Radial Singular Solutions of a Critical Problem in a Ball

Jean DOLBEAULT[1], Maria ESTEBAN[1]
& Mythily RAMASWAMY[2]

[1] CEREMADE, U.M.R. C.N.R.S. no. 7534,
Université Paris IX-Dauphine
Pl. du Maréchal de Lattre de Tassigny,
75775 Paris Cédex 16, France
dolbeaul, esteban@ceremade.dauphine.fr

[2] T.I.F.R. Center
P.O. Box No. 1234
IISc. Campus
Bangalore, 560012, India

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Abstract

This paper is devoted to a complete classification of the radial singular and possibly sign changing solutions of a semilinear elliptic equation with critical or subcritical growth.

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1 Introduction

The qualitative behavior of the branches of bounded radial solutions of

$$\Delta u + \lambda f(u) = 0 \quad \text{in} \;\; B \,, \qquad u = 0 \quad \text{on} \quad \partial B \,, \tag{1}$$

where $B \subset \mathbb{R}^d$ is the unit ball, and the singularity at the origin of the unbounded radial solutions of

$$\Delta u + \lambda f(u) = 0 \quad \text{in} \quad B^* = B(0,1) \setminus \{0\} \subset \mathbb{R}^d \tag{2}$$

have been extensively studied. However the structure of the set of the radial singular and possibly sign changing solutions of (2) is not so well-known. Actually, a complete classification of these solutions has been achieved only for power law nonlinearities in the whole space [23] or in the ball [5], even though in many cases all solutions of (2) are distributional solutions of (1).

For any real value of the parameter λ, an uncountable number of radial unbounded solutions exists, as can easily be shown by a shooting argument. The goal of this article is to provide for general nonlinearities such that

$$f(u) \sim |u|^{p-1}u \quad \text{as } u \to \pm\infty$$

a complete classification similar to the one given in [5] for the power law case:

$$-\Delta u = |u|^{p-1}u + \lambda u \ , \tag{3}$$

for $\lambda > 0$. Although the results of this paper are similar to those obtained in [5], we have to introduce a significantly different technique for the proof of our main result, which (in the critical case) essentially says that singular solutions with exactly k zeros exist if, for the same λ, there is also a bounded solution with k zeros. There are also radial unbounded solutions which are oscillating (and sign changing) near 0 (and such solutions become generic for $p > \frac{d+2}{d-2}$, but here we will not study this case, which involves other tools).

Actually, we will impose the following restriction on the range of p :

$$\frac{d}{d-2} < p \leq \frac{d+2}{d-2} \ ,$$

which corresponds to the most interesting case, and refer to [5] for a precise description of the other cases (for a power law nonlinearity).

Let us briefly mention a few key papers of the literature concerning mostly branches of bounded solutions of (3) and refer to [5] for more details. These branches have been constructed for instance in [22] and studied in the subcritical case in [6, 11, 15, 24, 17]. Concerning the behavior of the solutions at the singularity, we may refer to [16, 18, 19, 7, 21, 20]. The study of the branches of solutions in the critical case is slightly more delicate and the results strongly depend on the dimension: see for instance [8, 2, 3, 12, 13, 10, 4, 1].

2 Main results

We will distinguish the cases of critical and subcritical nonlinearities. In both cases, the results hold under the following hypothesis:

Assume that f is locally Lipschitz on $\mathbb{R}$ and such that

$$\left|\frac{f(u)}{|u|^{p-1}u} - 1\right| \leq g(|u|) \qquad (H)$$

for some $p \in (\frac{d}{d-2}, \frac{d+2}{d-2}]$ $(d > 2)$ and for a function g such that $\lim_{s\to+\infty} g(s) = 0$, $s \mapsto g(s)s^{-1} \in L^1(1, +\infty)$ and $s \mapsto g(s)s^{p-1}$ is nondecreasing.

