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Some elliptic semilinear indefinite problems on $\mathbb{R}^N$

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This paper deals with the existence and the behaviour of global connected branches of positive solutions of the problem

$$-\Delta u = \lambda f(x, u), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N).$$

The function f is allowed to change sign and has an asymptotically linear or a superlinear behaviour.

1. Introduction

In this paper we discuss the existence of positive solutions to the problem

$$-\Delta u = \lambda(a(x)u + b(x)r(u)), \quad u \in \mathcal{D}^{1,2}(\mathbb{R}^N), \tag{1.1}$$

where $N \geqslant 3$ and $\mathcal{D}^{1,2}(\mathbb{R}^N)$ is the completion of $\mathcal{C}_0^\infty(\mathbb{R}^N)$ under the norm

$$\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^{1/2}.$$

For the nonlinearity r, we assume the following.

(H1) We have

(i) $r \in C^0(\mathbb{R})$, $\lim_{s\to 0}(r(s)/s) = 0$;

(ii) $sr(s) \geqslant 0$ $\forall s \in \mathbb{R}$ and r is odd.

(H2) We have

(AL) $\lim_{s\to\infty}(r(s)/s) = r_\infty < \infty$ (*asymptotically linear case*); or

(SL) $r(s) = s^p$ for all $s \geqslant 0$ with $1 < p < (N+2)/(N-2)$ (*superlinear case*).

The weight functions can change sign and satisfy the following conditions.

(H3) $a \in \mathrm{L}^{N/2}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$.

(H4) We have

(AL) $b \in \mathrm{L}^{N/2}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$;
(SL) $b \in \mathrm{L}^{\beta_p}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$ with $\beta_p := 2^*/(2^* - p - 1)$.

(H5) We have

(i) $\operatorname{supp} a^+$ and $\operatorname{supp} b^+$ are compact;
(ii) $(\operatorname{supp} a^+ \cap \operatorname{supp} b^+)^0 \neq \emptyset$.

In addition, we will also use the following assumption in the superlinear case.

(H6) $b \in C^1(\mathbb{R}^N)$, and $x \in \Gamma := \{x \in \mathbb{R}^N : b(x) = 0\}$ implies that $\nabla b(x) \neq 0$. Furthermore, a is C^1 in a neighbourhood of Γ and at each point of Γ, a is decreasing in the outward normal direction.

Concerning the results in the framework of $\mathrm{L}^\infty(\mathbb{R}^N)$, we assume instead of (H4)(SL) and in addition to (H6), the following.

(H7) $\lim_{|x|\to+\infty} b(x) = -\infty$.

Since the assumption (H1) implies $r(0) = 0$, the function $u \equiv 0$ is a solution of problem (1.1) for all $\lambda \in \mathbb{R}$. We are thus interested in the existence of bifurcating branches from the set of trivial solutions $\{(\lambda, 0)\}$.

Remark 1.1. Hypothesis (H5) implies, in particular, that $a^+, b^+ \not\equiv 0$.
When $a, b \in C^0(\mathbb{R}^N)$, hypothesis (H5) is equivalent to the existence of $x \in \mathbb{R}^N$ such that $a(x), b(x) > 0$.

Example 1.2. A typical example of a nonlinearity r satisfying (H2) for the asymptotically linear case (AL) is given by $r(s) = (|s|/(1+|s|))s$.

On bounded domains, the semilinear problem with superlinear indefinite nonlinearities have been studied by Alama and Tarantello [2,3] and Ramos *et al.* [31] with the variational point of view. In the papers of Hess [23], Ambrosetti and Hess [5] and Hess and Kato [24], the authors investigate on bounded domains the existence of global branches using topological-degree arguments for autonomous nonlinearities that are allowed to change sign.

Global bifurcation for semilinear problems on $\mathbb{R}^N$ with indefinite and superlinear nonlinearity has been investigated, for example, in [9, 16] for $N \geqslant 3$ and in [1] for $N = 2$. In these papers, the authors study a local problem on a ball of radius R for which the existence of a global branch is proved. Then, using *a priori* estimates, they show that when $R \to \infty$, the branch converges to a branch for the initial problem set on $\mathbb{R}^N$. Recently, via variational methods, the existence of solutions to asymptotically linear problems with positive nonlinearities has been also studied on

$\mathbb{R}^N$ in [26,28,33]. But this approach does not seem suitable to treat indefinite nonlinearities. Moreover, the variational approach, in general, does not give the branch. Our approach here will be to use topological-degree arguments (more precisely, the global bifurcation theory of Rabinowitz) to get global branches, extending the earlier work in [5, 23, 24] to the case of unbounded domains. When this bifurcation theory cannot be used directly, we get the global branches by approaching the problem in the whole space by the sequence of problems posed in expanding balls.

As it is now well known that there are no positive solutions in $\mathbb{R}^N$ to the equation

$$-\Delta u = u^p, \quad 1 < p < \frac{N+2}{N-2}$$

(see [21]), it is natural to explore the solutions for the nonlinearity $b(x)u^p$, for example. We see, in contrast, the existence of a branch of positive solutions, as shown in our theorems.

One of the physical motivations for considering asymptotically linear problems arises from the study of guided modes of an electromagnetic field in a nonlinear medium, satisfying some suitable constitutive assumptions (see, for example, [32]). For example, positive nonlinearities of the form

$$r(s) = \frac{|s|^2}{1+\gamma|s|^2}s, \quad \gamma > 0, \tag{1.2}$$

were found to describe the variation of the dielectric constant of gas vapours where a laser beam propagates, and those of the form

$$r(s) = \left(1 - \frac{1}{\mathrm{e}^{\gamma|s|^2}}\right)s \tag{1.3}$$

were used in the context of laser beams in plasma (see [34] and the references therein).

Moreover, motivation for considering indefinite functions, also superlinear, arises, for example, from some selection–migration models in population genetics, where nonlinear problems of the kind

$$-\Delta u = \lambda a(x)u(1-u) \tag{1.4}$$

are considered, with a function a changing sign (see [11,20]).

In order to state our results, let us introduce some notation. The completion of $C_0^\infty(\Omega)$ with respect to the norm $(\int_\Omega |\nabla u|^2)^{1/2}$ is denoted by $\mathcal{D}_0^{1,2}(\Omega)$ and $\mathcal{D}^{1,2}(\mathbb{R}^N) := \mathcal{D}_0^{1,2}(\mathbb{R}^N)$, $\Pi_\mathbb{R}$ will mean the projection of $\mathbb{R}\times\mathcal{D}^{1,2}(\mathbb{R}^N)$ to $\mathbb{R}$, the positive and negative principle eigenvalues of the problem $-\Delta u = \lambda a(x)u$, $u \in \mathcal{D}_0^{1,2}(\Omega)$, will be denoted by $\lambda_1^+(a,\Omega), \lambda_1^-(a,\Omega)$ and, finally,

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R}\times\mathcal{D}^{1,2}(\mathbb{R}^N) : (\lambda, u) \text{ solution of (1.1)}, u \not\equiv 0\}. \tag{1.5}$$

Our main results are as follows.

THEOREM 1.3 (asymptotically linear case). *Assume that (H1), (H2)(AL), (H3) and (H4)(AL) hold and that $a^+ \not\equiv 0$ (respectively, $a^- \not\equiv 0$). Then there exists a global branch $\mathcal{C}^+$ (respectively, $\mathcal{C}^-$), connected in $\bar{\mathcal{S}}$, bifurcating from $\lambda_1^+(a,\Omega)$ (respectively, $\lambda_1^-(a,\Omega)$) and $\mathcal{C}^+\cap\mathcal{C}^- = \emptyset$.*

By further assuming (H5), the following holds for the branch $\mathcal{C}^+$.

(i) $\forall(\lambda, u) \in \mathcal{C}^+$, *we have* $|u| > 0$ *or* $u \equiv 0$ *and* $\mathcal{C}^+$ *is unbounded.*

(ii) $\Pi_{\mathbb{R}}\mathcal{C}^+ \subseteq (0, \Lambda^+]$, *where* $\Lambda^+ := \lambda_1^+(a, (\operatorname{supp} b^+)^0)$.

(iii) *Setting* $\lambda_\infty^+ := \lambda_1^+(a + br_\infty, \mathbb{R}^N)$, *we have that* $\mathcal{C}^+$ *bifurcates from infinity at* λ_∞^+ *and, moreover,*

$$(\lambda_n, u_n) \in \mathcal{C}^+, \quad \text{with } \|u_n\|_{\mathcal{D}^{1,2}} \to \infty \quad \Rightarrow \quad \lambda_n \to \lambda_\infty^+.$$

For the superlinear case, we have the following result.

THEOREM 1.4 (superlinear case). *Assume that (H1), (H2)(SL), (H3) and (H4)(SL) hold and that* $a^+ \not\equiv 0$ (*respectively,* $a^- \not\equiv 0$). *Then there exists a global branch* $\mathcal{C}^+$ (*respectively,* $\mathcal{C}^-$), *connected in* $\bar{\mathcal{S}}$, *bifurcating from* $\lambda_1^+(a, \Omega)$ (*respectively,* $\lambda_1^-(a, \Omega)$) *and* $\mathcal{C}^+ \cap \mathcal{C}^- = \emptyset$.

By further assuming (H5), the following holds for the branch $\mathcal{C}^+$.

(i) $\forall(\lambda, u) \in \mathcal{C}^+$, *we have* $|u| > 0$ *or* $u \equiv 0$ *and* $\mathcal{C}^+$ *is unbounded.*

(ii) $\Pi_{\mathbb{R}}\mathcal{C}^+ \subseteq (0, \Lambda^+]$, *where* $\Lambda^+ := \lambda_1^+(a, (\operatorname{supp} b^+)^0)$.

Supposing further that (H6) holds, we also have

(iii) $(\lambda_n, u_n) \in \mathcal{C}^+$ *and* $\|u_n\|_{\mathcal{D}^{1,2}(R^N)} \to +\infty \Leftrightarrow \lambda_n \to 0^+$.

Now, working in $\mathrm{L}^\infty(\mathbb{R}^N)$, we have the following result for the problem (1.1).

THEOREM 1.5 (superlinear case). *Assume that (H1), (H2)(SL), (H3), (H5), (H6) and (H7) are satisfied. Then there exists a global branch,* $\mathcal{C}^+$, *belonging to* $\bar{\mathcal{S}}$ *and connected in* $\mathbb{R} \times \mathrm{L}^\infty(\mathbb{R}^N)$, *such that the following hold.*

(i) $\mathcal{C}^+$ *bifurcates from* $\lambda_1^+(a, \mathbb{R}^N)$.

(ii) $\Pi_{\mathbb{R}}\mathcal{C}^+ \subseteq (0, \lambda_1^+(a, \operatorname{supp} b^+)]$.

(iii) *If* $(\lambda_n, u_n) \in \mathcal{C}^+$ *and* $\|u_n\|_{\mathcal{D}^{1,2}(R^N)} \to +\infty$ *or* $\|u_n\|_{\mathcal{C}_o(\mathbb{R}^N)} \to +\infty$, *then we have* $\lambda_n \to 0^+$.

The proof of theorem 1.5 is not a straightforward application of Rabinowitz global bifurcation theory, since the nonlinear operator is not compact in the framework of $\mathrm{L}^\infty(\mathbb{R}^N)$. To overcome this difficulty, we study first the problem in $B_R(0)$ and pass to the limit when $R \to +\infty$. This is done in the last section.

The paper is organized as follows.

In § 2, after setting up the functional framework, we recall some results about the principal eigenvalues, $\lambda_1^+(w, \Omega)$ and $\lambda_1^-(w, \Omega)$, of the linear problem with a sign-changing weight w,

$$-\Delta u = \lambda w(x) u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \quad N \geqslant 3. \tag{1.6}$$

In § 3, we prove the existence of at least one globally connected branch of solutions $\mathcal{C}^+$ in $\bar{\mathcal{S}}$. In § 4, we prove that the solutions of problem (1.1) belonging to the branch

$\mathcal{C}^+$ do not change sign. As a consequence, we get that $\mathcal{C}^+$ has to be unbounded. In § 5, we deal with the asymptotically linear case and prove that the branches $\mathcal{C}^+$ bifurcate from infinity at some value λ_∞^+. In the next section, we get suitable *a priori* estimate in $\mathcal{D}^{1,2}$ for the superlinear case to ascertain the global behaviour of the branch $\mathcal{C}^+$ in theorems 1.4. Finally, in the last section, we prove theorem 1.5, working in $L^\infty(\mathbb{R}^N)$.

