e^{-\lambda t}dt + NV(X_x(t_1)) \mid t_1 > \varepsilon \right\}.$$

Suppose not; then $\varepsilon = 0$, which implies $\xi_1 = 0$, which by (DPPC) implies $V(x) = NV(x)$; this is a contradiction of our hypothesis $V(x) < NV(x)$. Hence $\varepsilon > 0$. Choose $r < \min\{d(x, X_x(\varepsilon)), d(A, C)\}$. Then in the ball $B(x, r)$, no impulses are applied. So we can do the analysis in this ball around x and conclude as in the earlier case. This proves the QVI for the case $x \in C$. ∎

5. Uniqueness. We take up the issue of uniqueness of the viscosity solutions of (QVI) in this section. Inspired by the earlier work on impulse control problem (see [2], [9]), we prove the comparison between any two solutions of the QVI.

THEOREM 5.1. *Assume* (A1)–(A7) *and* (C1), (C2). *Let* $u_1, u_2 \in BC(\Omega)$, *bounded continuous functions on* Ω, *be two viscosity solutions of the QVI given by* (QVI). *Then* $u_1 = u_2$.

Proof. The idea of the proof is to show that $u_1(x) \le u_2(x)$ for all $x \in \Omega$. We define the following auxiliary function Φ on $\bigcup_{i=1}^{\infty} (\Omega_i \times \Omega_i)$ that is Φ^i on each $\Omega_i \times \Omega_i$ by

$$\Phi^i(x, y) = u_1(x) - u_2(y) - \frac{1}{\epsilon}|x - y|^2 - \kappa\big(|x|^2 + |y|^2\big), \tag{5.1}$$

where ε and κ are small positive parameters to be chosen suitably later on. Observe that for each i, Φ^i attains its supremum over $\Omega_i \times \Omega_i$, thanks to the last two terms, which become large negative as $|x|, |y|$ goes to 0. We prove the theorem in two steps. In the first step of the proof we show that $\sup_i \sup_{\Omega_i \times \Omega_i} \Phi^i(x, y) \le 0$. In the next step we prove the uniqueness using Step 1.

Step 1. Let

$$\sup_i \sup_{\Omega_i \times \Omega_i} \Phi^i(x, y) = C > 0.$$

Fix $\kappa > 0$ such that $\kappa < \min\{\frac{C}{2}, \frac{C'}{2}\}$. If the above supremum is achieved at some (x_0, y_0), the following proof gets simplified. If not, corresponding to this κ we can choose (x_κ, y_κ) in some $\Omega_i \times \Omega_i$, say, $\Omega_1 \times \Omega_1$, such that

$$\Phi^1(x_\kappa, y_\kappa) > C - \kappa > \frac{C}{2}. \tag{5.2}$$

Let Φ^1 attain its supremum at some finite point, say, at (x_0, y_0) in $\Omega_1 \times \Omega_1$. Then

$$\sup_{\Omega_1 \times \Omega_1} \Phi^1(x, y) = \Phi^1(x_0, y_0) > C - \kappa > \frac{C}{2}. \tag{5.3}$$

Since x_0 and y_0 can lie in different sets in Ω_1, $u_1(x_0)$ and $u_2(y_0)$ will satisfy different equations from the QVI. We list below the different cases which arise:

1. $(x_0, y_0) \in A \times C$ or $C \times A$.
2. $(x_0, y_0) \in \Omega \setminus (A \cup C) \times \Omega \setminus (A \cup C)$.
3. $x_0, y_0 \notin A$ and one of x_0 or $y_0 \in C$. This takes care of $(x_0, y_0) \in C \times \Omega \setminus (A \cup C), (x_0, y_0) \in \Omega \setminus (A \cup C) \times C,\ \ (x_0, y_0) \in C \times C$.
4. $x_0, y_0 \notin C$ and one of the x_0 or $y_0 \in A$, i.e., $(x_0, y_0) \in A \times A$ or $(x_0, y_0) \in A \times \Omega \setminus (A \cup C),\ \ (x_0, y_0) \in \Omega \setminus (A \cup C) \times A$.

Our idea is to show that in any of these cases, $u_1(x) - u_2(x)$ is arbitrarily small for ε and κ small. For this we will estimate $u_1(x_0) - u_2(y_0)$ at the maximum point (x_0, y_0) of Φ^1 or $u_1(x_n) - u_2(y_n)$ at the maximum point (x_n, y_n) of ψ_n, a suitably defined auxiliary function. The crucial point in our proof is that after at most finitely many steps, say n_0, at the maximum point of ψ_{n_0} both u_1 and u_2 satisfy the HJB equation. Then we can use the usual comparison principle available in the literature. We first list some standard estimates needed later in the proof.