Theorem 1 *Under assumption (H), if $d/(d-2) < p < (d+2)/(d-2)$, $\lambda > 0$, any radial solution of (2) has a finite number of zeros in the interval $(0, 1)$, and for any $\lambda > 0$ there exists an uncountable set of unbounded radial solutions of (2), which all behave at the origin like $a(|x|)\,|x|^{-2/(p-1)}$ for some bounded positive or negative function a. All radial solutions of (2) are also distributional solutions of (1).*

In the critical case, the situation is slightly more delicate.

Theorem 2 *Let $\lambda > 0$, $p = (d+2)/(d-2)$ and assume that f satisfies (H) and is such that*

$$uf(u) - \frac{2d}{d-2}\int_0^u f(s)\,ds \leq 0 \quad \forall\, u \in \mathbb{R}\,,$$

and for any $r > 0$, $sup_{|s|>r}\,|s|^{-4/(d-2)}\,|f'(s)| < +\infty$.

If (2) has no bounded radial solution, there is an uncountable number of unbounded radial solutions which are all oscillating near the origin (i.e. which have infinitely many zeros accumulating at 0). No solution of (2) with a finite number of zeros exists.

If there exists one bounded radial solution of (2) with k zeros in $(0, 1)$ and if λ is not a boundary point of the interval for which such solutions exist, then

- *there exists an uncountable number of unbounded radial solutions of (2) with k zeros in $(0, 1)$,*
- *there exists an uncountable number of unbounded radial oscillating solutions of (2).*

All these solutions are also distributional solutions of (1).

The rest of this paper is devoted to the proofs of these two results. First, we introduce a change of variables, which transforms the study of the singularity at the origin into the description of the asymptotic behaviour of a dynamical system. The key tool is an *asymptotic energy* which is now standard (see [9] and [5] in the power law case). The two new ingredients, which are crucial for the proof of Theorem 2 (critical case), are a property of order preservation in the phase space and a convenient parametrization of the set of the bounded solutions. It is noticeable that we completely avoid any use of uniqueness results for positive solutions in balls or annuli, which was a crucial tool in [5].

Notations. For any function f, defined in $\mathbb{R}$, f' denotes its derivative and we will write undistinctly $u(x)$ or $u(r)$, $r = |x|$, for any radially symmetric function u defined in $\mathbb{R}^d$ or in $B := \{x \in \mathbb{R}^d \,:\, |x| < 1\}$.

Throughout the paper, λ is a positive real parameter.

3 Preliminary results

In this section, we prove a series of intermediate results which are used in the proofs of Theorems 1 and 2. We shall detail the minimal assumptions needed in each case, which in several cases are weaker than (H).

Let u be a radially symmetric solution of (2) with $u(1) = 0$, $d > 2$ and $\lambda > 0$. Assuming that f is locally Lipschitz, u is uniquely defined as a solution of the O.D.E.

$$u'' + \frac{d-1}{r}\, u' + \lambda f(u) = 0 \;, \quad u(1) = 0 \;, \quad u'(1) = -\gamma \tag{4}$$

with $r \in (0, 1]$. We assume $d > 2$, $\frac{d}{d-2} < p \leq \frac{d+2}{d-2}$ (critical and subcritical cases) and

$$\lim_{u \to \pm\infty} \frac{f(u)}{|u|^{p-1}u} = 1 \;, \tag{$H1$}$$

and consider

$$w(t) = r^{\frac{2}{p-1}} u(r) \;, \quad \text{with } r = e^{-t} \tag{5}$$

which is the solution of the O.D.E. problem in $(0, +\infty)$

$$
\begin{cases} w'' + L_{p,d}\, w' + \lambda e^{-\frac{2pt}{(p-1)}} f(e^{\frac{2t}{(p-1)}}\, w) + 2\, \frac{p+1-(d-1)(p-1)}{(p-1)^2}\, w = 0 \\ w(0) = 0\ ,\ w'(0) = \gamma\ , \end{cases} \tag{6}
$$

with $L_{p,d} = \frac{4}{p-1} - d + 2$ (note that for $p = (d+2)/(d-2)$, $L_{p,d} = 0$). For the solution of (6), the *energy functional*

$$
E[t,w] = \frac{1}{2} w'^2(t) + V(t, w(t)) + L_{p,d} \int_0^t |w'(s)|^2 ds\,,
$$