2. Preliminaries

Let Ω be an open subset of $\mathbb{R}^N$ with $N \geqslant 3$ and let $\mathcal{C}_0^\infty(\Omega)$ be endowed with the scalar product

$$\langle u, v\rangle := \int_\Omega \nabla u \nabla v.$$

The completion of this space will be denoted by $\mathcal{D}_0^{1,2}(\Omega)$. When $\Omega = \mathbb{R}^N$, we simply write $\mathcal{D}^{1,2} = \mathcal{D}_0^{1,2}(\mathbb{R}^N)$ and denote by $\|\cdot\|$ the associated norm.

We recall the following facts:

$$\mathcal{D}^{1,2} \hookrightarrow \mathrm{L}^{2^*}(\mathbb{R}^N) \qquad \text{(continuously)}, \tag{2.1}$$

$$\mathcal{D}^{1,2} \hookrightarrow \mathrm{L}^p_{\mathrm{loc}}(\mathbb{R}^N) \quad \forall p \in [1, 2^*) \quad \text{(continuous and compact)}. \tag{2.2}$$

Using Riesz's theorem, we can show that the mapping

$$\mathcal{D}^{1,2} \to \mathrm{L}^{(2^*)'}(\mathbb{R}^N) = \mathrm{L}^{2N/(N+2)}(\mathbb{R}^N), \quad u \mapsto -\Delta u, \tag{2.3}$$

is one to one with continuous inverse.

Definition 2.1. We say that $\lambda \in \mathbb{R}$ is a principal eigenvalue for the problem

$$-\Delta u = \lambda w(x) u, \quad u \in \mathcal{D}_0^{1,2}(\Omega), \tag{2.4}$$

if there exists $u \in \mathcal{D}_0^{1,2}(\Omega)$ such that $u > 0$ and solves (2.4).

For a weight function w that changes sign, the problem of the existence of principal eigenvalue has been studied when Ω is bounded by Hess and Kato [24]. When $\Omega = \mathbb{R}^N$, sufficient conditions on w ensuring this existence of principal eigenvalues have been given by Brown *et al.* [12] and Allegretto [4]. In [35], Szulkin and Willem have studied the question on any Ω, with assumptions weaker than ours. Using their result (see [35, theorem 2.5]), we get the following.

Proposition 2.2. *Let Ω be an open subset of $\mathbb{R}^N$, $w \in L^{N/2}(\Omega)$ be such that $w^+ \not\equiv 0$ and consider*

$$\lambda_1^+(w, \Omega) := \inf_{\varphi \in \mathcal{D}_0^{1,2}(\Omega)} \left\{ \frac{\int_\Omega |\nabla \varphi|^2}{\int_\Omega w\varphi^2} : \int_\Omega w\varphi^2 > 0 \right\}. \tag{2.5}$$

Then $\lambda_1^+(w, \Omega)$ is a positive principal eigenvalue of problem (2.4).

If $w^- \not\equiv 0$, then, from the previous proposition applied to the weight function $-w$, we deduce that

$$\lambda_1^-(w, \Omega) := -\lambda_1(-w, \Omega)$$

is a negative principal eigenvalue of problem (2.4).

In the sequel, in order to get the existence of bifurcation branches, we shall know that the dimension of the eigenspace associated to the first principal eigenvalue is of dimension 1. Such a question, tightly related to the uniqueness of the principal eigenvalue, has been largely investigated in the literature on bounded or unbounded domains. On $\mathbb{R}^N$, which is the domain of our interest, we mention the works of [10,36].

PROPOSITION 2.3. *Let $w \in \mathrm{L}^{N/2}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$ be such that $w^+ \not\equiv 0$ and consider $\lambda_1^+(w,\mathbb{R}^N)$ defined by (2.5). Then we have the following.*

(i) *If (λ_2, ϕ_2) satisfies problem (2.4) with $\lambda_2 > \lambda_1$, then ϕ_2 changes sign. That is, $\lambda_1^+(w,\mathbb{R}^N)$ is the unique positive principal eigenvalue of (2.4).*

(ii) *$\lambda_1^+(w,\mathbb{R}^N)$ is a positive principal eigenvalue of multiplicity* 1.

Proof. For the sake of completeness, we sketch a proof following some arguments found in [10,18]. We write $\lambda_1 := \lambda_1^+(w,\mathbb{R}^N)$ and consider $\phi_1 > 0$ such that (λ_1, ϕ_1) solves (2.4).

From [22, proposition 8.17], we deduce that, for $i = 1, 2$ and $p > 1$, there exists a constant $C := C(R, \|w\|_{L^\infty}, p)$ such that

$$\sup_{B_R(y)} \phi_i \leqslant C\|\phi\|_{L^p(B_{2R}(y))}.$$

By choosing $p = 2^*$, we obtain $\phi_i \in L^\infty(\mathbb{R}^N)$.

Now, a direct calculation shows that

$$|\nabla\phi_1|^2 - \left\langle \nabla\phi_2, \nabla\left(\frac{\phi_1^2}{\phi_2+\epsilon}\right)\right\rangle = \left|\nabla\phi_1 - \left(\frac{\phi_1}{\phi_2+\epsilon}\right)\nabla\phi_2\right|^2. \tag{2.6}$$

Since $\phi_1/(\phi_2+\epsilon) \in L^\infty(\mathbb{R}^N)$, we get that $\phi_1^2/(\phi_2+\epsilon) \in \mathcal{D}^{1,2}$. Therefore, on the one hand, we are allowed to integrate (2.6) on $\mathbb{R}^N$ and, on the other hand, from the equation satisfied by ϕ_2, we have

$$\int_{\mathbb{R}^N} \left\langle \nabla\phi_2, \nabla\left(\frac{\phi_1^2}{\phi_2+\epsilon}\right)\right\rangle = \lambda_2 \int_{\mathbb{R}^N} w(x)\phi_2 \frac{\phi_1^2}{\phi_2+\epsilon}. \tag{2.7}$$

Hence, integrating (2.6) and using (2.7), we deduce that

$$\int_{\mathbb{R}^N} w(x)\phi_1^2\left(\lambda_1 - \lambda_2\frac{\phi_2}{\phi_2+\epsilon}\right) = \int_{\mathbb{R}^N}\left|\nabla\phi_1 - \left(\frac{\phi_1}{\phi_2+\epsilon}\right)\nabla\phi_2\right|^2$$

and, by letting $\epsilon \to 0$, we finally derive

$$(\lambda_1 - \lambda_2)\int_{\mathbb{R}^N} w(x)\phi_1^2 = \int_{\mathbb{R}^N}\left|\nabla\phi_1 - \left(\frac{\phi_1}{\phi_2}\right)\nabla\phi_2\right|^2. \tag{2.8}$$

Now, from the definition of λ_1, we note that $\lambda_2 \geqslant \lambda_1$ and $\int_{\mathbb{R}^N} w(x)\phi_1^2 > 0$. Thus

$$\lambda_1 = \lambda_2 \quad \text{and} \quad \nabla\phi_1 - \left(\frac{\phi_1}{\phi_2}\right)\nabla\phi_2 = 0.$$

But

$$\nabla\phi_1 - \left(\frac{\phi_1}{\phi_2}\right)\nabla\phi_2 = 0 \quad \Leftrightarrow \quad \nabla\left(\frac{\phi_1}{\phi_2}\right) = 0 \quad \Leftrightarrow \quad \phi_1 = C\phi_2$$

for some constant $C > 0$. □

REMARK 2.4. Note that, in [19], the result is extended to the p-Laplace operator and weight $w \in \mathrm{L}^{N/2}(\Omega) \cap \mathrm{L}^\infty(\Omega)$, which also allows them to conclude the regularity of the eigenfunction.

3. Global bifurcation

The existence of global branches of bifurcation for problem (1.1) is mainly based on Rabinowitz's theorem (see [30]) that we now recall.

THEOREM 3.1. *Let* $(B, \|\cdot\|)$ *be a Banach space and let us consider*

$$G : \mathbb{R} \times B \to B, \quad (\lambda, u) \mapsto \lambda L(u) + H(\lambda, u),$$

where $L : B \to B$ *is a compact linear operator,* $H(\lambda, \cdot) : B \to B$ *is continuous and compact and*

$$\lim_{\|u\|\to 0} \frac{\|H(\lambda, u)\|}{\|u\|} = 0.$$

We write

$$r(L) := \left\{\mu \in \mathbb{R} : \frac{1}{\mu} \text{ is an eigenvalue of } L \text{ with odd multiplicity}\right\},$$
$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R} \times B : (\lambda, u) \text{ is solution of } u = G(\lambda, u),\ u \not\equiv 0\}.$$

Then, given $\mu \in r(L)$, $\bar{\mathcal{S}}$ *has a connected branch* $\mathcal{C}_\mu$ *bifurcating from* $(\mu, 0)$ *and the following hold.*

(i) *Either,* $\mathcal{C}_\mu$ *is unbounded in* $\mathbb{R} \times B$*; or*

(ii) $\mathcal{C}_\mu \ni (\hat{\mu}, 0)$ *with* $\mu \neq \hat{\mu} \in r(L)$.

In order to apply Rabinowitz's theorem to problem (1.1), we define the following mappings:

$$L : \mathcal{D}^{1,2} \to \mathcal{D}^{1,2}, \quad u \mapsto (-\Delta)^{-1}(a(x)u), \tag{3.1}$$
$$H : \mathcal{D}^{1,2} \to \mathcal{D}^{1,2}, \quad u \mapsto (-\Delta)^{-1}(b(x)r(u)). \tag{3.2}$$

We will first prove that, under assumptions (H1), (H2), (H3) and (H4), those mappings are well defined and compact. It will follow that u is a solution of problem (1.1) if and only if

$$u = \lambda L(u) + \lambda H(u). \tag{3.3}$$

PROPOSITION 3.2. *Let* $f : \Omega \times \mathbb{R} \to \mathbb{R}$ *be such that the following hold.*

(1) f *is Carathéodory,* $f(\cdot, 0) \equiv 0$.

(2) *There exists* $g \in \mathrm{L}^{\gamma}(\Omega)$ *and* $\tilde{\gamma} \geqslant 2$ *such that*

$$|f(x,s)| \leqslant g(x)|s|^{\tilde{\gamma}-1} \quad a.e.\ x \in \Omega \quad \forall s \in \mathbb{R}. \tag{3.4}$$

Then the Nemitsky operator

$$F : \mathrm{L}^{p_1}(\Omega) \to \mathrm{L}^{p_2}(\Omega), \quad u \mapsto f(\cdot, u),$$

is well defined and continuous if

$$1 \leqslant p_2 < \gamma \quad and \quad \frac{1}{p_1} = \frac{1}{\tilde{\gamma}-1}\left(\frac{1}{p_2} - \frac{1}{\gamma}\right). \tag{3.5}$$

Proof. Let $u \in \mathrm{L}^{p_1}(\Omega)$. From condition (3.4) and Hölder's inequality, we get

$$\int_\Omega |f(x,u)|^{p_2} \leqslant \left(\int_\Omega |g(x)|^{p_2 q}\right)^{1/q} \left(\int_\Omega |u|^{(\tilde{\gamma}-1)p_2 p}\right)^{1/p},$$

where (p,q) are chosen in such a way to satisfy

$$\frac{1}{p} + \frac{1}{q} = 1, \quad p, q \geqslant 1, \quad p_2 q = \gamma, \quad (\tilde{\gamma}-1)p_2 p = p_1,$$

which is solvable if and only if (3.5) is satisfied.

The continuity follows immediately by the same inequality. □

PROPOSITION 3.3. *Let* $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ *be such that the following hold.*

(1) f *is Carathéodory,* $f(\cdot, 0) \equiv 0$.

