A STRONG MAXIMUM PRINCIPLE FOR A CLASS OF NON-POSITONE SINGULAR ELLIPTIC PROBLEMS

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Abstract. We prove that nonnegative solutions of quasilinear elliptic problems of the type

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \quad , \quad 1 < p \leq 2 \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{0.1}$$

are actually positive in Ω, under the following assumptions: Ω is a regular bounded strictly convex domain in $\mathbb{R}^N$, $N \geq 2$, symmetric with respect to an hyperplane, f is a locally Lipschitz continuous function in $[0, +\infty)$ with $f(0) < 0$, and u is a weak solution in $C^1(\overline{\Omega})$. The proof of this result uses the moving plane method as in [2] and can be adapted to more general geometric situations.

1. Introduction

In this paper we study the following quasilinear elliptic problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega \\ u \geq 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

wher Ω is a bounded regular domain in $\mathbb{R}^N$, $N \geq 2$, f is a locally Lipschitz continuous function in $[0, +\infty)$ with $f(0) < 0$ and Δ_p denotes the p-Laplace operator div $(|Du|^{p-2}Du)$, $p > 1$.

In the case $p = 2$, Castro and Shivaji in the papers [2] and [3] put in evidence an interesting phenomenum related to (1.1): while in dimension $N = 1$ there exist solutions of (1.1) with interior zeros ([3]), when N is bigger than 1 and Ω is a ball then any solution of (1.1) is positive in Ω and hence radially symmetric by the Gidas-Ni-Nirenberg's theorem ([2], [7]). To prove this result they use the Alexandrov-Serrin moving

1991 *Mathematics Subject Classification.* 35B05,35B50,35J70.

* Supported by MURST, Project "Metodi Variazionali ed Equazioni Differenziali Non Lineari".

plane method ([9]) following the approach of Gidas, Ni and Nirenberg ([7]) and exploiting, therefore, the C^2 regularity of the solution. In [2] the proof is detailed for the case of a ball, while it is stated, at the end of the paper, that it works also for other bounded domains which satisfy some geometric conditions.

It is useful to observe that when $f(0) > 0$ then the usual strong maximum principle for the Laplace operator gives immediately the positivity of the solution even if f is only continuous. This is also true for solutions of (1.1), for every $p > 1$, as a consequence of the strong maximum principle for the p-laplacian (see [10], [11]).

Instead if $f(0) = 0$ and f is Lipschitz continuous in $[0, +\infty)$, from (1.1) we have

$$-\Delta_p u - (f(u) - f(0)) = 0$$

and hence, exploiting the Lipschitz continuity of f, it turns out that u satisfies a differential inequality of the type

$$-\Delta_p u + c(x) u \geq 0\ , \quad c \in L^\infty(\Omega) \tag{1.2}$$

Then, if $1 < p \leq 2$ the strong maximum principle gives the positivity of u in Ω (see [12], [8], [4]). Let us remark that this is no longer true when $p > 2$.

Thus it is a natural question to ask whether solutions of (1.1) are positive in Ω, when $f(0) < 0$, Ω is a ball in $\mathbb{R}^N$, $N \geq 2$ and $1 < p < 2$.

In this paper we answer affirmatively this question and actually prove that, for any domain Ω with a strictly convex outer boundary, there is a region $O \subseteq \Omega$ which is defined by the moving plane method (see the precise definition in section 2) such that $u > 0$ in O. Then it is trivial to see that O coincides with Ω whenever Ω is strictly convex and also symmetric with respect to an hyperplane.

Note that when $p \neq 2$ the solutions of (1.1) must be understood only in a weak sense since they belong to $C^1(\overline{\Omega})$ because the second derivatives of u may not exist in the points where Du vanishes.

Therefore, though we also use the moving plane method, we need to prove our result in a way different from that of [2], avoiding any argument which requires the C^2 regularity of the solution. In doing that we also simplify the proof for the case $p = 2$.