where

$$
V(t,w) = \lambda e^{-\frac{2(p+1)t}{p-1}} F(e^{\frac{2t}{(p-1)}} w) + \tfrac{p+1-(d-1)(p-1)}{(p-1)^2}\, w^2,
$$

and

$$
F(u) = \int_0^u f(s)\; ds,
$$

satisfies

$$
\frac{d}{dt}\Big(E[t,w](t) \Big) = \tfrac{2\lambda}{p-1} e^{-\frac{2(p+1)t}{p-1}} H(e^{\frac{2t}{(p-1)}} w(t)) \tag{7}
$$

where

$$
H(u) = uf(u) - (p+1)F(u)\ .
$$

To emphasize the dependence on γ, we shall also note

$$
E_\gamma(t) = E[t, w_\gamma]
$$

this “energy” in the case of a solution w_γ of (6). It is straightforward that

$$
\lim_{s\to\pm\infty} \frac{H(s)}{|s|^{p+1}} = 0\,,
$$

because of $(H1)$. Next, we make a further assumption on f which makes the asymptotic behaviour of H more precise:

$$
\frac{|H(u)|}{|u|^{p+1}} \le \varphi(|u|), \ \text{ for all } \ u\,, \tag{$H2$}
$$

for some continuous function $\varphi : (0,+\infty) \to \mathbb{R}^+$ such that $\lim_{s\to+\infty} \varphi(s) = 0$, $s \mapsto \varphi(s)s^{-1} \in L^1(M,+\infty)$ for some $M > 0$ and the map $s \mapsto \varphi(s)\, s^{p+1}$ is nondecreasing in $(0,+\infty)$.

Remark 3 *Typical examples of functions* φ *satisfying* $(H2)$ *are:*

- $\varphi(s) = \alpha|s|^{-\epsilon}, \quad \alpha > 0\,, \quad 0 < \epsilon < p+1\,,$ *which corresponds in case of an equality in (H2) to*

$$f(u) = |u|^{p-1}u \pm \alpha\,\frac{q+1}{p-q}|u|^{q-1}u\ ,$$

  with $-1 < q = p - \epsilon < p$.

  Note that in all our results we asume that f *is locally Lipschitz in* $\mathbb{R}$, *which excludes the interval* $q \in (-1,1)$ *in this example.*

- $\varphi(s) = |\log(\beta{+}s)|^a, \quad \beta > 1,\ a < -1,$ *and either* $a{+}(p{+}1)(1{+}\log\beta) \geq 0$ *or* $a{+}(p{+}1)(1{+}\log\beta) < 0$ *and* $a{+}(p{+}1)(1{+}\log(|a|\beta/(p{+}1)))) \geq 0$.

Remark 4 *Notice that if assumption* $(H2)$ *is satisfied, then for all* $\gamma > 0$, *for all* $0 \leq s < t \leq +\infty$,

$$\begin{aligned} |E_\gamma(t) - E_\gamma(s)| \ &\leq\ \frac{2\lambda}{p-1}\int_s^t |w(\sigma)|^{p+1}\varphi\left(e^{\frac{2\sigma}{p-1}}|w(\sigma)|\right)d\sigma \\ &\leq\ \lambda\|w\|^{p+1}_{L^\infty(s,t)}\int_{e^{\frac{2s}{(p-1)}}\|w\|_{L^\infty(s,t)}}^{e^{\frac{2t}{(p-1)}}\|w\|_{L^\infty(s,t)}} \varphi(x)x^{-1}dx\,. \end{aligned}$$

So denoting $\phi(y) = \int_y^{+\infty}\varphi(x)\,x^{-1}\,dx$,

$$|E_\gamma(t) - E_\gamma(s)| \leq \lambda\|w\|^{p+1}_{L^\infty(s,t)}\left(\phi(e^{\frac{2s}{p-1}}\|w\|_{L^\infty(s,t)}) - \phi(e^{\frac{2t}{p-1}}\|w\|_{L^\infty(s,t)})\right)\,.$$