LEMMA 5.2. *Let Φ and (x_0, y_0) be as above. Then*

(i) $\frac{|x_0-y_0|^2}{\epsilon} \leq C$ *for some C independent of κ and ε;*

(ii) $\sqrt{\kappa}|x_0|, \sqrt{\kappa}|y_0| \leq \hat{C}$ *for some $\hat{C}$ independent of κ and ε;*

(iii) $\frac{|x_0-y_0|^2}{\epsilon} \leq \omega_\kappa^1(\sqrt{C\epsilon})$, *where ω_κ^1 is the local modulus of continuity of both u_1 and u_2 in the ball of radius R, dependent on κ but independent of ε, $R = R(\kappa) = \hat{C}/\sqrt{\kappa}$ in Ω_1.*

Proof. By our assumption

$$2\Phi^1(x_0, y_0) \geq \Phi^1(x_0, x_0) + \Phi^1(y_0, y_0). \tag{5.4}$$

Hence

$$\frac{2}{\epsilon}|x_0 - y_0|^2 \leq u_1(x_0) - u_1(y_0) + u_2(x_0) - u_2(y_0). \tag{5.5}$$

Since u_1 and u_2 are bounded,

$$\frac{|x_0 - y_0|^2}{\epsilon} \leq C,$$

which proves (i). This also implies

$$|x_0 - y_0| \leq \sqrt{C\epsilon}.$$

To prove (ii), fix some $z \in \Omega_1$ such that $|z| = 1$; then $\Phi^1(x_0, y_0) \geq \Phi^1(z, z)$, which implies

$$\begin{aligned}\kappa\big(|x_0|^2 + |y_0|^2\big) &\leq u_1(x_0) - u_1(z) - u_2(y_0) + u_2(z) - \frac{1}{\epsilon}|x_0 - y_0|^2 + 2\kappa|z|^2 \\ &\leq C + 2\kappa \leq C + 2.\end{aligned}$$

Hence $\sqrt{\kappa}|x_0| \leq \hat{C}$, where $\hat{C}$ is independent of κ and ε. Similarly, $\sqrt{\kappa}|y_0| \leq \hat{C}$. This proves (ii). Hence x_0 and y_0 lie in some ball B_R of radius $R = R(\kappa)$.

Now using the estimate in (i) and the modulus of continuity of u_1 and u_2 in the compact set $\bar{B}_{R(\kappa)}$ in Ω_1, we get

$$\frac{|x_0 - y_0|^2}{\epsilon} \leq \omega_\kappa^1(\sqrt{C\epsilon}).$$

This proves (iii). ☐

Now we consider the different cases listed earlier.

Case 1. $(x_0, y_0) \in A \times C$ or $C \times A$.

Claim. This case does not occur.

Without loss of generality let $(x_0, y_0) \in A \times C$. Since $d(A, C) > \beta$,

$$\Rightarrow |x_0 - y_0| > \beta.$$

On the other hand by Lemma 5.2(i),

$$|x_0 - y_0| < \sqrt{C\epsilon}.$$

So choosing ϵ such that $\sqrt{C\epsilon} < \frac{\beta}{2}$,

$$|x_0 - y_0| < \frac{\beta}{2},$$

which is a contradiction. Hence Case 1 does not occur, for small ε.

Case 2. $(x_0, y_0) \in \Omega \setminus (A \cup C) \times \Omega \setminus (A \cup C)$.

In this case at $(x_0, y_0) \in \Omega_1 \times \Omega_1$, u_1, u_2 both satisfy the HJB equation. Hence we do all the calculations in Ω_1. Let us define the test functions ϕ_1 and ϕ_2 on Ω_1 as follows:

$$\phi_1(x) = u_2(y_0) + \frac{1}{\epsilon}|x - y_0|^2 + \kappa\big(|x|^2 + |y_0|^2\big), \tag{5.6}$$

$$\phi_2(y) = u_1(x_0) - \frac{1}{\epsilon}|x_0 - y|^2 - \kappa\big(|x_0|^2 + |y|^2\big). \tag{5.7}$$

Then, since (x_0, y_0) is point of supremum for Φ^1, $u_1 - \phi_1$ attains its maximum at x_0 and $u_2 - \phi_2$ attains its minimum at y_0. Also observe

$$D\phi_1(x_0) = \frac{2}{\epsilon}(x_0 - y_0) + 2\kappa x_0, \tag{5.8}$$

$$D\phi_2(y_0) = \frac{2}{\epsilon}(x_0 - y_0) - 2\kappa y_0, \tag{5.9}$$

and by Lemma 5.2

$$|D\phi_2(y_0)| \le \frac{2}{\epsilon}|x_0 - y_0| + \sqrt{\kappa}\hat{C}. \tag{5.10}$$

Now by definition of the viscosity sub- and supersolutions, and using u_1 as the subsolution and u_2 as the supersolution,

$$\begin{aligned} &u_1(x_0) + H(x_0, D\phi_1(x_0)) \le 0 \le u_2(y_0) + H(y_0, D\phi_2(y_0)) \\ &\quad \Rightarrow u_1(x_0) - u_2(y_0) \le H(y_0, D\phi_2(y_0)) - H(x_0, D\phi_1(x_0)). \end{aligned}$$