Our argument uses the moving plane method following the approach of Berestycki and Nirenberg ([1]) and essentially relies on the use of the comparison principle for solutions of differential inequalities in domains with small measure ([4]) and on the procedure used in [5] and [6] to derive the monotonicity and symmetry of solutions of nonlinear elliptic equations involving the p-laplacian.

Finally we also prove that, as for the case $p = 2$, a similar strong maximum princile cannot hold in dimension $N = 1$, for any $p > 1$, by constructing some counterexamples which generalize those of [3].

The outline of the paper is the following.

In Section 2 we give some notations and state the results while in Section 3 we present the proofs recalling some preliminaries. In Section 4 we describe a counterexample for the 1- dimensional case and in Section 5 we make some geometrical considerations.

2. Notations and statement of the results

Let Ω be a bounded, regular, domain in $\mathbb{R}^N$, $N \geq 2$, with a strictly convex outer boundary. For a direction ν in $\mathbb{R}^N$, i.e. a vector $\nu \in \mathbb{R}^N$ with $|\nu| = 1$, and $\lambda \in \mathbb{R}$, we define

$$a(\nu) = \inf_{x \in \Omega} x \cdot \nu \tag{2.1}$$

$$T_\lambda^\nu = \{x \in \mathbb{R}^N : x \cdot \nu = \lambda\} \tag{2.2}$$

$$\Omega_\lambda^\nu = \{x \in \Omega : x \cdot \nu < \lambda\} \tag{2.3}$$

$$x_\lambda^\nu = R_\lambda^\nu(x) = x + 2(\lambda - x \cdot \nu)\nu, \quad x \in \mathbb{R}^N \tag{2.4}$$

(i. e. R_λ^ν is the reflection through the hyperplane T_λ^ν). If $\lambda > a(\nu)$ then Ω_λ^ν is nonempty, thus we set

$$(\Omega_\lambda^\nu)' = R_\lambda^\nu(\Omega_\lambda^\nu) \tag{2.5}$$

Because of the regularity of Ω, we have that $(\Omega_\lambda^\nu)' \subseteq \Omega$ for λ close to $a(\nu)$ and bigger than $a(\nu)$. Then we set

$$\lambda_1(\nu) = \sup\{\lambda' > a(\nu) : (\Omega_\lambda^\nu)' \subset \Omega \text{ for every } \lambda \leq \lambda'\} \tag{2.6}$$

and it is easy to see that, by the strict convexity of the outer boundary of Ω, $\lambda_1(\nu)$ is characterized by the property of being the smallest value of λ such that at least one of the following occurs:

(i) $(\Omega_\lambda^\nu)'$ becomes internally tangent to $\partial\Omega$ at some point not on T_λ^ν

(ii) T_λ^ν is orthogonal to $\partial\Omega$ at some point

Note that, by the regularity of $\partial\Omega$, $a(\nu)$ is a continuous function of ν, while $\lambda_1(\nu)$ is a lower semicontinuous function.

Now we consider the set B given by the union of the maximal caps $\Omega_{\lambda_1(\nu)}^\nu$ and their reflections, namely

$$B = \bigcup_{\nu \in S^{N-1}} \left(\Omega_{\lambda_1(\nu)}^\nu \cup (\Omega_{\lambda_1(\nu)}^\nu)' \cup (T_{\lambda_1(\nu)}^\nu \cap \Omega)\right) \tag{2.7}$$

where S^{N-1} is the unit sphere in $\mathbb{R}^N$.

With these notations we can state our result.