(2) *There exists* $\tilde{\gamma} \in [2, 2^*)$, $g \in \mathrm{L}^{2^*/(2^*-\tilde{\gamma})}(\mathbb{R}^N) \cap \mathrm{L}^{\Gamma}_{\mathrm{loc}}(\mathbb{R}^N)$ *with* $\Gamma > 2^*/(2^*-\tilde{\gamma})$ *such that*

$$|f(x,s)| \leqslant g(x)|s|^{\tilde{\gamma}-1} \quad a.e.\ x \in \Omega \quad \forall s \in \mathbb{R}. \tag{3.6}$$

Then the Nemitsky operator

$$F : \mathcal{D}^{1,2} \to \mathrm{L}^{(2^*)'}(\mathbb{R}^N), \quad u \mapsto f(\cdot, u),$$

is well defined and weakly continuous.

Proof. The fact that the mapping F is well defined follows from the property that $\mathcal{D}^{1,2} \hookrightarrow \mathrm{L}^{2^*}(\mathbb{R}^N)$ and by applying proposition 3.2 with $p_1 := 2^*$, $p_2 := (2^*)'$.

Consider now a sequence $(u_n)_{n=1}^{\infty}$ with $u_n \rightharpoonup u$ weakly in $\mathcal{D}^{1,2}$, and let us verify that $\|f(\cdot,u_n) - f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}} \to 0$. With this aim, let $\varepsilon > 0$ and $B := B(0,R)$ be a ball of $\mathbb{R}^N$.

On the complement B^{c} of the ball, we have

$$\begin{aligned}\int_{B^{\mathrm{c}}} |f(\cdot,u_n)|^{(2^*)'} &\leqslant \int_{B^{\mathrm{c}}} |g|^{(2^*)'} |u_n|^{(\tilde{\gamma}-1)(2^*)'} \\ &\leqslant \|g\|^{(2^*)'}_{\mathrm{L}^{2^*/(2^*-\tilde{\gamma})}(B^{\mathrm{c}})} \|u_n\|^{(2^*)'}_{\mathrm{L}^{2^*}(B^{\mathrm{c}})}.\end{aligned} \tag{3.7}$$

Since (u_n) is bounded in $\mathcal{D}^{1,2}$, it is also bounded in L^{2^*}. Thus, inequality (3.7) gives

$$\|f(\cdot,u_n)\|_{\mathrm{L}^{(2^*)'}(B^{\mathrm{c}})} \leqslant C\|g\|_{\mathrm{L}^{2^*/(2^*-\tilde{\gamma})}(B^{\mathrm{c}})}.$$

In the same way, we deduce that

$$\|f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}(B^c)} \leqslant C\|g\|_{\mathrm{L}^{2^*/(2^*-\tilde{\gamma})}(B^c)}.$$

Since $\lim_{R\to\infty}\|g\|_{\mathrm{L}^{2^*/(2^*-\tilde{\gamma})}(B_R^c)} = 0$, we deduce then the existence of some sufficiently large ball $B_R = B$ such that

$$\|f(\cdot,u_n) - f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}(B^c)} \leqslant \|f(\cdot,u_n)\|_{\mathrm{L}^{(2^*)'}(B^c)} + \|f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}(B^c)} \leqslant \varepsilon. \quad (3.8)$$

On the ball B, we first observe that

$$u_n \to u \quad \text{strongly in } \mathrm{L}^p(B) \quad \forall p \in [1, 2^*). \quad (3.9)$$

Now, using $g \in \mathrm{L}^{\Gamma}(B)$ with $\Gamma > 2^*/(2^* - \tilde{\gamma})$, and applying proposition 3.2 with $\Omega := B$, $\gamma := \Gamma$, $p_2 := (2^*)'$, we deduce that

$$\mathrm{L}^{p_1}(B) \to \mathrm{L}^{(2^*)'}(B), \quad u \mapsto f(\cdot,u), \quad (3.10)$$

is continuous with

$$\frac{1}{p_1} = \frac{1}{\tilde{\gamma}-1}\left(\frac{1}{(2^*)'} - \frac{1}{\Gamma}\right).$$

We easily check that $p_1 < 2^*$. Thus, from (3.9) and (3.10), we deduce the existence of $n_0 \in \mathbb{N}$ such that

$$\|f(\cdot,u_n) - f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}(B)} \leqslant \tfrac{1}{2}\varepsilon \quad \forall n \geqslant n_0. \quad (3.11)$$

From (3.8) and (3.11), we deduce that

$$\|f(\cdot,u_n) - f(\cdot,u)\|_{\mathrm{L}^{(2^*)'}(\mathbb{R}^N)} \leqslant \varepsilon \quad \forall n \geqslant n_0.$$

□

PROPOSITION 3.4. *Assume (H1)–(H4). Then the mappings (3.1) and (3.2) are well defined and weakly continuous. Moreover,*

$$\lim_{\|u\|\to 0} \frac{\|H(u)\|}{\|u\|} = 0. \quad (3.12)$$

Proof. The property for the mapping (3.1) to be weakly continuous is a consequence of proposition 3.3 applied with $\tilde{\gamma} := 2$, $g := a$ (note that $2^*/(2^* - 2) = \frac{1}{2}N$) and from the continuity of $(-\Delta)^{-1}$.

Similarly, we deduce that (3.2) is weakly continuous by applying proposition 3.3 with $g := b$ and $\tilde{\gamma} := 2$ when (H2)(AL) holds and $p + 1$ when (H2)(SL) holds.

Let us now prove (3.12). We first consider the superlinear case when (SL) of (H2) and (H4) hold,

$$\begin{aligned}\|H(u)\| &\leqslant C\|b(x)(u)^p\|_{\mathrm{L}^{(2^*)'}} \\ &\leqslant C\|b\|_{\mathrm{L}^{2^*/(2^*-p-1)}}\|u\|^p_{\mathrm{L}^{(2^*)}} \\ &\leqslant C\|u\|^p.\end{aligned}$$

Thus

$$\frac{\|H(u)\|}{\|u\|} \leqslant C\|u\|^{p-1},$$

and (3.12) follows immediately in this case.

Assume now that (H2)(AL) and (H4)(AL) hold. Let $\varepsilon > 0$, $u \in \mathcal{D}^{1,2}$ and set

$$C_1 := \sup_{s\in\mathbb{R}}\left|\frac{r(s)}{s}\right|, \qquad C_2 := \sup_{u\in\mathcal{D}^{1,2}\setminus\{0\}} \frac{\|u\|_{\mathrm{L}^{(2^*)}}}{\|u\|}.$$

From (H4)(AL), there exists a ball B such that $\|b\|_{\mathrm{L}^{N/2}(B^c)} < \varepsilon/3C_1C_2$. Thus, on the set B^c, we deduce

$$\begin{aligned}\|br(u)\|_{\mathrm{L}^{(2^*)'}(B^c)} &\leqslant \|b\|_{\mathrm{L}^{N/2}(B^c)}\|r(u)\|_{\mathrm{L}^{2^*}(B^c)}\\ &\leqslant \|b\|_{\mathrm{L}^{N/2}(B^c)}C_1C_2\|u\|\\ &\leqslant \tfrac{1}{3}\varepsilon\|u\|. \end{aligned} \tag{3.13}$$

To get an estimate on B, we note that, by (H1), there exists $s_0 > 0$ such that

$$\left|\frac{r(s)}{s}\right| < \frac{\varepsilon}{3C_2\|b\|_{\mathrm{L}^{N/2}}} \quad \forall s < s_0. \tag{3.14}$$

We split B as

$$E := B \cap \{u(x) < s_0\} \quad \text{and} \quad F := B \cap \{u(x) \geqslant s_0\}.$$

On E, by Hölder's inequality and (3.14), we then get

$$\begin{aligned}\|br(u)\|_{\mathrm{L}^{(2^*)'}(E)} &\leqslant \|b\|_{\mathrm{L}^{N/2}(E)}\|r(u)\|_{\mathrm{L}^{2^*}(E)}\\ &\leqslant \|b\|_{\mathrm{L}^{N/2}(E)} \sup_{|s|<s_0}\left|\frac{r(s)}{s}\right|C_2\|u\|\\ &\leqslant \tfrac{1}{3}\varepsilon\|u\|. \end{aligned} \tag{3.15}$$

To obtain an estimate on F, we note that, since $|r(s)/s\,|$ is bounded, for each $\eta > 1$, there exists $M := M(\varepsilon,\eta) > 0$ such that

$$|r(s)| \leqslant M|s|^{\eta} \quad \forall |s| \geqslant s_0. \tag{3.16}$$

Thus, from Holder's inequality and (3.16), we have

$$\int_F |br(u)|^{(2^*)'} \leqslant M^{(2^*)'}\left(\int_F |b|^{(2^*)'p}\right)^{1/p}\left(\int_F |u|^{(2^*)'\eta q}\right)^{1/q},$$

where (p,q) are chosen such that

$$\frac{1}{p}+\frac{1}{q} = 1, \qquad (2^*)'\eta q = 2^*, \qquad (2^*)'p = \beta > \tfrac{1}{2}N,$$

where β is a fixed constant chosen such that $\beta > \frac{1}{2}N$. By calculation, we see that those three equations are solvable with

$$\eta = 2^*\left(1-\frac{1}{\beta}\right) - 1 \tag{3.17}$$

and, moreover, the assumption $\beta > \frac{1}{2}N$ implies $\eta > 1$. Taking (η, M) satisfying (3.17) and (3.16), we deduce that

$$\begin{aligned}\|br(u)\|_{\mathrm{L}^{(2^*)'}(F)} &\leqslant M\|b\|_{\mathrm{L}^{\beta}(B)}\|u\|^{\eta}_{\mathrm{L}^{2^*}(B)}\\ &\leqslant MC_2\|b\|_{\mathrm{L}^{\beta}(B)}\|u\|^{\eta}. \end{aligned} \tag{3.18}$$

The relations (3.13), (3.15), (3.18) imply then that

$$\frac{\|br(u)\|_{\mathrm{L}^{(2^*)'}}}{\|u\|} \leqslant \tfrac{2}{3}\varepsilon + MC_2\|b\|_{\mathrm{L}^\beta(B)}\|u\|^{(\eta-1)}. \tag{3.19}$$

Inequality (3.19) shows that, for each $\epsilon > 0$,

$$\|u\| \leqslant \left(\frac{\epsilon}{3MC_2\|b\|_{\mathrm{L}^\beta(B)}}\right)^{\eta-1} \quad \Rightarrow \quad \frac{\|br(u)\|_{\mathrm{L}^{(2^*)'}}}{\|u\|} \leqslant \epsilon.$$

Thus

$$\lim_{\|u\|\to 0} \frac{\|br(u)\|_{\mathrm{L}^{(2^*)'}}}{\|u\|} = 0$$

and, from the continuity of $(-\Delta)^{-1} : \mathrm{L}^{(2^*)'} \to \mathcal{D}^{1,2}$, we finally get (3.12). □

By applying Rabinowitz's theorem (theorem 3.1) and proposition 3.4, we immediately deduce the following result.

Proposition 3.5. *Assume that (H1)(i), (H2), (H3), (H4) hold and that $a^+ \not\equiv 0$ (respectively, $a^- \not\equiv 0$). Then there exists a global branch $\mathcal{C}^+$ (respectively, $\mathcal{C}^-$), connected in $\bar{\mathcal{S}}$ and bifurcating from $\lambda_1^+(a,\mathbb{R}^N)$ (respectively, $\lambda_1^-(a,\mathbb{R}^N)$). Moreover, $\mathcal{C}^+$ (respectively, $\mathcal{C}^-$) is*

(i) *either unbounded; or*

(ii) *contains a point $(\lambda, 0)$ with $\lambda \neq \lambda_1^+(a,\mathbb{R}^N)$ (respectively, $\lambda \neq \lambda_1^-(a,\mathbb{R}^N)$).*

4. Property of the branch $\mathcal{C}^+$

In this section, we show that when a, b are negative outside a ball, then the branch $\mathcal{C}^+$ obtained in proposition 3.5 is constituted of solutions that do not change sign. In such situation, we then derive that this branch is necessarily unbounded. To this end, we follow some arguments found in [10]. To state the results of this section, we introduce the following sets:

$$\mathcal{Z} := \{(\lambda, u) \in \mathbb{R} \times \mathcal{D}^{1,2} : (\lambda, u) \text{ solves } (1.1)\}, \tag{4.1}$$

$$\mathcal{S} := \{(\lambda, u) \in \mathbb{R} \times \mathcal{D}^{1,2} : (\lambda, u) \text{ solves } (1.1),\ u \not\equiv 0\}, \tag{4.2}$$

$$\mathcal{S}_0 := \{(\lambda, u) \in \mathcal{S} : \lambda > 0,\ |u| > 0\}, \tag{4.3}$$

and we endow each of this set with the topology induced by $\mathbb{R} \times \mathcal{D}^{1,2}$.