By our assumptions (A1)–(A7) and the definition of Hamiltonian H, one can easily prove that H satisfies the structural condition

$$|H(x,p) - H(y,q)| \le F|p - q| + L|q||x - y| + K_1|x - y|, \tag{5.11}$$

where K_1 is the Lipschitz constant for the running cost k. Using (5.11) we get

$$\begin{aligned} u_1(x_0) - u_2(y_0) \le L\,|D\phi_2(y_0)|\,|x_0 - y_0| + K_1|x_0 - y_0| \\ + F\,|D\phi_2(y_0) - D\phi_1(x_0)|. \end{aligned}$$

Substituting from (5.8), (5.9), and (5.10),

$$u_1(x_0) - u_2(y_0) \le \frac{2L}{\epsilon}|x_0 - y_0|^2 + \sqrt{\kappa}L\hat{C}|x_0 - y_0| + K_1|x_0 - y_0| + 2\kappa F|x_0 + y_0|.$$

By Lemma 5.2 we then get

$$u_1(x_0) - u_2(y_0) \le 2L\omega_\kappa^1(\sqrt{C\epsilon}) + L\hat{C}\sqrt{C\kappa\varepsilon} + K_1(\sqrt{C\epsilon}) + 4F\hat{C}\sqrt{\kappa}. \tag{5.12}$$

Also observe that by (5.2)

$$\begin{aligned}\frac{C}{2} < C-\kappa < \Phi^1(x_\kappa, x_\kappa) \\ &\le \Phi^1(x_0,y_0) \\ &\le u_1(x_0)-u_2(y_0) \\ &\le 2L\omega_\kappa^1\left(\sqrt{C\epsilon}\right)+2L\hat{C}\sqrt{C\kappa\varepsilon}+K_1\left(\sqrt{C\epsilon}\right)+4F\hat{C}\sqrt{\kappa}.\end{aligned}$$

Now fixing κ and sending ε to 0 and then choosing κ such that $4F\hat{C}\sqrt{\kappa} < \frac{C}{4}$ we will have

$$\frac{C}{2} < \frac{C}{4}.$$

This is a contradiction. Hence,

$$\sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x,y) \le 0.$$

Case 3. $x_0, y_0 \notin A$, and one of $x_0, y_0 \in C$. Without loss of generality let $y_0 \in C$. $x_0 \notin A$ and u_1 is a subsolution of the QVI implies

$$u_1(x_0)+H(x_0, Du_1(x_0)) \le 0,$$

$$y_0 \in C \Rightarrow \max\left\{u_2(y_0)+H(y_0,Du_2(y_0)), u_2(y_0)-Nu_2(y_0)\right\}=0,$$

and u_2 is a solution of the QVI, in particular it is a supersolution. Hence either $u_2+H\ge 0$ or $u_2-Nu_2\ge 0$ at y_0.

If $u_2(y_0)+H(y_0,Du_2(y_0))\ge 0$, we can proceed as in Case 2 and get a contradiction. Otherwise assume $u_2(y_0)-Nu_2(y_0)\ge 0$. Since u_2 is also a subsolution

$$u_2(x)\le Nu_2(x) \quad \forall x\in C.$$

Therefore,

$$u_2(y_0)=Nu_2(y_0)=\inf_{y'\in D} u_2(y')+c_c(y_0,y')=\inf_i \inf_{D_i} u_2(y')+c_c(y_0,y').$$

As each D_i is compact, the infimum is attained on each D_i. If the infimum over i is not attained, then we can choose y_0' in, say, D_2 such that

$$u_2(y_0)=Nu_2(y_0)>u_2(y_0')+c_c(y_0,y_0')-\kappa, \quad y_0'\in D_2.$$

Also $y_0'\notin A$. We estimate the difference $\Phi^1(x_0,y_0)$ and $\Phi^2(y_0',y_0')$ in the following lemma, which we will use to define another auxiliary function ψ_1, and consider the maximum point (x_1,y_1) of ψ_1, in the same spirit as in the earlier work on the impulse control problem (see [2], [7], [9]). We will show that after at most a finite number of such auxiliary functions, we necessarily arrive at Case 2.

Recall that y_0' lies in D, hence by (A2), $|y_0'|<R$. We will also need that x_0 and y_0 are not too close to y_0' in case $y_0'\in\Omega_1$. The following lemma proves this fact. More generally we prove here that whenever $u(x)=Nu(x)$ or $u(x)=Mu(x)$ the destination point is at a certain positive distance away from the point of supremum.

LEMMA 5.3. *Let $u \in BC(\Omega)$ be a solution of* (QVI). *If x, x', and $g(x,v')$ belong to $D_1 \subseteq \Omega_1$ and if*

$$\begin{aligned} & u(x) = Nu(x) > u(x') + c_c(x,x') - \kappa \\ \text{or} \quad & u(x) = Mu(x) = u(g(x,v')) + c_a(x,v'), \end{aligned}$$

then there exists an $\alpha_1 > 0$ depending only on the uniform continuity of u on $D_1 \subseteq \Omega_1$ but independent of ε and κ such that

$$|x - x'| > \alpha_1 \tag{5.13}$$

$$\text{or} \quad |x - g(x,v')| > \alpha_1, \tag{5.14}$$

depending on which equation $u(x)$ satisfies.