THEOREM 2.1. *Let $u \in C^1(\overline{\Omega})$ be a (nonnegative) weak solution of (1.1). If Ω has a strictly convex outer boundary and $1 < p \leq 2$, then $u > 0$ in B.*

If Ω is strictly convex and also symmetric with respect to an hyperplane $T = T_{\lambda_0}^{\nu_0}$ orthogonal to a direction ν_0, then obviously $\lambda_0 = \lambda_1(\nu_0)$ i.e. $T = T_{\lambda_1(\nu_0)}^{\nu_0}$ and $B = \Omega$. Therefore we have the following

COROLLARY 2.1. *Let $u \in C^1(\overline{\Omega})$ be a nonnegative weak solution of (1.1). If $1 < p \leq 2$ and Ω is strictly convex and symmetric with respect to an hyperplane $T = T_{\lambda_0}^{\nu_0}$, then $u > 0$ in Ω. Moreover u is symmetric* *with respect to T, i. e. $u(x) = u(x_{\lambda_0}^{\nu_0})$, and strictly increasing in the ν_0-direction in the set $\Omega_{\lambda_0}^{\nu_0}$.*

Note that in the case of the previous corollary the symmetry of the solution follows from the Gidas-Ni-Nirenberg's theorem [7] when $p = 2$ and by a theorem in [5] when $1 < p < 2$ (see also [6] for the strict monotonicity of the solution).

From the definition of B it is clear that the positivity of u can be deduced also for domains Ω which are not strictly convex but have other geometrical properties, such as an annulus. We postpone the discussion of this to Section 5.

3. Preliminaries and proofs

We start by recalling a version of the strong maximum principle and of the Hopf's lemma for the p-laplacian. It is a particular case of a result proved in [12].

THEOREM 3.1. (Strong Maximum Principle and Hopf's Lemma) *Let Ω be a domain in $\mathbb{R}^N$ and suppose that $u \in C^1(\Omega)$, $u \geq 0$ in Ω, weakly solves*

$$-\Delta_p u + cu^q = g \geq 0 \qquad \text{in } \Omega \tag{3.1}$$

with $1 < p < \infty$, $q \geq p-1$, $c \geq 0$ and $g \in L^\infty_{loc}(\Omega)$. If $u \not\equiv 0$ then $u > 0$ in Ω. Moreover for any point $x_0 \in \partial\Omega$ where the interior sphere condition is satisfied, and such that $u \in C^1(\Omega \cup \{x_0\})$ and $u(x_0) = 0$ we have that $\frac{\partial u}{\partial s} > 0$ for any inward directional derivative (this means that if y approaches x_0 in a ball $B \subseteq \Omega$ that has x_0 on its boundary then $\lim_{y \to x_0} \frac{u(y)-u(x_0)}{|y-x_0|} > 0$).

Next we recall some weak and strong comparison principles, whose proofs can be found in [4].

Let Ω be a domain in $\mathbb{R}^N$, let $f : \mathbb{R} \to \mathbb{R}$ be a locally Lipschitz continuous function and suppose that $u, v \in C^1(\overline{\Omega})$ weakly solve

$$\begin{cases} -\Delta_p u \le f(u) & \text{in } \Omega \\ -\Delta_p v \ge f(v) & \text{in } \Omega \end{cases} \tag{3.2}$$

For any set $A \subseteq \Omega$ we define

$$M_A = M_A(u,v) = \sup_A (|Du| + |Dv|) \tag{3.3}$$

and denote by $|A|$ its Lebesgue measure.

THEOREM 3.2. (Weak Comparison Principle) *Suppose that $1 < p < 2$ and Ω is bounded. Then there exist α, $M > 0$, depending on p, $|\Omega|$, M_Ω and the L^∞ norms of u and v such that: if an open set $\Omega' \subseteq \Omega$ satisfies $\Omega' = A_1 \cup A_2$, $|A_1 \cap A_2| = 0$, $|A_1| < \alpha$, $M_{A_2} < M$ then $u \le v$ on $\partial\Omega'$ implies $u \le v$ in Ω'.*

Remark 3.1. The previous comparison principle is a stronger version of the better known comparison (or maximum) principle for the Laplace operator in domains with small measure which would assert that if $|\Omega'| < \alpha$ ($\alpha > 0$ depending on $|\Omega|$) then $u \le v$ on $\partial\Omega'$ implies $u \le v$ in Ω'. It is important to stress that Theorem 3.2 does not hold for $p = 2$ (see [4] for further remarks).