We start with the following result.

Proposition 4.1. *Let (H1)–(H4) be satisfied. Then the following hold.*

(1) *$\mathcal{Z} \subset \mathbb{R} \times L^\infty(\mathbb{R}^N)$ and the embedding $\mathcal{Z} \hookrightarrow \mathbb{R} \times L^\infty(\mathbb{R}^N)$ is continuous.*

(2) *For all $(\lambda, u) \in \mathcal{Z}$, we have $\lim_{|x|\to\infty} u(x) = 0$.*

Proof. Let us write the equation (1.1) as

$$-\Delta u - \lambda a(x) u = \lambda f(x), \tag{4.4}$$

where $f(x) = b(x)r(u)$. We note that $u \in \mathrm{L}^{2^*}(\mathbb{R}^N)$, $a, b \in \mathrm{L}^\infty(\mathbb{R}^N)$ and that f is in $\mathrm{L}^{2^*}(\mathbb{R}^N)$ in the asymptotically linear case and is in $\mathrm{L}^{2^*/p}(\mathbb{R}^N)$ in the superlinear case.

In any case, if $f \in \mathrm{L}^q$ for $q > \frac{1}{2}N$, then, from [22, theorem 8.17], we obtain, for any $s > 1$,

$$\sup_{B_R(y)} u \leqslant C(R, s)\{\|u\|_{L^s(B_{2R}(y))} + |\lambda|\|f\|_{\mathrm{L}^q(B_{2R}(y))}\}.$$

Taking $s = 2^*$, we get that $u \in L^\infty_{\mathrm{loc}}$. The same estimate shows that, for y large, the right-hand side goes to 0, and hence $u \in L^\infty(\mathbb{R}^N)$. Thus (2) follows in this case.

If $q \leqslant \frac{1}{2}N$, then we use L^p regularity theory (see [22, theorem 9.11]) to conclude that $u \in W^{2,q}_{\mathrm{loc}}$. By Sobolev embedding theorem, $u \in \mathrm{L}^{q_1}_{\mathrm{loc}}$ for $q_1 = (N - 2q)/Nq$. Since $q_1 > q$, we can continue this boot strap argument to get that $f \in L^{q_m}_{\mathrm{loc}}$ for some $q_m > \frac{1}{2}N$ and proceed as in the previous case.

To prove that the embedding is continuous, let us consider $(\bar{\lambda}, \bar{u}) \in \mathcal{Z}$. Given any $(\lambda, u) \in \mathcal{Z}$, the function $(u - \bar{u})$ satisfies the following equation,

$$-\Delta(u - \bar{u}) - \lambda a(x)(u - \bar{u}) = f, \tag{4.5}$$

where

$$f = (\lambda - \bar{\lambda})\{a(x)\bar{u} + b(x)r(\bar{u})\} + \lambda b(x)\{r(u) - r(\bar{u})\}.$$

We note that if (λ, u) is close to $(\bar{\lambda}, \bar{u})$ in the topology of $\mathbb{R} \times \mathcal{D}^{1,2}$, then f is small in $L^q(\mathbb{R}^N)$, where q is as in the above proof. Then, following the same arguments as we did before for the equation (4.4), but now for (4.5), we conclude the proof. □

Proposition 4.2. *Let (H1)–(H4) be satisfied. Then the following hold.*

(1) *If $a^+ \not\equiv 0$, then $\mathcal{S}_0 \cup \{(\lambda_1^+(a), 0)\}$ is a closed set of $\mathbb{R} \times \mathcal{D}^{1,2}$.*

(2) *If $\operatorname{supp} a^+$, $\operatorname{supp} b^+$ are compact, then $\mathcal{S}_0$ is an open set of $\mathcal{Z}$.*

Proof. (1) Let $(\lambda_n, u_n) \in \mathcal{S}_0 \cup \{(\lambda_1^+(a), 0)\}$ be a sequence converging to $(\lambda, u) \in \mathbb{R} \times \mathcal{D}^{1,2}$. Then, up to a subsequence, we can suppose that $u_n > 0$. Hence $u \geqslant 0$ and satisfies

$$-\Delta u + V^+(x)u = V^-(x)u,$$

where

$$V(x) = \lambda\left(a(x) + b(x)\frac{r(u)}{u}\right).$$

By our assumptions and proposition 4.1, we have $V \in L^\infty(\mathbb{R}^N)$. By applying the strong maximum principle, we get that

(i) $u > 0$; or

(ii) $u \equiv 0$.

In case (i), we immediately have $(\lambda, u) \in \mathcal{S}_0$.

If case (ii) occurs, we shall prove that $\lambda_n \to \lambda_1^+(a)$. To this end, we note that u_n is a positive solution of problem (1.1). Hence λ_n is a positive principal eigenvalue of the problem (2.4) with $\Omega = \mathbb{R}^N$ and

$$w_n(x) := a(x) + b(x)\frac{r(u_n)}{u_n}.$$

From our assumptions and proposition 4.1, we get $w_n \in \mathrm{L}^{N/2}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$, and therefore, by proposition 2.3, we have

$$\lambda_n = \inf_{\varphi \in \mathcal{D}^{1,2}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2}{\int_{\mathbb{R}^N} (a + b(r(u_n)/u_n))\varphi^2} : \int_{\mathbb{R}^N} \left(a + b\frac{r(u_n)}{u_n} \right) \varphi^2 > 0 \right\}. \tag{4.6}$$

Since $\|u_n\| \to 0$, we deduce from proposition 4.1 and (H1) that

$$\lim_{n\to\infty} \frac{r(u_n)}{u_n} = 0.$$

Hence

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} b\frac{r(u_n)}{u_n}\varphi^2 = 0.$$

Therefore,

$$\lambda_n \to \inf_{\varphi \in \mathcal{D}^{1,2}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla \varphi|^2}{\int_{\mathbb{R}^N} a\varphi^2} : \int_{\mathbb{R}^N} a\varphi^2 > 0 \right\} = \lambda_1(a, \mathbb{R}^N).$$

(2) Let $(\lambda_0, u_0) \in \mathcal{S}_0$. We shall prove the existence of a ball $B \subset \mathbb{R} \times \mathcal{D}^{1,2}$ centred at (λ_0, u_0) such that $B \cap \mathcal{Z} \subset \mathcal{S}_0$. We assume $u_0 > 0$ (the same arguments hold if $u_0 < 0$).

By assumption, there exists $R > 0$ such that

$$a(x), b(x) \leqslant 0 \quad \text{a.e. } |x| > R. \tag{4.7}$$

Let us set

$$\epsilon := \tfrac{1}{2} \min \left\{ \lambda_0, \min_{|x| \leqslant R} \{u_0(x)\} \right\}.$$

Since the embedding $\mathcal{Z} \hookrightarrow \mathbb{R} \times L^\infty(\mathbb{R}^N)$ is continuous (proposition 4.1), we deduce the existence of $\delta_0 > 0$ such that

$$\text{any } (\lambda, u) \in \mathcal{Z} \text{ satisfying } |\lambda - \lambda_0| + \|u - u_0\| < \delta_0 \tag{4.8}$$

has the property $|\lambda - \lambda_0| + \|u - u_0\| < \epsilon$.

Given (λ, u) satisfying (4.8), we shall show that $|u| > 0$. Indeed, $\lambda > 0$ (since $\lambda_0 > 0$, by the definition of $\mathcal{S}_0$) and, on the other hand, $u(x) > 0$ for all $|x| \leqslant R$.

For $|x| \geqslant R$, we note that (λ, u) satisfies

$$-\Delta u + V(x)u = 0 \quad \text{on } |x| > R,$$
$$\lim_{|x|\to\infty} u(x) = 0,$$

where

$$V(x) := -\lambda \left(a(x) + b(x)\frac{r(u)}{u} \right).$$

Since $\lambda > 0$, $r(u)/u \geqslant 0$ (by (H1)) and, using (4.7), we get that $V \geqslant 0$ on $|x| > R$. Hence, by applying the maximum principle (see [22, theorem 8.1]), we get $u \geqslant 0$ on $|x| \geqslant R$. Hence $u \geqslant 0$ on $\mathbb{R}^N$. The strong maximum principle then implies $u > 0$, showing that $(\lambda, u) \in \mathcal{S}_0$. □

PROPOSITION 4.3. *Assume (H1)–(H4) and (H5)(ii) hold. Let (λ, u) be a solution of (1.1) such that $|u| > 0$. Then $\lambda \leqslant \lambda_1^+(a, (\operatorname{supp} b^+)^0) < \infty$.*

Proof. The proof is based on standard arguments (see, for example, [16]). Since u does not change sign and solves problem (1.1), we have that λ is a positive principal eigenvalue of the problem (2.4) with $\Omega = \mathbb{R}^N$ and $w(x) := a(x) + b(x)(r(u)/u)$. Our assumption on a, b and proposition 4.1 shows that $w \in \mathrm{L}^{N/2}(\mathbb{R}^N) \cap \mathrm{L}^\infty(\mathbb{R}^N)$, and thus, by proposition 2.3, we have

$$\begin{aligned}
\lambda &= \inf_{\varphi \in \mathcal{D}^{1,2}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2}{\int_{\mathbb{R}^N} (a + b(r(u)/u))\varphi^2} : \int_{\mathbb{R}^N} \left(a + b\frac{r(u)}{u} \right) \varphi^2 > 0 \right\} \\
&\leqslant \inf_{\varphi \in \mathcal{D}^{1,2}} \left\{ \frac{\int_{\mathbb{R}^N} |\nabla\varphi|^2}{\int_{\mathbb{R}^N} (a + b(r(u)/u))\varphi^2} : \int_{\mathbb{R}^N} a\varphi^2 > 0 \text{ and } \int_{\mathbb{R}^N} b\frac{r(u)}{u}\varphi^2 > 0 \right\} \\
&\leqslant \inf_{\varphi \in \mathcal{D}_0^{1,2}((\operatorname{supp} b^+)^0)} \left\{ \frac{\int_{(\operatorname{supp} b^+)^0} |\nabla\varphi|^2}{\int_{(\operatorname{supp} b^+)^0} (a + b(r(u)/u))\varphi^2} : \int_{(\operatorname{supp} b^+)^0} a\varphi^2 > 0 \right\} \\
&\leqslant \inf_{\varphi \in \mathcal{D}_0^{1,2}((\operatorname{supp} b^+)^0)} \left\{ \frac{\int_{(\operatorname{supp} b^+)^0} |\nabla\varphi|^2}{\int_{(\operatorname{supp} b^+)^0} a\varphi^2} : \int_{(\operatorname{supp} b^+)^0} a\varphi^2 > 0 \right\} \\
&= \lambda_1^+(a, (\operatorname{supp} b^+)^0),
\end{aligned}$$

and since $(\operatorname{supp} a^+ \cap \operatorname{supp} b^+)^0 \neq \emptyset$ by (H5)(ii), this last quantity is finite. □

PROPOSITION 4.4. *Let (H1)–(H4) be satisfied. then we have the following.*

1. *If (H5)(i) holds, then, for all $(\lambda, u) \in \mathcal{C}^+ \setminus \{(\lambda_1^+(a), 0)\}$, we have $|u| > 0$ and $\mathcal{C}^+$ is unbounded.*
2. *If (H5) is assumed, then $\Pi_{\mathbb{R}}\mathcal{C}^+ \subseteq (0, \Lambda^+]$, where $\Lambda^+ := \lambda_1^+(a, (\operatorname{supp} b^+)^0)$.*

Proof. (1) Let us prove that the solutions of problem (1.1) belonging to $\mathcal{C}^+$ do not change sign. With this aim, we introduce the set

$$\tilde{\mathcal{C}} := (\mathcal{S}_0 \cap \mathcal{C}^+) \cup \{ (\lambda_1^+(a), 0)\} = \{(\lambda, u) \in \mathcal{C}^+ : |u| > 0\} \cup \{ (\lambda_1^+(a), 0)\}.$$

This set is non-empty and it follows from proposition 4.2 that it is a closed set of $\mathcal{C}^+$ for the induced topology. We claim that it is also open in $\mathcal{C}^+$. From the fact that $\mathcal{C}^+$ is connected, we will then get $\tilde{\mathcal{C}} = \mathcal{C}^+$.