Proof. We claim that there exists $\alpha_1 > 0$ such that $|x - x'| > \alpha_1$. Suppose the contrary. That is, there exists sequence $x_n,\ x'_n \in \Omega_1$ such that

$$u(x_n) > u(x'_n) + c_c(x_n, x'_n) - \kappa \ \text{ and } \ |x_n - x'_n| \to 0.$$

Then by continuity of u, $|u(x_n) - u(x'_n)| \to 0$. But

$$|u(x_n) - u(x'_n)| = c_c(x_n, x'_n) - \kappa > C' - \kappa > \frac{C'}{2} > 0,$$

which is a contradiction. Hence given $\frac{C'}{4}$ choose the corresponding α_1 given by uniform continuity of u on $D_1 \subseteq \Omega_1$ such that $|y - z| < \alpha_1 \Rightarrow |u(y) - u(z)| < \frac{C'}{4}$. Then

$$|x - x'| > \alpha_1.$$

This proves (5.13).

To prove that $|x - g(x,v')| > \alpha_1$, we proceed with arguments similar to those above and choose α_1 corresponding to the $\frac{C'}{4}$ in the definition of uniform continuity of u on D_1. ∎

In the next lemma we estimate the difference $\Phi^1(x_0, y_0)$ and $\Phi^2(y'_0, y'_0)$, which we are going to use to define new auxiliary function ψ_1.

LEMMA 5.4. *Let Φ be as defined in* (5.1) *and let $(x_0, y_0) \in \Omega_1 \times \Omega_1$ be as in* (5.3), *the point where Φ^1 attains supremum. Let $y'_0 \in D_2$ be such that*

$$u_2(y_0) = Nu_2(y_0) > u_2(y'_0) + c_c(y_0, y'_0) - \kappa. \tag{5.15}$$

Then

$$\Phi^1(x_0, y_0) - \Phi^2(y'_0, y'_0) \le \kappa K$$

for some constant $K > 1$ depending only on the constants of the problem and independent of ϵ and κ.

Proof.

$$\begin{aligned} \Phi^1(x_0, y_0) - \Phi^2(y'_0, y'_0) = {} & u_1(x_0) - u_2(y_0) - \frac{1}{\epsilon}|x_0 - y_0|^2 - \kappa(|x_0|^2 + |y_0|^2) \\ & - u_1(y'_0) + u_2(y'_0) + 2\kappa|y'_0|^2. \end{aligned}$$

Using (5.15) we get

$$\Phi^1(x_0,y_0) - \Phi^2\big(y_0',y_0'\big) < u_1(x_0) - c_c\big(y_0,y_0'\big) - \frac{1}{\epsilon}|x_0-y_0|^2 - \kappa\big(|x_0|^2+|y_0|^2\big) - u_1\big(y_0'\big) + 2\kappa|y_0'|^2 + \kappa.$$

Also $u_1(y_0) \le Nu_1(y_0) \le u_1(y_0') + c_c(y_0,y_0')$. Hence,

$$\begin{aligned}\Phi^1(x_0,y_0) - \Phi^2\big(y_0',y_0'\big) &\le u_1(x_0) - u_1(y_0) - \frac{1}{\epsilon}|x_0-y_0|^2 - \kappa\big(|x_0|^2+|y_0|^2\big) + 2\kappa|y_0'|^2 \\ &\quad + \kappa \le u_1(x_0) - u_1(y_0) + 2\kappa|y_0'|^2 + \kappa \\ &\le u_1(x_0) - u_1(y_0) + 2\kappa R^2 + \kappa \\ &\le \omega_\kappa^1\big(\sqrt{C\epsilon}\big) + 2\kappa R^2 + \kappa.\end{aligned}$$

Using the modulus of continuity of u_1, on $\bar{B_R}$ in Ω_1 for a given $\kappa > 0$ choose $\epsilon > 0$ such that

$$\omega_\kappa^1\big(\sqrt{C\epsilon}\big) < \kappa \Rightarrow \Phi^1(x_0,y_0) - \Phi^2\big(y_0',y_0'\big) \le \kappa K.$$

This proves the lemma. ▯

We use the above difference to define another auxiliary function ψ_1. We further restrict α_2 given by Lemma 5.3, if necessary, so that $\alpha_2 < \frac{\beta}{2}$. Define

$$\psi_1^2(x,y) = \Phi^2(x,y) + 2\kappa K\ \zeta_1(x,y),$$

$$\psi_1^i(x,y) = \Phi^i(x,y) \quad \forall i \neq 2,$$

where, $\zeta_1(x,y) \in C_0^\infty(\Omega_2 \times \Omega_2)$, such that

$$\zeta_1\big(y_0',y_0'\big) = 1; \quad 0 \le \zeta_1 \le 1; \quad |D\zeta_1| \le \frac{2}{\alpha_2};$$

$$\zeta_1(x,y) < 1 \text{ if } (x,y) \neq \big(y_0',y_0'\big);$$

$$\text{and} \quad \zeta_1(x,y) = 0 \quad \forall (x,y) \ \text{ such that } |x-y_0'|^2 + |y-y_0'|^2 > \alpha_1,$$

i.e., ζ_1 has support in the α_1 ball around $(y_0',y_0') \in \Omega_2 \times \Omega_2$, having maximum at (y_0',y_0') and it vanishes on all $\Omega_i \times \Omega_i$ other than $i = 2$.