THEOREM 3.3. (Strong Comparison Principle) *Suppose that $1 < p < \infty$ and define $Z_u^v = \{x \in \Omega : Du(x) = Dv(x) = 0\}$. If $u \le v$ in Ω and there exists $x_0 \in \Omega \setminus Z_u^v$ with $u(x_0) = v(x_0)$, then $u \equiv v$ in the connected component of $\Omega \setminus Z_u^v$ containing x_0.*

Now we prove the following lemma.

LEMMA 3.1. *Let* $u \in C^1(\overline{\Omega})$ *be a (nonnegative) weak solution of* (1.1) *and assume that* Ω *has a strictly convex outer boundary* Γ *and* $1 < p \leq 2$. *Then there exists a neighborhood* N_δ *of* Γ, $N_\delta = \{x : \ dist\ (x, \Gamma) < \delta\}$, $\delta > 0$, *such that* u *is positive in* $N_\delta \cap \Omega$.

Proof. Let P be any point on Γ and denote by ν_P the inner normal to $\partial\Omega$ at P. Because of the strict convexity of Γ we know that the map $P \mapsto \nu_P \in S^{N-1}$ is bijective, i. e. any direction in $\mathbb{R}^N$ is the inner normal to $\partial\Omega$ at exactly one point of Γ. Moreover it is obvious that the hyperplane $T_{a_{\nu_P}} = \{x : x \cdot \nu_P = a_{\nu_P}\}$ (a_{ν_P} defined as in Section 2) is precisely the hyperplane tangent to $\partial\Omega$ at P.

We claim that there exists $\delta_1 > 0$ such that for all $P \in \Gamma$ the reflected cap $(\Omega_{\mu_1}^{\nu_P})'$, $\mu_1 = a_{\nu_P} + \delta_1$, is contained in Ω.

In fact, if this is not true then there exist sequences $\delta_n \to 0$ and $P_n \in \Gamma$ such that $(\Omega_{\mu_n}^{\nu_{P_n}})'$, $\mu_n = a_{\nu_{P_n}} + \delta_n$, is not contained in Ω. By i) and ii) of the previous section (see also [7]) this implies that either $(\Omega_{\mu_n}^{\nu_{P_n}})'$ is internally tangent to Ω at some point not on $T_{\mu_n}^{\nu_{P_n}}$ or $T_{\mu_n}^{\nu_{P_n}}$ is orthogonal to Γ at some point. Passing to the limit, up to a subsequence, and using the continuity of a_ν and the strict convexity of Γ, we get a point $\overline{P} \in \Gamma$ such that either $T_{a_{\nu_{\overline{P}}}}$ is orthogonal to Γ at some point or none of the caps $(\Omega_{\lambda_\varepsilon}^{\nu_{\overline{P}}})'$, $\lambda_\varepsilon = a_{\nu_{\overline{P}}} + \varepsilon$, are contained in Ω, for any $\varepsilon > 0$. Both situations contradict the strict convexity of Γ.

Now by Theorem 3.2, for the case $1 < p < 2$, or by the weak comparison principle in domains with small measure for the Laplace operator (see [1]) we know that there exists $\alpha > 0$ (depending only on the data of the problem) such that if $\Omega' \subset \Omega$ and $|\Omega'| > \alpha$, then the weak comparison principle holds, for solutions of (1.1) in Ω'. Because Ω is bounded we can choose a number $\delta_2 > 0$ such that for any $P \in \Gamma$ the cap $\Omega_{\mu_2}^{\nu_P}$, $\mu_2 = a_{\nu_P} + \delta_2$, has measure smaller than α. Let us denote by δ the minimum between δ_1 and δ_2 and fix a point $P \in \Gamma$. For simplicity we denote by ν the inner normal at P, instead of ν_P. Then we consider in Ω^ν_λ, $a(\nu) < \lambda < a(\nu) + \delta$, the function $u^\nu_\lambda(x) = u(x^\nu_\lambda)$, which satisfies in Ω^ν_λ the same equation as u. On $\partial\Omega^\nu_\lambda$ the inequality $u \le u^\nu_\lambda$ holds, since $0 = u \le u^\nu_\lambda$ on $\partial\Omega \cap \partial\Omega^\nu_\lambda$, while $u = u^\nu_\lambda$ on $\partial\Omega^\nu_\lambda \cap T^\nu_\lambda$, by definition. Therefore, since for $a(\nu) < \lambda < a(\nu) + \delta$ the cap Ω^ν_λ has measure smaller than α, by the weak comparison principle we get $u \le u^\nu_\lambda$ in Ω^ν_λ. Doing the same for all point $P \in \Gamma$ we get that u is nondecreasing in the direction ν_P in every cap $\Omega^{\nu_P}_{a_{\nu_P}+\delta}$.