Let $(\lambda, u) \in \tilde{\mathcal{C}}$. If $\lambda \neq \lambda_1^+(a)$, then it follows from proposition 4.2 that $\mathcal{S}_0 \cap \mathcal{C}^+$ is an open set of $\mathcal{C}^+$ containing (λ, u). Therefore, we are reduced to the case $(\lambda, u) = (\lambda_1^+(a), 0)$. That is, we have to show the existence of a ball $B \subset \mathbb{R} \times \mathcal{D}^{1,2}$ such that

$$(\lambda_1^+(a), 0) \in B \quad \text{and} \quad B \cap \mathcal{C}^+ \subset \tilde{\mathcal{C}}.$$

To this end, let $(\lambda_1^+(a), \phi_1)$ satisfying $-\Delta\phi_1 = \lambda_1^+(a)a\phi_1$ with $\phi_1 > 0$. By Rabinowitz's theorem, there exists a neighbourhood V of $\mathbb{R} \times \mathcal{D}^{1,2}$ and a continuous function

$$\mu \times \eta : (-\delta, \delta) \to \mathbb{R}, \quad \text{with } (\mu(0), \eta(0)) = (\lambda_1^+(a), 0), \quad \lim_{t\to 0} \|\eta(t)\| = 0, \tag{4.9}$$

such that $V \cap \mathcal{C}^+ = \{(\mu(t), t[\phi_1 + \eta(t)])\}$. Let us denote $u_t := t[\Phi + \eta(t)]$.

By (H5), there exists $R > 0$ such that

$$a(x), b(x) < 0 \quad \text{on } |x| > R. \tag{4.10}$$

CLAIM 4.5. *By choosing δ small enough in (4.9), we have*

$$[\phi_1 + \eta(t)](x) > 0 \quad \textit{on } |x| \leqslant R \quad \forall t \in (-\delta, \delta). \tag{4.11}$$

Indeed, a simple calculation shows that, for each $t \in (-\delta, \delta)$, the function $\eta(t)$ satisfies the following equation,

$$-\varDelta[\eta(t)] - \mu(t)b(x)\frac{r(u_t)}{u_t}\eta(t) = f_t,$$

where

$$f_t := [\mu(t) - \lambda_1^+(a)]a(x)\phi_1 + \mu(t)b(x)\frac{r(u_t)}{u_t}\phi_1.$$

Since $(\mu(t), u_t) \to (\lambda_1^+(a), 0)$ in $\mathbb{R}\times\mathcal{D}^{1,2}$, we deduce from proposition 4.1 that $u_t \to 0$ in $L^\infty(\mathbb{R}^N)$. Hence, noting that $\lim_{t\to 0}(r(u_t)/u_t) = 0$ (by (H1)), $f_t \in L^\infty(\mathbb{R}^N)$ (from our assumptions and proposition 4.1), we deduce $\lim_{t\to 0}\|f_t\|_{L^\infty(|x|<R)} = 0$. Using [22, theorem 8.1], we deduce that

$$\sup_{|x|<R/2}\{\eta(t)\} \leqslant C(R)\{\|\eta(t)\|_{L^{2^*}(|x|<R)} + \|f_t\|_{L^\infty(|x|<R)}\}.$$

Hence $\lim_{t\to 0}\|\eta(t)\|_{L^\infty(|x|<R)} = 0$. Since $\phi_1 > 0$ on $|x| < R$, we deduce (4.11).

CLAIM 4.6. *For δ chosen as in claim 4.5, we have*

$$|u_t| > 0 \quad \textit{on } |x| > R \quad \forall t \in (-\delta, \delta). \tag{4.12}$$

Indeed, using proposition 4.1, we see that u_t solves

$$\begin{aligned} -\Delta u_t + V(x)u_t &= 0 \quad \textit{on } |x| > R, \\ u_t &\geqslant 0 \quad \textit{on } |x| = R, \\ \lim_{|x|\to\infty} u_t(x) &= 0, \end{aligned}$$

where

$$V(x) := -\mu(t)\left(a(x) + b(x)\frac{r(u_t)}{u_t}\right).$$

Now, by (4.10) and the fact that $\mu(t) > 0$, we get $V(x) \geqslant 0$ on $|x| > R$. By applying the maximum principle [22], we deduce (4.12).

Claims 4.5 and 4.6 show that the functions belonging to $\mathcal{C}^+$ do not change sign. Now, by applying proposition 4.3, we deduce that $\mathcal{C}^+$ cannot satisfy the alternative (ii) of proposition 3.5. Hence $\mathcal{C}^+$ has to be unbounded.

(2) The second statement follows from (1) and proposition 4.3. □

Proposition 4.4 (part (2)) implies the existence of a sequence $(\lambda_n, u_n) \in \mathcal{C}^+$ such that $\|u_n\| \to \infty$ and $\lambda_n \to \lambda_0$. In other words, the branch $\mathcal{C}^+$ bifurcates from infinity at some λ_0, value that we would like to characterize. With this aim, in the next two Clarify sentence?

sections, we will prove some *a priori* estimates to solutions of problem (1.1) that do not change sign. Since r is assumed to be odd, we note that (λ, u) solves (1.1) if and only if the same holds for $(\lambda, -u)$. Hence we are reduced to giving *a priori* estimates for a positive solution.

5. Bifurcation from infinity: the asymptotically linear case

In this section, we study the behaviour of the branch $\mathcal{C}^+$ for the asymptotically linear case.

PROPOSITION 5.1. *If (H1), (H2)(AL), (H3), (H4)(AL) and (H5) hold, then, for every solution* (λ, u) *of problem (1.1), with* $\lambda > 0$ (*respectively,* $\lambda < 0$) *and* $u > 0$, *we have*

$$0 < \alpha^+ \leqslant \lambda \quad (\textit{respectively, } \lambda \leqslant \alpha^- < 0), \tag{5.1}$$

where

$$\alpha^+ := \lambda_1^+\biggl(a + b^+\biggl\|\frac{r(s)}{s}\biggr\|_\infty, \mathbb{R}^N\biggr), \qquad \alpha^- := -\lambda_1^+\biggl(-a + b^-\biggl\|\frac{r(s)}{s}\biggr\|_\infty, \mathbb{R}^N\biggr). \tag{5.2}$$

Proof. Assume first that $\lambda > 0$. We have

$$\int_{\mathbb{R}^N} |\nabla u|^2 = \lambda \int_{\mathbb{R}^N} \biggl(a + b\frac{r(u)}{u}\biggr)u^2 \leqslant \lambda \int_{\mathbb{R}^N} \biggl(a + b^+\biggl\|\frac{r(u)}{u}\biggr\|_\infty\biggr)u^2.$$

Thus

$$\lambda \geqslant \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\int_{\mathbb{R}^N} (a + b^+\|r(u)/u\,\|_\infty)u^2} \geqslant \lambda_1^+\biggl(a + b^+\biggl\|\frac{r(s)}{s}\biggr\|_\infty, \mathbb{R}^N\biggr), \tag{5.3}$$

which proves the first inequality of (5.1).

If $\lambda < 0$, we argue in the same way, with $-a$ and $-b$ instead of a and b. □

PROPOSITION 5.2. *Under the same assumptions as in the previous proposition, let* (λ_n, u_n) *be a sequence of solutions of problem (1.1) satisfying*

$$\lambda_n > 0 \quad (\textit{respectively, } \lambda_n < 0), \quad \lambda_n \textit{ bounded}, \quad u_n > 0, \quad \|u_n\|_{\mathcal{D}^{1,2}} \to \infty.$$

Then, up to a subsequence,

$$\lambda_n \to \lambda_\infty = \lambda_1^+(a + br_\infty, \mathbb{R}^N) \quad (\textit{respectively, } \lambda_1^-(a + br_\infty, \mathbb{R}^N)).$$

Proof. We set $w_n := u_n/\|u_n\|$. Then, up to a subsequence,

$$\lambda_n \to \lambda_\infty \geqslant 0, \quad w_n \rightharpoonup w \quad \text{in } \mathcal{D}^{1,2}.$$

Moreover, w_n satisfies the following equation:

$$-\Delta w_n = \lambda_n\biggl(a(x)w_n + b(x)\frac{r(u_n)}{u_n}w_n\biggr). \tag{5.4}$$

By (H1) and (H2)(AL), we have $r(u_n)/u_n \leqslant C$. Therefore, since $L^\infty(\mathbb{R}^N)$ is the dual of $L^1(\mathbb{R}^N)$, we deduce from Alaoglu's theorem that $r(u_n)/u_n$ converges in the weak* topology of $L^\infty(\mathbb{R}^N)$ to some $\tilde{r} \in L^\infty(\mathbb{R}^N)$.

Using again the fact that $r(u_n)/u_n \leqslant C$, we deduce that the right-hand side of (5.4) defines a compact map $\mathcal{D}^{1,2} \to \mathrm{L}^{(2^*)'}$. Since Δ^{-1} is continuous, we deduce that (w_n) converges strongly in $\mathcal{D}^{1,2}$ to some w. We note that $\|w\| = 1$, $w \not\equiv 0$, $w_n(x) \to w(x)$ a.e. $x \in \mathbb{R}^N$ (up to a subsequence), and, moreover, w satisfies

$$-\Delta w = \lambda_\infty(a(x) + b(x)\tilde{r}(x))w. \tag{5.5}$$

Let us prove that $\tilde{r} = r_\infty$. By setting $V(x) := \lambda_\infty(a(x) + b(x)\tilde{r}(x))$ and writing $V := V^+ - V^-$, we get from (5.5) that

$$-\Delta w + V^- w = V^+ w \geqslant 0 \quad \text{in } \mathbb{R}^N.$$

From the strong maximum principle, we deduce that $w > 0$ a.e. (since $w \not\equiv 0$). Therefore,

$$w(x) = \lim_{n\to\infty} w_n(x) = \lim_{n\to\infty} \frac{u_n(x)}{\|u_n\|} > 0 \quad \text{a.e. } x \in \mathbb{R}^N. \tag{5.6}$$

Since, by assumption, $\|u_n\| \to \infty$, relation (5.6) implies that $\lim_{n\to\infty} u_n(x) = +\infty$ for a.e. $x \in \mathbb{R}^N$. Hence

$$\lim_{n\to\infty} \frac{r(u_n)}{u_n} = r_\infty \quad \text{a.e. } x \in \mathbb{R}^N \quad \text{and} \quad \frac{r(u_n)}{u_n} \rightharpoonup \tilde{r}.$$

Therefore, we must have $\tilde{r} = r_\infty$ and, in particular, w has to satisfy

$$-\Delta w = \lambda_\infty(a(x) + b(x)r_\infty)w, \quad w > 0.$$

By proposition 2.3, this implies $\lambda_\infty = \lambda_1^+(a + br_\infty, \mathbb{R}^N)$. □

We are now able to prove our result on the existence and behaviour of branches of solutions in the asymptotically linear case.