Observe that by the definition of ψ_1^i,

$$\begin{aligned}\psi_1^2\big(y_0',y_0'\big) &= \Phi^2\big(y_0',y_0'\big) + 2\kappa K \\ &\ge \Phi^1(x_0,y_0) - K\kappa + 2\kappa K \\ &\ge \sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x,y) + \kappa K - \kappa \\ &\ge \psi_1^2(x,y) - 2\kappa K\ \zeta_1(x,y) + \kappa(K-1).\end{aligned}$$

As ζ_1 is 0 for all $(x,y) \in \Omega_i \times \Omega_i$, $i \neq 2$, and for (x,y) outside the α_1 ball around (y_0',y_0') in $\Omega_2 \times \Omega_2$, we have for all such (x,y)

$$\psi_1^2\big(y_0',y_0'\big) > \psi_1^2(x,y).$$

Hence ψ_1^2 has the supremum over $\Omega_2 \times \Omega_2$ in the α_1 ball around (y_0', y_0'). Let (x_1, y_1) be such that

$$\sup_{\Omega_2 \times \Omega_2} \psi_1^2 = \psi_1^2(x_1, y_1).$$

Then

$$\psi_1^2(x_1, y_1) \geq \psi_1^1(x_0, y_0) = \Phi^1(x_0, y_0) > C - \kappa. \tag{5.16}$$

Since $\alpha_1 < \frac{\beta}{2}$, $x_1, y_1 \notin A$. We remark here that by using the technique of Lemma 5.2, we can prove that

$$\frac{|x_1 - y_1|^2}{\epsilon} \leq \omega_\kappa^2\big(\sqrt{C\epsilon}\big) + 2K\kappa \quad \text{and} \quad |x_1|, |y_1| < \hat{C}\sqrt{\kappa}.$$

Thus either $x_1, y_1 \notin C$ or one of them is in C. If $x_1, y_1 \notin C$, we are in Case 2 or Case 4. If we are in Case 2, we can get the comparison by working with ψ_1 instead of Φ as in Case 2. We will show in the next step of the proof how to handle Case 4. Now if one of $x_1, y_1 \in C$, we are again in Case 3. So without loss of generality let $y_1 \in C$ and y_1 be such that $u_2(y_1) - Nu_2(y_1) \geq 0$. Then, as earlier, the approximate infimum will be attained at some point, say, $y_1' \in D$, some D_i which we call D_3. That is

$$u_2(y_1) = Nu_2(y_1) > u_2\big(y_1'\big) + c_c\big(y_1, y_1'\big) - \kappa.$$

We define ψ_2 on $\bigcup \Omega_i \times \Omega_i$, that is, ψ_2^i on $\Omega_i \times \Omega_i$, by

$$\psi_2^i(x, y) = \Phi^i(x, y) + 2\kappa K \sum_{j=1}^{2} \zeta_j(x, y),$$

where $\zeta_2(y_1', y_1') = 1$ and ζ_2 has support in the α_3 ball around (y_1', y_1') in $\Omega_3 \times \Omega_3$ with the properties $\zeta_2 \in C_0^\infty(\Omega \times \Omega)$, $0 \leq \zeta_2 \leq 1$, $|D\zeta_2| \leq \frac{2}{\alpha_3}$, $\zeta_2(x, y) < 1$ if $(x, y) \neq (y_1', y_1')$. Hence as before we can show that the supremum of ψ_2 is attained in the α_3 ball around (y_1', y_1'). Also we can show that ψ_2^3 satisfies the inequality similar to (5.16), namely,

$$\psi_2^2(x_1, y_1) \geq \psi_1^2(x_1, y_1) = \Phi^1(x_0, y_0) > C - \kappa.$$

Thus we can proceed to define $\psi_3, \psi_4, \dots, \psi_n$ and so on, in case $u_2(y_i) = Nu_2(y_i)$. We now claim that this process has to terminate in finitely many steps, which is the content of the following lemma.