We claim that $u > 0$ in $\bigcup_{P\in\Gamma} \Omega^{\nu_P}_{a_{\nu_P}+\delta}$, which coincides with $N_\delta \cap \Omega$ by the strict convexity of the domain. In fact, if this is not true there exists a point $\overline{x}$ in some cap $\Omega^{\nu_{\overline{P}}}_{a_{\nu_{\overline{P}}}+\delta}$ such that $u(\overline{x}) = 0$. Then, by the monotonicity of u in the $\nu_{\overline{P}}$- direction, u would be zero on the segment, parallel to $\nu_{\overline{P}}$, connecting $\overline{x}$ to Γ. Then $\overline{x}$ would be interior to $\Omega^{\nu_P}_{a_{\nu_P}+\delta}$ for any P close to $\overline{P}$ and hence, by the monotonicity in the ν_P-direction, u would vanish on all the segments, parallel to ν_P, connecting $\overline{x}$ to Γ. The union of all these segments gives a cone with nonempty interior and vertex in $\overline{x}$, where $u \equiv 0$. This is impossible because $f(0) < 0$. $\square$

Now we prove Theorem 2.1.

Proof of Theorem 2.1. For any direction ν, we define

$$\Lambda_0(\nu) = \left\{\lambda \in (a(\nu), \lambda_1(\nu)] : u \le u^\nu_\mu \text{ in } \Omega^\nu_\mu\,,\ \text{for any } \mu \in (a(\nu), \lambda]\right\}$$

By the proof of the previous lemma we know that $\Lambda_0(\nu)$ is nonempty and hence we define

$$\lambda_0(\nu) = \sup \Lambda_0(\nu)$$

The assertion will be proved if we show that $\lambda_0(\nu) = \lambda_1(\nu)$. In fact, if this happens, u will be nondecreasing in the ν- direction in the whole cap $\Omega^\nu_{\lambda_1(\nu)}$ and hence positive, because in the previous lemma we have already proved that $u > 0$ near $\partial\Omega$. Moreover u will also be positive in $(\Omega^\nu_{\lambda_1(\nu)})'$ by the inequality $0 < u \le u^\nu_{\lambda_1(\nu)}$ in $\Omega^\nu_{\lambda_1(\nu)}$.

To prove that $\lambda_0(\nu) = \lambda_1(\nu)$ we argue by contradiction and assume that $\lambda_0(\nu) < \lambda_1(\nu)$. By continuity, in the maximal cap $\Omega^\nu_{\lambda_0(\nu)}$ the inequality $u \leq u^{\nu}_{\lambda_0(\nu)}$ holds. On the other hand $u \not\equiv u^{\nu}_{\lambda_0(\nu)}$ in $\Omega^{\nu}_{\lambda_0(\nu)}$ and this is the point where we exploit the fact that the dimension is strictly greater than 1. In fact by the strict convexity of Γ and the inequality $\lambda_0(\nu) < \lambda_1(\nu)$, the reflection of $\Gamma \cap \partial\Omega^{\nu}_{\lambda_0(\nu)}$ will lie inside Ω, and there will be points $x^{\nu}_{\lambda_0(\nu)}$ on it, $x^{\nu}_{\lambda_0(\nu)}$ being the reflection of some point $x \in \Gamma \cap \partial\Omega^{\nu}_{\lambda_0(\nu)}$, sufficiently close to $\partial\Omega$, where $u(x^{\nu}_{\lambda_0(\nu)}) > 0$ by Lemma 3.1, while $u(x) = 0$, since $x \in \partial\Omega$.