Proof of theorem 1.3. The existence of a global branch follows from proposition 3.5. Properties (i) and (ii) of $\mathcal{C}^+$ are proved in proposition 4.4, while the property (iii) is a consequence of proposition 5.2. □

REMARK 5.3. In theorem 1.3, the asymptotic bifurcation of the branch $\mathcal{C}^+$ at $\lambda_1^+(a + br_\infty, \mathbb{R}^N)$ is made possible through the hypotheses given in (H5). Indeed, condition (H5)(i) ensures that the branch is unbounded, while (H5)(ii) gives an *a priori* bound on λ. If (H5)(ii) is not satisfied, the bifurcation from infinity is in general not true. Consider, for example, the problem

$$-\Delta u = \lambda a(x)(u - r(u)), \quad u \in \mathcal{D}^{1,2}, \tag{5.7}$$

with

(1) r satisfying (H1) but $r_\infty := \lim_{s\to\infty}(r(s)/s) \geqslant 1$; and

(2) $a \in C_0^\infty(\mathbb{R}^N)$, $a \geqslant 0$, $a \not\equiv 0$.

From proposition 3.5, there exists a branch of solutions $\mathcal{C}$ bifurcating from $(\lambda_1, 0)$, where $\lambda_1 := \lambda_1(a, \mathbb{R}^N)$. Moreover, proposition 4.4 (part (i)) shows that this branch is unbounded and constituted of solutions having constant sign. If $\mathcal{C}$ bifurcates from infinity at some λ^∞, by proposition 5.2, we must have $\lambda^\infty = \lambda_1^+(a(1 - r_\infty))$. But, since $a(1 - r_\infty) \leqslant 0$, we have $\lambda^\infty = \infty$, showing that $\mathcal{C}$ cannot bifurcate from infinity.

6. Superlinear case: branches in $\mathcal{D}^{1,2}$

In this section, we prove the theorem 1.4 on the existence and behaviour of global branches in $\mathcal{D}^{1,2}$ for the superlinear case. For this, we need an *a priori* bound in $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ for positive solutions to (1.1) for λ away from 0. To this end, as in [15], by considering

$$\Omega^+ = \{x : b(x) > 0\}, \qquad \Omega^- = \{x : b(x) < 0\},$$

we first look for L^p *a priori* estimates in a ball B_ϵ contained in $\Omega^+ \cup \Omega^-$. More precisely, we have the following result.

PROPOSITION 6.1. *Let (λ, u) be a solution of (1.1) with $u > 0$. Given $x_0 \in \Omega^\pm$, $\epsilon > 0$ and $B_\epsilon(x_0) \subset\subset \Omega^\pm$, there exists $C = C(\epsilon, \|a\|_\infty)$ such that*

$$\int_{B_{\epsilon/2}(x_0)} u^p \,\mathrm{d}x \leqslant \left(\frac{C}{|\lambda| \inf_{B_\epsilon} |b|}\right)^{p/(p-1)}. \tag{6.1}$$

Proof. We consider on the ball $B_\epsilon = B_\epsilon(x_0)$ an eigenfunction ϕ associated to the first eigenvalue $\lambda_1(\epsilon)$, which satisfies

$$\begin{aligned} -\Delta\phi &= \lambda_1(\epsilon)\phi && \text{in } B_\epsilon,\\ \phi &= 0 && \text{on } \partial B_\epsilon,\\ \phi &> 0 && \text{in } B_\epsilon,\\ \|\phi\|_{C^1} &\leqslant 1. \end{aligned}$$

Multiply the equation in (1.1) by ϕ^α and choose $\alpha \geqslant 2p/(p-1)$. We obtain

$$\int_{B_\epsilon} (-\Delta u)\phi^\alpha = \lambda \int_{B_\epsilon} \{a(x)u + b(x)u^p\}\phi^\alpha. \tag{6.2}$$

Since

$$\phi|_{\partial B_\epsilon} = \frac{\partial \phi^\alpha}{\partial n}\bigg|_{\partial B_\epsilon} = 0$$

(note that $\alpha > 1$), the left-hand side of (6.2) gives

$$\begin{aligned} \int_{B_\epsilon} (-\Delta u)\phi^\alpha &= -\int_{B_\epsilon} u\Delta(\phi^\alpha)\\ &= -\alpha \int_{B_\epsilon} u(\Delta\phi)\phi^{\alpha-1} - \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2}\\ &= \alpha\lambda_1(\epsilon)\int_{B_\epsilon} u\phi^\alpha - \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2}. \end{aligned} \tag{6.3}$$

From (6.2) and (6.3), we have

$$\int_{B_\epsilon} bu^p\phi^\alpha = \frac{\alpha\lambda_1(\epsilon)}{\lambda}\int_{B_\epsilon} u\phi^\alpha - \frac{\alpha(\alpha-1)}{\lambda}\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} - \int_{B_\epsilon} au\phi^\alpha. \tag{6.4}$$

If $B_\epsilon \subset \Omega^+$, we have

$$\int_{B_\epsilon} bu^p\phi^\alpha \leqslant \begin{cases} \dfrac{\alpha\lambda_1(\epsilon)}{\lambda}\displaystyle\int_{B_\epsilon} u\phi^\alpha - \int_{B_\epsilon} au\phi^\alpha & \text{if } \lambda > 0, \\ \dfrac{\alpha(\alpha-1)}{|\lambda|}\displaystyle\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} - \int_{B_\epsilon} au\phi^\alpha & \text{if } \lambda < 0. \end{cases} \tag{6.5}$$

If $B_\epsilon \subset \Omega^-$, we have

$$\begin{aligned} \int_{B_\epsilon} |b|u^p\phi^\alpha &= \int_{B_\epsilon} (-b)u^p\phi^\alpha \\ &= -\frac{\alpha\lambda_1(\epsilon)}{\lambda}\int_{B_\epsilon} u\phi^\alpha + \frac{\alpha(\alpha-1)}{\lambda}\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} + \int_{B_\epsilon} a(x)u\phi^\alpha \\ &\leqslant \begin{cases} \dfrac{\alpha(\alpha-1)}{\lambda}\displaystyle\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} + \int_{B_\epsilon} au\phi^\alpha & \text{if } \lambda > 0, \\ \dfrac{\alpha\lambda_1(\epsilon)}{|\lambda|}\displaystyle\int_{B_\epsilon} u\phi^\alpha + \int_{B_\epsilon} au\phi^\alpha & \text{if } \lambda < 0. \end{cases} \end{aligned} \tag{6.6}$$

Now, the right-hand side of (6.5) and (6.6) can be estimated using Hölder's inequality $(1/p + 1/q = 1)$ as follows:

$$\int_{B_\epsilon} u\phi^\alpha = \int_{B_\epsilon} u\phi^{\alpha/p}\,\phi^{\alpha/q} \leqslant \left(\int_{B_\epsilon} u^p\phi^\alpha\right)^{1/p}\left(\int_{B_\epsilon} \phi^\alpha\right)^{1/q},$$

$$\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} = \int_{B_\epsilon} u\phi^{\alpha/p}\,\phi^{\alpha/q\,-2}|\nabla\phi|^2 \leqslant \left(\int_{B_\epsilon} u^p\phi^\alpha\right)^{1/p}\left(\int_{B_\epsilon} \phi^{\alpha-2q}|\nabla\phi|^{2q}\right)^{1/q}.$$

Therefore, by choosing $\alpha \geqslant 2q = 2p/(p-1)$ and $\|\phi\|_{C^1} \leqslant 1$, we deduce the existence of a constant $C_0 := C(\epsilon)$ such that

$$\int_{B_\epsilon} u\phi^\alpha, \int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} \leqslant C_0\left(\int_{B_\epsilon} u^p\phi^\alpha\right)^{1/p}. \tag{6.7}$$

Hence, from equations (6.5), (6.6) and (6.7), we get the existence of a constant $C_1 := C(\epsilon, \|a\|_\infty)$ such that

$$\int_{B_\epsilon} |b|u^p\phi^\alpha \leqslant \frac{C_1}{|\lambda|}\left\{\int_{B_\epsilon} u^p\phi^\alpha\right\}^{1/p},$$

which implies

$$\left\{\inf_{B_\epsilon} |b|\right\}\left\{\int_{B_\epsilon} u^p\phi^\alpha\right\}^{1-1/p} \leqslant \frac{C_1}{|\lambda|}.$$

Thus we finally obtain

$$\left\{\inf_{B_{\epsilon/2}(x_0)} \phi^\alpha\right\}\int_{B_{\epsilon/2}(x_0)} u^p \leqslant \left\{\frac{C_1}{|\lambda|\inf_{B_\epsilon(x_0)} |b|}\right\}^{p/(p-1)},$$

from which we immediately get (6.1). □

Now we look for $L^\infty_{\rm loc}(\mathbb{R}^N)$ bound for the positive solutions of problem (1.1).

PROPOSITION 6.2. *Assume that (H1), (H2)(SL), (H3) and (H6) are satisfied. Let $\Lambda > 0$ and $R > 0$ large enough. Then there exists a constant $C := C(\Lambda, R)$ such that, for any positive solution (λ, u) to (1.1) with $\lambda \geqslant \Lambda$, we have*

$$\|u\|_{L^\infty(B_R)} \leqslant C.$$

Proof. Let $\delta > 0$. As in [15], we divide the domain $\mathbb{R}^N$ in three regions.

(1) $\Omega_\delta^- := \Omega^- \cap \{x : \operatorname{dist}(x, \Gamma) \geqslant \delta\}$, where $\Omega^- := \{x \in \mathbb{R}^N : b(x) < 0\}$.

(2) $\Gamma_\delta := \{x : \operatorname{dist}(x, \Gamma) \leqslant \delta\}$.

(3) $\Omega_\delta^+ := \Omega^+ \cap \{x : \operatorname{dist}(x, \Gamma) \geqslant \delta\}$, where $\Omega^+ := \{x \in \mathbb{R}^N : b(x) > 0\}$.

To prove the proposition, we will show that, in each region, the set of solutions is bounded in $L^\infty_{\rm loc}$.

STEP 1. *A priori* bound in $\Omega_{\delta,R}^- = \Omega_\delta^- \cap B_R$.

We have two possibilities: either $-\Delta u(x) \geqslant 0$ or $-\Delta u(x) \leqslant 0$.

In the first case, we just note that

$$-\Delta u = \lambda(a(x)u + b(x)u^p) \geqslant 0 \quad \Rightarrow \quad a(x)u + b(x)u^p \geqslant 0 \quad (\lambda > 0).$$

Since $\inf_{\Omega_{\delta,R}^-} |b| > 0$ by (H6), we get, in this case, the following bound:

$$u \leqslant \left(\frac{\|a\|_\infty}{\inf_{\Omega_{\delta,R}^-} |b|}\right)^{1/(p-1)}. \tag{6.8}$$

If the second case occurs, by using [22, theorem 8.17], we deduce

$$\sup_{B_{\epsilon/2}(y)} u \leqslant C(n, p, \epsilon)\left(\frac{1}{|B|}\int_B (u^+)^p\right)^{1/p}. \tag{6.9}$$

Now, for each $y \in \Omega_\delta^-$, consider the ball $B_{\delta/8}(y)$. Combining (6.1) and (6.9), we get

$$\sup_{B_{\delta/8}(y)} u \leqslant C(n, p, \delta, \|a\|_\infty)\left\{\frac{1}{|\lambda| \inf_{B_{\delta/2}(y)} |b|}\right\}^{1/(p-1)}. \tag{6.10}$$

Therefore, by setting

$$\beta := \left\{\inf_{\Omega_{\delta/2}^- \cap B_{R+\delta/2}} |b|\right\}^{1/(p-1)}$$

(and by (H6), $\beta > 0$), we get from (6.8) and (6.10) the *a priori* bound

$$\sup_{\Omega_{\delta,R}^-} u \leqslant m := \frac{1}{\beta}\left\{\|a\|_\infty^{1/(p-1)} + \frac{C(n, p, \delta, \|a\|_\infty)}{|\lambda|^{1/(p-1)}}\right\}. \tag{6.11}$$

STEP 2. *A priori* bound in a neighbourhood of Γ.