Lemma 5.5. *Suppose $(x_n, y_n) \in \Omega_{n+1} \times \Omega_{n+1}$, $y_n' \in D_{n+2}$ are sequences such that*

$$u_2(y_n) = Nu_2(y_n) > u_2(y_n') + c_c(y_n, y_n') - \kappa, \quad y_n \in B(y_{n-1}', \alpha_{n+1});$$
$$\psi_n(x, y) = \psi_{n-1}(x, y) + 2\kappa K\, \zeta_n(x, y); \quad \psi_n(x_n, y_n) = \sup_{\Omega_{n+1} \times \Omega_{n+1}} \psi_n(x, y);$$

where ζ_n is such that $\zeta_n \in C_0^\infty(\Omega \times \Omega)$; actually ζ_n has support in the α_{n+1} ball around $(y_n', y_n') \in \Omega_{n+2} \times \Omega_{n+2}$. $0 \leq \zeta_n \leq 1$; $|D\zeta_n| < \frac{2}{\alpha_{n+1}}$; $\zeta_n(y_{n-1}', y_{n-1}') = 1$, $n = 1, 2, \dots$. Then $n < n_0 = [\frac{8\hat{C}}{C'}]$, where $\hat{C}$ is a bound on u_1 and u_2 and C' is the lower bound on c_c.

Proof. Observe that y_i', $y_{i+1} \in D_{i+2}$. By uniform continuity of u_2 on $D_{i+2} \subseteq \Omega_{i+2}$, for all i,

$$|y_{i+1} - y_i'| < \alpha_{i+1} \Rightarrow |u_2(y_{i+1}) - u_2(y_i')| < \frac{C'}{4}.$$

By assumption,

$$\begin{aligned} u_2(y_0) &> u_2(y_0') + c_c(y_0, y_0') - \kappa \\ &> u_2(y_0') + C' - \kappa\ ; \quad \text{because } c_c \geq C' > 0 \\ &> u_2(y_1) - \frac{C'}{4} + C' - \kappa = u_2(y_1) + \frac{3}{4}C' - \kappa \\ &> u_2(y_1') + c_c(y_1, y_1') + \frac{3}{4}C' - 2\kappa \ > \ u_2(y_1') + C' + \frac{3}{4}C' - 2\kappa \\ &> u_2(y_2) - \frac{C'}{4} + C' + \frac{3}{4}C' - 2\kappa = u_2(y_2) + \frac{6}{4}C' - 2\kappa. \end{aligned}$$

Therefore, at the nth stage we will get

$$\hat{C} \geq u_2(y_0) > u_2(y_n) + \frac{3}{4}nC' - n\kappa.$$

By using $\kappa < \frac{C'}{2}$, if $n > n_0 = [\frac{8\hat{C}}{C'}]$, then $u_2(y_0) > \hat{C}$, which is a contradiction, because $|u_2| < \hat{C}$. ∎

Thus we have only a finite sequence of $\{y_n\}$ such that $u_2(y_n) = Nu_2(y_n)$. So, for $n > n_0 = [\frac{8\hat{C}}{C'}]$ necessarily $u_2(y_n) < Nu_2(y_n)$ and hence

$$u_2(y_n) + H(y_n, Du_2(y_n)) \geq 0.$$

Hence both u_1 and u_2 satisfy the HJB at the supremum point of auxiliary function ψ_n. Now we proceed as in Case 2 taking care of the extra terms.

In this case we define test functions ϕ_1 and ϕ_2 by

$$\phi_1(x) = u_2(y_n) + \frac{1}{\epsilon}|x - y_n|^2 + \kappa(|x|^2 + |y_n|^2) - 2\kappa K \sum_{j=1}^{n} \zeta_j(x, y_n), \tag{5.17}$$

$$\phi_2(y) = u_1(x_n) - \frac{1}{\epsilon}|x_n - y|^2 - \kappa(|x_n|^2 + |y|^2) + 2\kappa K \sum_{j=1}^{n} \zeta_j(x_n, y). \tag{5.18}$$

Then by the definition of (x_n, y_n), $u_1 - \phi_1$ has maximum at x_n and $u_2 - \phi_2$ has minimum at y_n. Using u_1 as the viscosity subsolution and u_2 as the viscosity supersolution, we get

$$u_1(x_n) - u_2(y_n) \leq H(y_n, D\phi_2(y_n)) - H(x_n, D\phi_1(x_n)).$$

Let $\alpha = \min\{\alpha_1, \ldots, \alpha_{n+1}\}$. Also, whenever $(x_n, y_n) \in \Omega_{j+1} \times \Omega_{j+1}$ we can write

$$D\phi_1(x_n) = \frac{2}{\epsilon}(x_n - y_n) + 2\kappa x_n - 2K\kappa \sum_{j=1}^{n} D\zeta_j(x_n, y_n), \tag{5.19}$$

$$D\phi_2(y_n) = \frac{2}{\epsilon}(x_n - y_n) - 2\kappa y_n + 2K\kappa \sum_{j=1}^{n} D\zeta_i(x_n, y_n), \tag{5.20}$$

$$|D\phi_1(y_n)| \leq \frac{2}{\epsilon}(x_n - y_n) + 2\kappa|y_n| + \frac{4nK\kappa}{\alpha}. \tag{5.21}$$