Proposition 5 *Let* $d/(d-2) < p \leq (d+2)/(d-2)$, $\lambda > 0$. *Under assumptions* $(H1)$ *and* $(H2)$, *the following properties hold.*

1. *Any solution* w_γ *of (6) is bounded on* $\mathbb{R}^+$.
2. $E_\gamma(t)$ *has a finite limit as* $t \to +\infty$. *We will denote it by* $\mathcal{E}(\gamma,\lambda)$.
3. *The map* $(\gamma,\lambda) \mapsto \mathcal{E}(\gamma,\lambda)$ *is continuous.*

Proof. Assume by contradiction that w_γ is not bounded on $\mathbb{R}^+$ for some $\gamma > 0$. Hence there exist a sequence $(t_m)_{m\in\mathbb{N}}$ of positive numbers such that

$$|w_\gamma(t_m)| \underset{m}{\to} +\infty, \quad |w_\gamma(t_m)| \geq |w_\gamma(s)|, \quad \forall\ s \in [0,t_m]\,.$$

By Remark 4,

$$|E_\gamma(t_m) - E_\gamma(0)| = |E_\gamma(t_m) - \frac{\gamma^2}{2}| \leq \lambda\, |w_\gamma(t_m)|^{p+1}\phi(|w_\gamma(t_m)|),$$

and $\phi(|w_\gamma(t_m)|) \to 0$ as $m \to +\infty$. But on the other hand, by $(H1)$, we have

$$E_\gamma(t_m) \geq \frac{\lambda}{2(p+1)}|w_\gamma(t_m)|^{p+1},$$

for m large enough, a contradiction. Hence, $w_\gamma \in L^\infty(\mathbb{R}^+)$ for all $\gamma > 0$.

That $E_\gamma(t)$ has a limit as t goes to $+\infty$, follows from $(H2)$ and Remark 4. Denoting by $\mathcal{E}(\gamma,\lambda)$ the limit of $E_\gamma(t)$ as t goes to $+\infty$, and using again $(H2)$, we may write

$$|E_\gamma(t) - \mathcal{E}(\gamma,\lambda)| \leq \lambda\, \|w_\gamma\|_\infty^{p+1}\phi\left(e^{\frac{2t}{p-1}}\, \|w_\gamma\|_\infty\right).$$

But this, together with $(H2)$ and the continuity of w_γ with respect to γ and λ in any fixed interval $(0,\bar{t})$, proves the continuity of the map $(\gamma,\lambda) \to \mathcal{E}(\gamma,\lambda)$. Indeed, $\|w_\gamma\|_\infty$ depends continuously on (γ,λ) and $\phi\left(e^{\frac{2t}{p-1}}\, \|w_\gamma\|_\infty\right) \underset{t\to+\infty}{\longrightarrow} 0$. □

We now investigate the asymptotic behaviour of $w_\gamma(t)$ for t large.

Proposition 6 *Let $d/(d-2) < p \leq (d+2)/(d-2)$, $\lambda > 0$ and $\gamma > 0$. Under assumptions $(H1)$ and $(H2)$, there exists a solution $\widetilde{w}$ of the autonomous equation*

$$w'' + L_{p,d}\, w' + \lambda|w|^{p-1}w + 2\, \tfrac{p+1-(d-1)(p-1)}{(p-1)^2}\, w = 0, \qquad (8)$$

such that

$$\limsup_{s\to+\infty}\, \|w_\gamma - \widetilde{w}\|_{W^{1,\infty}(s,+\infty)} = 0\,.$$

If $d/(d-2) < p < (d+2)/(d-2)$, $\widetilde{w}$ is constant. If $p = (d+2)/(d-2)$, we may define the energy *functional $E^\infty[w]$ by*

$$E^\infty[w] := \frac{|w'|^2}{2} + \frac{c\,|w|^2}{2} + \frac{\lambda|w|^{p+1}}{p+1}$$

with $c = -(d-2)^2/4$. Then the map $t \to E^\infty[\widetilde{w}](t)$ is constant in $\mathbb{R}^+$ and the function $\widetilde{w}$ is periodic.