Let us fix $x_0 \in \Gamma$. Since Γ is compact, it is sufficient to give an *a priori* bound in a neighbourhood of x_0. The sketch of the proof is as follows.

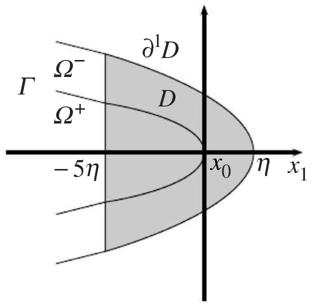

Figure 1.

(1) By first making a transformation preserving some properties of the coefficients of the equation, we construct a convex neighbourhood of x_0.

(2) Applying in this domain the moving-plane method to an auxiliary function (similar to [15]), we show a 'Harnack inequality' satisfied by u in a cone with x_0 as vertex. Combining this inequality with the integral estimate (6.1), we get the *a priori* bound.

A strict convex neighbourhood of x_0

Figure 1 not cited in text!

Up to some rotation or translation, we can suppose that $x_0 = 0$ and that Γ is tangent to the hyperplane $x_1 = 0$. Doing a Kelvin transform (take the centre of the inversion on the x_1-axis such that the sphere is tangent to $x_1 = 0$), we can suppose that Ω^+ is at the left of Γ and also strictly convex in the x_1 direction in a neighbourhood of x_0. But, contrary to the case of [14], the equation is not preserved by Kelvin transform. Indeed, let K be the Kelvin transform, with y_0 as the centre of the inversion, that is,

$$K : \mathbb{R}^N \setminus \{y_0\} \to \mathbb{R}^N, \quad x \mapsto y_0 + \frac{x - y_0}{|x - y_0|^2}|y_0|^2,$$

and let $\bar{u}$ be the Kelvin transform of u, that is,

$$\bar{u}(x) = \left(\frac{|y_0|}{|x - y_0|}\right)^{N-2} u(K(x)).$$

Then $\bar{u}$ satisfies the following equation,

$$-\Delta \bar{u} = \lambda(\tilde{a}(x)\bar{u} + \tilde{b}(x)\bar{u}^p), \tag{6.12}$$

where

$$\tilde{a}(x) = a(K(x))\left(\frac{|y_0|}{|x - y_0|}\right)^4,$$

$$\tilde{b}(x) = b(K(x))\left(\frac{|y_0|}{|x - y_0|}\right)^{(N+2)-p(N-2)}.$$

Given $\eta > 0$, consider the convex domain D containing x_0 enclosed by the surfaces

$$\partial^1 D := \{x \in \Omega^- : \mathrm{dist}(x, \Gamma) = \eta\} \quad \text{and} \quad \partial^2 D := \{x : x_1 = -5\eta\}.$$

Since $y_0 \neq 0$, by choosing η such that $5\eta < |y_0|$, we have $\tilde{a}, \tilde{b} \in L^\infty(D)$. Moreover, the assumptions made on a, b in (H6) are inherited by $\tilde{a}$, $\tilde{b}$ in a neighbourhood of $K(x_0) = 0$. In the sequel, for notational convenience, we will denote $\tilde{a}$ by a and $\tilde{b}$ by b.

With the aim of applying a moving-plane method to some auxiliary function in the domain D, we are led to choose η small enough in such a way that

$$\lambda_1(-\Delta - \lambda a(x), D) > 0, \tag{6.13}$$

$$\frac{\partial a}{\partial x_1}(x) \leqslant 0, \quad x \in D, \tag{6.14}$$

$$\sup_D \left\{ \frac{\partial b}{\partial x_1} \right\} < 0. \tag{6.15}$$

Condition (6.13) can be realized by considering η small enough to ensure that $\lambda_1(-\Delta, D) > \lambda \|a\|_\infty$, while the conditions (6.14) and (6.15) are made possible by (H6).

Moving-plane method and Harnack inequality

Let $\tilde{u}$ be a continuous extension of u on all ∂D such that $0 \leqslant \tilde{u} \leqslant \sup_{\partial^1 D} u$. Since $\partial^1 D \subset \Omega_\eta^-$, the results of step 1 show that $\tilde{u} \leqslant m$, where m is defined by (6.11).

Let $C_0 > 0$ be a constant to be fixed later and $g \in C^1(\bar{D})$ a function satisfying

$$g(x) < 0 \quad \text{and} \quad \frac{\partial g}{\partial x_1}(x) > 0 \quad \forall x \in \bar{D} \tag{6.16}$$

(for example, $g(x) = -A + x_1$ with $A > 0$ chosen to ensure $g < 0$ in $\bar{D}$).

We consider the function w solution of the following problem (which is well defined thanks to (6.13)): Clarify sentence?

$$\begin{aligned} -\Delta w - \lambda a(x) w &= C_0 g \quad \text{in } D, \\ w &= \tilde{u} \quad\ \ \text{on } \partial D. \end{aligned}$$

We introduce the auxiliary function v,

$$v = u - w. \tag{6.17}$$

From (6.17), one can see that v satisfies

$$\left.\begin{aligned} -\Delta v &= f(x, v) \quad \text{in } D, \\ v &= 0 \quad\quad\ \ \text{on } \partial^1 D, \end{aligned}\right\} \tag{6.18}$$

where

$$f(x, v) = \lambda a(x) v + \lambda b(x)(v + w)^p - C_0 g.$$

We claim that, by choosing C_0 large enough and $\eta_1 \in (0, \eta)$ small enough, the following conditions can be realized:

$$v \geqslant 0 \quad \text{on } D \cap \{-\eta < x_1 < \eta\}, \tag{6.19}$$

$$\frac{\partial f}{\partial x_1}(x, v) \leqslant 0 \quad \forall x \in D \cap \{-2\eta_1 < x_1 < \eta\} \quad \forall v > 0. \tag{6.20}$$

To prove (6.19), we are going to estimate w and $\partial w/\partial x_1$ in D. To this end, let us consider (H,G) solutions of

$$\begin{aligned} \Delta H + \lambda a(x) H &= 0 \quad \text{in } D,\\ H &= \tilde{u} \quad \text{on } \partial D \end{aligned}$$

and

$$\begin{aligned} \Delta G + \lambda a(x) G &= g \quad \text{in } D,\\ G &= 0 \quad \text{on } \partial D, \end{aligned}$$

allowing to split w as Clarify sentence?

$$w = H - C_0 G. \tag{6.21}$$

Since $\lambda_1(-\Delta - \lambda a(x), D) > 0$ (see (6.13)), the maximum principle holds for the operator $-\Delta - \lambda a(x)$. Therefore, on the one hand, by applying [7, theorem I.3], which extends the Alexandrov–Bakelman–Pucci estimate for narrow domains, we obtain

$$\|H\|_{L^\infty(D)} \leqslant C \sup_{\partial D} H \leqslant Cm. \tag{6.22}$$

On the other hand, since $g \leqslant 0$ (see (6.16)), we get

$$G > 0 \quad \text{on } D, \tag{6.23}$$

and from Hopf's lemma,

$$\frac{\partial G}{\partial x_1} < 0 \quad \text{on } \partial D \cap \{-\eta \leqslant x_1 \leqslant \eta\}. \tag{6.24}$$

Let $D_\eta \subset\subset D \cap \Omega^-$ be a tubular neighbourhood of $\partial^1 D \cap \{-\eta \leqslant x_1 \leqslant \eta\}$ such that

$$\sup_{D_\eta} \frac{\partial G}{\partial x_1} < 0. \tag{6.25}$$

Let us first show that (6.19) holds on D_η. Since $v = 0$ on $\partial^1 D$, it is sufficient to prove that $\partial v/\partial x_1 \leqslant 0$ in D_η. Clearly, by the definition of v,

$$\frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial H}{\partial x_1} + C_0 \frac{\partial G}{\partial x_1}. \tag{6.26}$$

Since $D_\eta \subset\subset \Omega^-$, by the estimates obtained in the previous step and by standard elliptic estimates, we have

$$\sup_{x \in D_\eta} \left| \frac{\partial u}{\partial x_1} \right| \leqslant Cm. \tag{6.27}$$

From (6.22) and [22, theorem 8.33], we have

$$\left\| \frac{\partial H}{\partial x_1} \right\|_{L^\infty(D_\eta)} \leqslant C\left(\sup_D H + \sup_{\partial D \cap \{-\eta \leqslant x_1 \leqslant \eta\}} \left| \frac{\partial \tilde{u}}{\partial x_1} \right| \right) \leqslant Cm. \tag{6.28}$$

From (6.26), (6.27) and (6.28), it follows that

$$\frac{\partial v}{\partial x_1} \leqslant Cm + C_0 \sup_{D_\eta} \frac{\partial G}{\partial x_1}. \tag{6.29}$$

Now, using (6.25), the right-hand side of (6.29) can be made negative on D_η by choosing C_0 large enough. Combining (6.29) with $v = 0$ for x in $\partial^1 D$, we obtain $v \geqslant 0$ in D_η.

On the compact set $K := D \cap \{-\eta \leqslant x_1 \leqslant \eta\} \setminus D_\eta$, by using $u \geqslant 0$ and (6.22), we get

$$v \geqslant -w = -H + C_0 G \geqslant -Cm + C_0 \inf_K G. \tag{6.30}$$

Using (6.23), we can choose C_0 large enough and make the right-hand side of (6.30) positive. This concludes the proof that $v \geqslant 0$ in $D \cap \{-\eta < x_1 < \eta\}$.

Let us now prove (6.20). A simple computation yields

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x, v) = \lambda \frac{\partial a}{\partial x_1}(x) v + \lambda \frac{\partial b}{\partial x_1}(x)(v + w(x))^p \\ + \lambda b(x) \frac{\partial w}{\partial x_1}(x) p (v + w(x))^{p-1} - C_0 \frac{\partial g}{\partial x_1}(x). \end{aligned}$$

Using (6.14) and the assumption $\lambda > 0$, we get

$$\frac{\partial f}{\partial x_1}(x, v) \leqslant \lambda \frac{\partial b}{\partial x_1}(x)(v + w)^p + \lambda b(x) \frac{\partial w}{\partial x_1}(x) p (v + w)^{p-1} - C_0 \frac{\partial g}{\partial x_1}. \tag{6.31}$$

We consider now two cases.

First, $b(x) \leqslant 0$. In this case, since $\partial b/\partial x_1 \leqslant 0$ in D, it suffices to prove that $\partial w/\partial x_1 \geqslant 0$ (for C_0 large). From (6.21) and taking into account (6.28), we obtain

$$\frac{\partial w}{\partial x_1} = -\frac{\partial H}{\partial x_1} + C_0 \frac{\partial G}{\partial x_1} \leqslant Cm + C_0 \frac{\partial G}{\partial x_1}. \tag{6.32}$$

Now, since $\partial a/\partial x_1 \leqslant 0$ on D, we can apply the moving plane to the equation satisfied by G and derive $\partial G/\partial x_1 < 0$ on $D \cap \{-\eta < x_1 < \eta\}$ (see [27]). Hence, by choosing C_0 large enough, the right-hand side can be made negative.

Now, let us consider the case where $b(x) > 0$. Since

$$b(x) \leqslant C\eta_1 \quad \text{for } -\eta_1 < x < 0, \tag{6.33}$$

we get from (6.31) that

$$\frac{\partial f}{\partial x_1}(x, v) \leqslant -F_1(v + w(x))^p + F_2(v + w(x))^{p-1} - F_3, \tag{6.34}$$

where F_i are strictly positive reals defined as

$$F_1 = \lambda \left| \sup_D \left\{ \frac{\partial b}{\partial x_1} \right\} \right|, \qquad F_2 := C\eta_1, \qquad F_3 = C_0 \inf_D \left\{ \frac{\partial g}{\partial x_1} \right\}.$$

Now, the function

$$F : [0, \infty) \to \mathbb{R}, \quad \xi \mapsto -F_1 \xi^p + F_2 \xi^{p-1} - F_3,$$

satisfies

$$F(0) < 0, \quad F' > 0 \quad \text{near } \xi = 0, \quad \lim_{\xi \to \infty} F(\xi) = -\infty.$$

Therefore, the function F has a maximum that is negative as soon as F_2 is small enough, a condition that can be realized by choosing η_1 small enough. Hence, going back to (6.34) with this choice of η_1, we conclude that Word added – OK?

$$\frac{\partial f}{\partial x_1}(x, v) \leqslant 0 \quad \forall x \in D \cap \{-2\eta_1 < x_1 < \eta\} \quad \forall v > 0. \tag{6.35}$$

Since $v \geqslant 0$, $v = 0$ in $\partial^1 D$ and (6.35) is satisfied, we can apply the moving-plane method to the equation (6.18) to prove that v is monotone decreasing in the x_1 direction on the domain $D \cap \{-\eta_1 < x_1 < \eta\}$ (see, for instance, [27]). At this point, we conclude as in [15, § 3, step 4, deriving the *a priori* bound]. Let us just sketch the proof.