Hence by structural condition on H given by (5.11),

$$u_1(x_n) - u_2(y_n) \le L|D\phi_2(y_n)|\,|x_n - y_n| + K_1|x_n - y_n| + F|D\phi_2(y_n) - D\phi_2(x_n)|. \tag{5.22}$$

By using (5.19), (5.20), (5.21) in the above we get

$$\begin{aligned} u_1(x_n) - u_2(y_n) \le{}& \frac{2L}{\epsilon}|x_n - y_n|^2 + 2\kappa L|y_n|\,|x_n - y_n| + \left(\frac{4K\kappa n}{\alpha}\right)|x_n - y_n| \\ &+ K_1|x_n - y_n| + 4F\kappa(|x_n| + |y_n|) + \frac{8\kappa K n}{\alpha}. \end{aligned} \tag{5.23}$$

Now by using the technique of Lemma 5.2 for ψ_n, we can prove that

$$\begin{aligned} |x_n - y_n| &< \sqrt{C\epsilon}, \\ \frac{|x_n - y_n|^2}{\epsilon} &\le \omega_\kappa^n\big(\sqrt{C\epsilon}\big) + 2\kappa K, \\ |x_n|\,|y_n| &\le \sqrt{\kappa}\hat{C}, \end{aligned}$$

where $\hat{C}, K$, and C are independent of ε and κ. Using these estimates in (5.23) we will get

$$\begin{aligned} u_1(x_n) - u_2(y_n) \le{}& 2L\omega_\kappa^n\big(\sqrt{C\epsilon}\big) + 4L\kappa K + 2L\hat{C}\sqrt{C\kappa\varepsilon} + \left(\frac{4K\kappa n}{\alpha}\right)\sqrt{C\epsilon} \\ &+ K_1\big(\sqrt{C\epsilon}\big) + 8F\hat{C}\sqrt{\kappa} + \frac{8\kappa K n}{\alpha}. \end{aligned} \tag{5.24}$$

Also observe that from (5.3),

$$\frac{C}{2} < C - \kappa < \Phi^1(x_0, y_0) \le \psi_n^{n+1}(x_n, y_n).$$

Hence

$$\begin{aligned} \frac{C}{2} < C - \kappa &\le u_1(x_n) - u_2(y_n) - \frac{|x_n - y_n|^2}{\varepsilon} - \big(|x_n|^2 + |y_n|^2\big) + 2\kappa K\sum_{j=1}^{n}\zeta_j(x^n, y^n) \\ &\le u_1(x_n) - u_2(x_n) + 2\kappa K n. \end{aligned}$$

By using (5.24) in the above, with $n \le n_0$ given by Lemma 5.5, we get

$$\begin{aligned} \frac{C}{2} \le{}& 2L\omega_\kappa^n\big(\sqrt{C\epsilon}\big) + 4L\kappa K + 2L\hat{C}\sqrt{C\kappa\varepsilon} + \left(\frac{4K\kappa n_0}{\alpha}\right)\sqrt{C\epsilon} \\ &+ K_1(\sqrt{C\epsilon}) + 8F\hat{C}\sqrt{\kappa} + \frac{8\kappa K n_0}{\alpha} + 2\kappa K n_0. \end{aligned}$$

Now first fixing κ and sending ε to 0 we get

$$\frac{C}{2} \le 8F\hat{C}\sqrt{\kappa} + 4L\kappa K + \frac{8\kappa K n_0}{\alpha} + 2\kappa K n_0.$$

Now we can choose κ so that the RHS of the above expression is strictly less than $\frac{C}{4}$ and hence we will get $\frac{C}{2} \le \frac{C}{4}$. This is a contradiction; hence, $\sup_i \sup_{\Omega_i\times\Omega_i} \psi_n^i(x, y) \le 0$. This implies that

$$\sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x, y) \le \sup_i \sup_{\Omega_i\times\Omega_i} \psi_n^i(x, y) \le 0.$$

Thus in this case also we have $\sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x,y) \le 0$.

Case 4. Now consider the last case where one of the x_0 or y_0 is in A. Without loss of generality we assume that $y_0 \in A$.

LEMMA 5.6. *Let Φ be as defined by* (5.1) *and let (x_0, y_0) be as in* (5.24), *that is, $\Phi^1(x_0,y_0) = \sup_{\Omega_1\times\Omega_1}\Phi^1$. Moreover, let y_0 be such that $u_2(y_0) = Mu_2(y_0) = u_2(g(y_0,v_0)) + c_a(y_0,v_0)$, where $g(y_0,v_0) \in \Omega_2$. Then*

$$\Phi^1(x_0,y_0) - \Phi^2(g(y_0,v_0), g(y_0,v_0)) < \kappa K$$

for some constant $K > 1$ depending only on the constants of the problem and independent of ϵ and κ.