Then, in the case $p = 2$, by the usual strong comparison principle for the Laplace operator we get $u < u^{\nu}_{\lambda_0(\nu)}$ in $\Omega^{\nu}_{\lambda_0(\nu)}$ and then, arguing exactly as in [1], it is possible to prove that $u \leq u^{\nu}_{\lambda'}$ for $\lambda' > \lambda_0(\nu)$ close to $\lambda_0(\nu)$, contradicting the definition of $\lambda_0(\nu)$.

In the case $p < 2$, even if $u \not\equiv u^{\nu}_{\lambda_0(\nu)}$ in $\Omega^{\nu}_{\lambda_0(\nu)}$, Theorem 3.3 does not exclude the existence of connected components C^{ν} of $\Omega^{\nu}_{\lambda_0(\nu)} \setminus Z^{\nu}_{\lambda_0(\nu)}$ ($Z^{\nu}_{\lambda_0(\nu)} = \{x \in \Omega^{\nu}_{\lambda_0(\nu)} : Du(x) = Du^{\nu}_{\lambda_0(\nu)}(x) = 0\}$) where $u \equiv u^{\nu}_{\lambda_0(\nu)}$.

Then, arguing exactly as in the proof of Theorem 1.1 of [5] we exclude the presence of these components C^{ν} and hence, by Theorem 3.3, we get $u < u^{\nu}_{\lambda_0(\nu)}$ in $\Omega^{\nu}_{\lambda_0(\nu)} \setminus Z^{\nu}_{\lambda_0(\nu)}$. Then, the same proof as Theorem 3.1 of [5] shows (as for the case $p = 2$) that $u \leq u^{\nu}_{\lambda'}$, for $\lambda' > \lambda_0(\nu)$ close to $\lambda_0(\nu)$, contradicting the definition of $\lambda_0(\nu)$. $\square$

4. A counterexample in dimension $N = 1$

In this section we consider the one-dimensional problem

$$
\begin{cases} -\left(|u'|^{p-2}u'\right)' = f(u) & \text{in } I \quad , \quad p > 1 \\ u = 0 & \text{on } \partial I \end{cases} \tag{4.1}
$$

where I is a bounded interval in $\mathbb{R}$ and give some example of nonnegative C^1 weak solutions of (4.1) which vanish in some interior point.

Let $f : [0, \infty) \to \mathbb{R}$ be a continous function such that there exists $\beta > 0$ satisfying

i) $f(s) < 0 \;\; \forall\, s \in [0, \beta)$ and $f(\beta) = 0$

ii) $f(s) > 0 \;\; \forall\, s \in (\beta, \infty)$

iii) $\int_\beta^\infty f(s)\,ds = +\infty$

We consider the primitive of f: $F(s) = \int_0^s f(t)\,dt$. By the hypotheses on f we get the existence of a number $\vartheta > \beta$ such that

iv) $F(0) = F(\vartheta) = 0$

v) $F(s) < 0 \quad \forall\, s \in (0, \vartheta)$

It is important to stress that the integral $\int_0^\vartheta \frac{du}{[-F(u)]^{\frac{1}{p}}}$ converges for every $p > 1$. In fact, by the continuity of f, there exists $s_0 > 0$ such that $f(s) \le \frac{1}{2}f(0) < 0$ $\forall s \le s_0$, and this implies that $-F(u) = \int_0^u -f(s)\,ds \ge -\frac{1}{2}f(0)u$ from which we get $\frac{1}{[-F(u)]^{\frac{1}{p}}} \le \frac{C}{u^{\frac{1}{p}}}$ for $u < s_0$. Analogously, since $f(\vartheta) > 0$ there exists $\vartheta_0 < \vartheta$ such that $f(s) \ge \frac{1}{2}f(\vartheta) > 0$ $\forall s \in (\vartheta_0, \vartheta)$. Moreover we can write $F(u) = -\int_u^\vartheta f(t)\,dt$, because $F(\vartheta) = 0$. Hence for $u \in (\vartheta_0, \vartheta)$ we have $-F(u) = \int_u^\vartheta f(t)\,dt \ge \frac{1}{2}f(\vartheta)(\vartheta - u)$ from which we get $\frac{1}{[-F(u)]^{\frac{1}{p}}} \le \frac{C}{(\vartheta-u)^{\frac{1}{p}}}$ for $u \in (\vartheta_0, \vartheta)$.