Since the function v is monotone decreasing in the x_1 direction, this is still true if we rotate the x_1-axis by a small angle. Therefore, for any $x_0 \in \Gamma$, there exists Δ_{x_0}, a cone of vertex x_0 and staying to the left of x_0, such that

$$v(x) \geqslant v(x_0) \quad \text{for } x \in \Delta_{x_0}. \tag{6.36}$$

From (6.36), we obtain

$$u(x) + C \geqslant u(x_0) \quad \text{for } x \in \Delta_{x_0}. \tag{6.37}$$

By a similar argument, one can prove that equation (6.37) is true for any point x in a small neighbourhood of Γ. Remarking that the intersection of Δ_{x_0} with the set $\{x \mid b(x) \geqslant \delta_0 > 0\}$ has a positive measure, and combining with the integral estimate (6.1), we get the *a priori* bound in the neighbourhood of Γ.

STEP 3. The *a priori* bound in the region Ω_δ^+.

In this region, the *a priori* bound is obtained by a technique of blow-up introduced in [21] and used in [6, 15]. Since the linear term (i.e. $\lambda a(x) u$) vanishes in the blow-up analysis (see [6] for more details), the proof is as in [15] (see particularly pp. 339, 340). Note that, by step 2, u is bounded on the boundary of Ω_δ^+.

The proof of proposition 6.2 is now completed. □

Proof of theorem 1.4. As in theorem 1.3, the existence of a global branch follows from proposition 3.5 and properties (i) and (ii) of $\mathcal{C}^+$ are proved in proposition 4.4. To prove the property (iii) satisfied by the branch $\mathcal{C}^+$, let (λ_n, u_n) be a sequence of solutions to problem (1.1) such that

$$0 < \Lambda \leqslant \lambda_n, \quad u_n > 0.$$

Proposition 4.3, together with proposition 6.2, implies that (λ_n, u_n) is bounded $\mathbb{R} \times L^\infty_{\text{loc}}(\mathbb{R}^N)$. We shall show that u_n is also bounded in $\mathcal{D}^{1,2}$. Indeed,

$$\int_{\mathbb{R}^N} |\nabla u_n|^2 = \lambda_n \biggl(\int_{\mathbb{R}^N} a(x) u_n^2 + \int_{\mathbb{R}^N} b(x) u_n^{p+1} \biggr). \tag{6.38}$$

By (H3), proposition 6.2 and Hölder's inequality,

$$\begin{aligned}\lambda_n \int_{\mathbb{R}^N} a(x)u_n^2 &= \lambda_n\biggl(\int_{B_R} a(x)u_n^2 + \int_{\mathbb{R}^N\setminus B_R} a(x)u_n^2\biggr)\\ &\leqslant C(C(\Lambda,R)\|a\|_\infty + \|a\|_{L^{N/2}(\mathbb{R}^N/B_R)}\|u\|^2_{L^{2^*}(\mathbb{R}^N)})\\ &\leqslant C_0(\Lambda,R) + C\|a\|_{L^{N/2}(\mathbb{R}^N/B_R)}\|u\|^2.\end{aligned}$$

Hence, given any $\epsilon > 0$, we can choose R large enough to ensure

$$\lambda_n \int_{\mathbb{R}^N} a(x)u_n^2 \leqslant C_0(\Lambda,R) + \epsilon\|u\|^2. \tag{6.39}$$

By (H5), $\{b > 0\}$ is bounded. Then, from proposition 6.2,

$$\int_{\mathbb{R}^N} b(x)u_n^{p+1} \leqslant \int_{\{b>0\}} b(x)u_n^{p+1} \leqslant C_0. \tag{6.40}$$

From (6.38)–(6.40) and choosing $\epsilon < 1$, we get

$$\|u_n\|_{\mathcal{D}^{1,2}} \leqslant C_0 \quad \text{if } \lambda_n \geqslant \Lambda > 0. \tag{6.41}$$

Therefore, on the one hand, $\mathcal{C}^+$ is unbounded and, on the other hand, equation (6.41) shows that $\mathcal{C}^+ \cap \{\lambda \geqslant \Lambda\}$, proving (iii). The proof of theorem 1.4 is now completed. □

7. Superlinear case: branches in L^∞

Let us prove now theorem 1.5. Working in $\mathrm{L}^\infty(\mathbb{R}^N)$, we can not apply theorem 3.1 directly, due to the lack of proper functional framework for a compact operator. Following the same approach as in [9], the method we use involves studying a 'local problem', (P_{B_R}), in a ball B_R centred at 0 and with radius R,

$$\left.\begin{aligned}-\Delta u = \lambda(a(x)u + b(x)u^p) \quad &\text{in } B_R,\\ u \in H_0^1(B_R), \quad u \geqslant 0,&\end{aligned}\right\} \tag{P_{B_R}}$$

and then we pass to the limit when R goes to $+\infty$. From proposition 6.2, theorem 3.1 and results in previous sections, we have the following.

PROPOSITION 7.1. *Suppose that (H1), (H2)(SL), (H3), (H5) and (H6) are satisfied and that B_R is large enough that $\Gamma \subset B_R$. Let $\lambda_1^+(a,B_R)$, $\lambda_1^-(a,B_R)$ be the eigenvalues defined in* § 2 *(see definition 2.1). Then there exist two global branches, $\mathcal{C}_R^+$ and $\mathcal{C}_R^-$, belonging to $\bar{\mathcal{S}}$ and connected in $\mathbb{R} \times \mathcal{C}_o(B_R)$ (the space of continuous functions vanishing on the boundary of B_R), such that the following hold.*

(i) *$\mathcal{C}_R^+$ (respectively, $\mathcal{C}_R^-$) bifurcates from $\lambda_1^+(a,B_R)$ (respectively, $\lambda_1^-(a,B_R)$).*

(ii) *$\forall u \in \mathcal{C}_R^+ \cup \mathcal{C}_R^-$, we have $|u| > 0$ or $u \equiv 0$,*

$$\begin{aligned}&\mathcal{C}_R^+ \cap \mathcal{C}_R^- = \emptyset,\\ &\Pi_{\mathbb{R}}\mathcal{C}_R^+ \subseteq (0, \lambda_1^+(a, \operatorname{supp} b^+ \cap B_R)],\\ &\Pi_{\mathbb{R}}\mathcal{C}_R^- \subseteq (\lambda_1^-(a, \operatorname{supp} b^- \cap B_R), 0].\end{aligned}$$

(iii) *If* $(\lambda_n, u_n) \in \mathcal{C}_R^+$ *and* $\|u_n\|_{L^\infty(R^N)} \to +\infty$*, then* $\lambda_n \to 0^+$*.*

Consider the branches $\mathcal{C}_R^+$ and $\mathcal{C}_R^-$ given by proposition 7.1. We will pass to the limit $\mathcal{C}_R^+$ and $\mathcal{C}_R^-$, letting $R \to +\infty$. For this, we need the following results.

Definition 7.2 (Whyburn). Let G be any infinite collection of point sets of a topological space X. The set of all points $x \in X$ such that every neighbourhood of x intersects infinitely many sets of G is called the superior limit of G ($\limsup G$).

The set of all points $y \in X$ such that every neighbourhood of y intersects all but a finite number of sets of G is called the inferior limit of G ($\liminf G$).

Theorem 7.3 (Whyburn). *Let* $\{A_n\}_{n\in\mathbb{N}}$ *be a sequence of connected closed sets of a complete metric space such that*

$$\liminf\{A_n\} \not\equiv \emptyset \quad \textit{and} \quad \bigcup_{n\in\mathbb{N}} A_n \textit{ is relatively compact.}$$

Then $\limsup\{A_n\}$ *is a non-empty closed connected set.*

We apply theorem 7.3 as follows. Set $\Lambda > 0$, $R_n \to +\infty$ and let A_n be the connected component in

$$\{\Lambda \leqslant \lambda\} \times \mathrm{L}^\infty(\mathbb{R}^N)$$

bifurcating from $\lambda_1^+(a, B_{R_n})$ (therefore, $A_n \subseteq \mathcal{C}_{R_n}^+$).

Proving that $\bigcup_{n\in\mathbb{N}} A_n$ is relatively compact in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$ and applying theorem 7.3, we obtain that $\limsup_{n\to\infty} A_n = \mathcal{C}_\Lambda$ is a connected set of non-trivial solutions of (1.1) in $\mathbb{R} \times L^\infty(\mathbb{R}^N)$. Passing to the limit $\Lambda \to 0$, we prove that $\mathcal{C}^+ := \lim_{\Lambda\to 0} \mathcal{C}_\Lambda$ is a global unbounded branch of non-trivial solutions of (1.1) bifurcating from $\lambda_1^+(a, \mathbb{R}^N)$. The important step in this process is to prove that the *a priori* bound, proved in proposition 6.2, does not depend on R, so that the relative compactness of $\bigcup_{n\in\mathbb{N}} A_n$ can be deduced. More precisely, we have the following result.

Proposition 7.4. *Assume that (H2)(SL), (H3), (H6) and (H7) are satisfied. Let* $\Lambda > 0$ *and* R *large enough. Then, for any solution* (λ, u) *to* (P_{B_R}) *such that* $\lambda \geqslant \Lambda$ *satisfies*

$$\|u\|_{L^\infty(\mathbb{R}^N)} \leqslant C(\Lambda)$$

and, independent of R*,*

$$u(x) \to 0,$$

when $|x|$ *tends to* ∞*.*

Proof. By the estimate in Ω_δ (see (6.1) and (6.8) in the proof of proposition 6.2), we can prove that

$$\int_{B_\epsilon(y)} u^p \leqslant C := \frac{C(\|a\|_\infty, \epsilon)}{|\lambda| \inf_{B_\epsilon(y)} |b|^{p/(p-1)}}.$$

If (H7) holds, then, for any $\mu > 0$, there exists $M > 0$ such that $|y| \geqslant M$ implies that

$$\int_{B_\epsilon(y)} u^p \leqslant \mu. \tag{7.1}$$

Then proposition 7.4 follows from proposition 6.2 and (7.1). □

Using proposition 7.4, we prove theorem 1.5.

Proof of theorem 1.5. Assertion (i) follows from theorem 7.3 and proposition 7.4, which ensures the compactness of $\bigcup_{n\in\mathbb{N}} A_n$. Note that $\lambda_1^+(a, B_{R_n}) \to \lambda_1^+(a, \mathbb{R}^N)$ when $n \to +\infty$, which implies that

$$(\lambda_1^+(a, B_{R_n}), 0) \in \liminf_{n\to+\infty} A_n \neq \emptyset.$$

The proof of assertion (ii) is the same as in theorem 1.4. Let us prove (iii). From proposition 7.4, for any $\Lambda > 0$, the solutions in $\mathcal{C}^+$ are uniformly bounded for $\lambda \geqslant \Lambda$. Moreover, from proposition 7.1 and assertion (i), $\mathcal{C}^+$ is unbounded. Then, if $\|u_n\|_{L^\infty} \to +\infty$, assertion (iii) follows. Now, if $\|u_n\|_{\mathcal{D}^{1,2}} \to +\infty$, it is easy to prove that $\|u_n\|_{L^\infty} \to +\infty$ (see, for instance, equations (6.39), (6.40), (6.41)). This proves assertion (iii) and the proof of theorem 1.5 is now complete. □

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