Proof.

$$\begin{aligned}\Phi^1(x_0,y_0) - \Phi^2(g(y_0,v_0),(g(y_0,v_0)) &= u_1(x_0) - u_2(y_0) - \frac{1}{\epsilon}|x_0-y_0|^2 - \kappa(|x_0|^2+|y_0|^2)\\ &\quad - u_1(g(y_0,v_0)) + u_2(g(y_0,v_0)) + 2\kappa|g(y_0,v_0)|^2\\ &= u_1(x_0) - c_a(y_0,v_0) - \frac{1}{\epsilon}|x_0-y_0|^2\\ &\quad - \kappa(|x_0|^2+|y_0|^2) - u_1(g(y_0,v_0)) + 2\kappa|g(y_0,v_0)|^2.\end{aligned}$$

We add and subtract $u_1(y_0)$ in the above, and observing that $u_1(y_0) \le Mu_1(y_0) \le u_1(g(y_0,v_0)) + c_a(y_0,v_0)$, we get

$$\begin{aligned}\Phi^1(x_0,y_0) - \Phi^2(g(y_0,v_0), g(y_0,v_0)) &\le u_1(x_0) - u_1(y_0) - c_a(y_0,v_0)\\ &\quad - u_1(g(y_0,v_0)) + u_1(y_0) + 2\kappa|g(y_0,v_0)|^2\\ &\le u_1(x_0) - u_1(y_0) + 2\kappa|g(y_0,v_0)|^2\\ &\le \omega_\kappa^1(|x_0-y_0|) + 2\kappa R^2.\end{aligned}$$

We can choose ϵ such that $\omega_\kappa^1(\sqrt{C\epsilon}) < \kappa$. Then by the Lemma 5.2,

$$\begin{aligned}&\omega_\kappa^1(|x_0-y_0|) \le \omega_\kappa^1\big(\sqrt{C\epsilon}\big) < \kappa\\ &\quad\Rightarrow \Phi^1(x_0,y_0) - \Phi^2(g(y_0,v_0), g(y_0,v_0)) \le K\kappa,\end{aligned}$$

where K depends on the modulus of continuity of u_1 and R. This proves the lemma. □

To proceed, if necessary, we restrict $\alpha_2 < \frac{\beta}{2}$, where α_2 is as in Lemma 5.3 and define a C_0^∞ function ζ_1 on $\Omega\times\Omega$ by

$$\zeta_1(g(y_0,v_0), g(y_0,v_0)) = 1; \quad 0 \le \zeta_1 \le 1; \quad |D\zeta_1| < \frac{2}{\alpha_2};$$

$$\zeta_1(x,y) < 1 \text{ if } (x,y) \ne (g(y_0,v_0), g(y_0,v_0));$$

$$\text{and} \quad \operatorname{supp} \zeta_1 \subseteq B((g(y_0,v_0), g(y_0,v_0)), \alpha_2).$$

Note that ζ_1 is nonzero only on $\Omega_2\times\Omega_2$ and it vanishes on all other $\Omega_i\times\Omega_i$. Define a new auxiliary function ψ_1 on $\Omega\times\Omega$ denoted by ψ_1^i on $\Omega_i\times\Omega_i$ such that

$$\begin{aligned}\psi_1^2(x,y) &= \Phi^i(x,y) + 2K\kappa\zeta_1(x,y),\\ \psi_1^i(x,y) &= \Phi^i(x,y) \quad \text{for } i \ne 2.\end{aligned}$$

Then arguing as in Case 3 we can conclude that ψ_1^2 attains its maximum in the α_2 ball around $(g(y_0, v_0), g(y_0, v_0))$. Let (x_1, y_1) be such that $\psi_1^2(x_1, y_1) = \sup_{\Omega_2\times\Omega_2} \psi_1^2$. Since $\alpha_2 < \frac{\beta}{2}$, $\quad x_1, y_1 \notin A$. Using techniques similar to those of Lemma 5.2 we can prove that

$$\begin{aligned}\frac{|x_1 - y_1|^2}{\epsilon} &\leq \omega_\kappa^2\big(\sqrt{C\epsilon}\big) + 2K\kappa,\\ |x_1|,\ |y_1| &< \hat{C}\sqrt{\kappa}.\end{aligned}$$

Now either $(x_1, y_1) \in \Omega \setminus (A \cup C) \times \Omega \setminus (A \cup C)$ or one of x_1 or $y_1 \in C$. In both cases, we are either in Case 2 or in Case 3. Thus in any case, after finitely many steps, we will arrive at Case 2 and get that $\sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x, y) \leq 0$. This proves the claim in Step 1.

Step 2. In Step 2 we show the uniqueness. For any $x \in \Omega$,

$$u_1(x) - u_2(x) \leq \Phi(x, x) + 2\kappa|x|^2.$$

Sending κ to 0, we get

$$\begin{aligned}u_1(x) - u_2(x) &\leq \Phi(x, x)\\ &\leq \sup_i \sup_{\Omega_i\times\Omega_i} \Phi^i(x, y)\\ &\leq 0,\end{aligned}$$

where the last inequality follows by Step 1. Now interchanging the roles of u_1 and u_2, we get other way inequality, which proves that $u_1 = u_2$ for all $x \in \Omega$, and hence the uniqueness. □

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