Let us define

$$L = \int_0^\vartheta \frac{ds}{[-\frac{p}{p-1}F(s)]^{\frac{1}{p}}} \tag{4.2}$$

and consider the function

$$x = x(u) = \int_0^u \frac{ds}{[-\frac{p}{p-1}F(s)]^{\frac{1}{p}}} \quad , \quad 0 < u < \vartheta \tag{4.3}$$

which is a C^1 function with positive derivative $(-\frac{p}{p-1}F(u))^{\frac{-1}{p}}$ and it is continuous in $[0, \vartheta]$ with $x(0) = 0$, $x(\vartheta) = L$. We denote by $u : [0, L] \to [0, \vartheta]$ the inverse function which belongs to $C^2((0, L))$ with

$$u'(x) = (-\frac{p}{p-1}F(u(x)))^{\frac{1}{p}} \quad , \quad x \in (0, L) \tag{4.4}$$

It is easy to see that u can be extended to a $C^1([0, L])$ function defining $u(0) = 0$, $u(L) = \vartheta$, $u'(0) = u'(L) = 0$ and that u satisfies in $(0, L)$ the equation

$$-(|u'|^{p-2}u')' = f(u)$$

in the classical sense. Since $u'(L) = 0$, reflecting u about the point L, i. e. defining $u(x) = u(2L - x)$ for $x \in (L, 2L)$, we get a C^1 weak solution in $(0, 2L)$ of the equation, with $u'(0) = u'(2L) = 0$. Finally reflecting again about 0 we get a C^1 nonnegative weak solution of (4.1) in $I = (-2L, 2L)$ with $u(0) = 0$.

5. **Some geometrical remarks**

As we have seen in section 2, for any bounded domain Ω the moving plane method defines naturally a set inside Ω which is given by all points of Ω which belong to some maximal cap $\Omega^{\nu}_{\lambda_1(\nu)}$ or to its reflection. Namely this is the set B defined in (2.7).

Of course this set can be defined even if the outer boundary of Ω is not strictlyy convex or $\partial\Omega$ is not smooth.

The importance of this set in the study of of solutions of differential problems in Ω is enlighted by Theorem 2.1, but we think that there are other qualitative properties of solutions, such as the study of the location of the critical points of solutions, which could be related to the set B. Therefore it is interesting to see what geometrical properties Ω must have to get that the set B coincides with Ω.

Let us immediately observe that if Ω is simply connected but not convex then it is easy to construct examples of domains Ω for which $\overline{B} \subset \Omega$. One of these could be a deformation of a dumb-bell, as in figure 5.1 .

Figure 5.1

On the other hand for any domain Ω, convex in the x_1-direction and symmetric with respect to the hyperplane $x_1 = 0$, we have that $B = \Omega$. Of course such a domain need not be convex.

Thus a natural and important question is whether $\Omega = B$ for any convex set Ω. In virtue of the properties of this kind of sets we believe that the answer should be positive, but we have not been able to get a proof so far.

Finally it is interesting to observe that if Ω is not simply connected, i.e. very far from being convex, than the equality $\Omega = B$ can still hold as for the case of an annulus. Again it is easy to construct examples of non-simply connected domains with strictly convex outer boundary, but nonconvex inner boundary, for which $\overline{B} \subset \Omega$. One example could be a domain obtained by removing from a ball a region with the shape of the set in figure 5.1 .

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