Remark 7 *When $p \in (\frac{d}{d-2}, \frac{d+2}{d-2})$, $L_{p,d}$ is positive and the only possible periodic solutions of (8) are the constant functions, $w \equiv 0$ and $w \equiv \pm(-\frac{c}{\lambda})^{1/(p-1)}$, which are the only critical points of the function $w \to V^\infty(t,w) := \frac{c\,|w|^2}{2} + \frac{\lambda |w|^{p+1}}{p+1}$. In the critical case $p = \frac{d+2}{d-2}$, there are periodic solutions of (8) which are not constant.*

Proof. Let us choose $\gamma \in \mathbb{R}$ and define for every $m \in \mathbb{N}$,

$$w^m(\cdot) := w_\gamma(\cdot + m).$$

The sequence $\{w^m\}$, is uniformly bounded in $L^\infty(\mathbb{R}^+)$. Furthermore, since $E_\gamma(t) \underset{t\to+\infty}{\longrightarrow} \mathcal{E}(\gamma,\lambda) \in \mathbb{R}$, the sequence $\{(w^m)'\}$ is also uniformly bounded in $L^\infty(\mathbb{R}^+)$ and the same holds for $\{(w^m)''\}$ by Equation (6).

Hence Ascoli-Arzela's Theorem implies the existence of some function $\widetilde{w}$ such that a subsequence of $\{w^m\}$, still denoted by $\{w^m\}$, converges locally in $C^{1,\alpha}(\mathbb{R}^+)$, for all $\alpha \in (0,1)$. To identify $\widetilde{w}$, for any $\eta > 0$, we define the set $A_\eta := \{x \in \mathbb{R}^+ \;:\; |\widetilde{w}(x)| \geq \eta\}$. Then for any $\chi \in \mathcal{D}(\mathbb{R}^+)$,

$$0 = \int_0^{+\infty} \chi(t) \left((w^m)'' + L_{p,d}\,(w^m)' + c\,w^m + \lambda e^{-\frac{2p(t+m)}{p-1}} f(e^{\frac{2(t+m)}{p-1}}\,w^m) \right) dt.$$

with $c = 2\,\frac{(p+1)-(d-1)(p-1)}{(p-1)^2}$. Clearly

$$\int_0^{+\infty} \chi(t) \Big((w^m)''(t) \;+\; L_{p,d}\,(w^m)' + c\,w^m(t) \Big) dt \tag{9}$$

$$\overset{m\to+\infty}{\longrightarrow} \int_0^{\infty} \chi(t) \Big(\widetilde{w}''(t) + L_{p,d}\,\widetilde{w}'(t) + c\,\widetilde{w}(t) \Big) dt\,.$$

Now, for any $\eta > 0$, by $(H1)$, $|f(t)| \leq C\,(1 + t^p)$ for any $t > 0$, for some C large enough:

$$\left| \int_{\mathbb{R}^+ \setminus A_\eta} \chi(t)\, e^{-\frac{2p(t+m)}{(p-1)}} f(e^{\frac{2(t+m)}{(p-1)}}\,w^m(t))\,dt \right| \leq C e^{-\frac{2pm}{(p-1)}} + C\eta^p.$$

On the other hand, on A_η, $|w^m| \geq \frac{\eta}{2}$ for m large enough, and so using $(H1)$,

$$\int_{A_\eta} \chi(t)\, e^{-\frac{2p(t+m)}{(p-1)}} f(e^{\frac{2(t+m)}{(p-1)}}\,w^m(t))\,dt \;=\; \int_{A_\eta \cap\, \mathrm{supp}\,\chi} |w^m|^{p-1} w^m\,\chi\,dt$$

$$+\; c_m \int_{A_\eta \cap\, \mathrm{supp}\,\chi} |w^m|^p\,\chi\,dt\,,$$

with $c_m \to 0$ as $m \to +\infty$. So, the uniform local convergence of w^m towards $\widetilde{w}$ and Lebesgue's Theorem imply that for m large enough,

$$0 = \lim_{m \to +\infty} \int_0^{+\infty} \chi(t) \left[e^{-\frac{2p\,(t+m)}{p-1}} f\left(e^{\frac{2\,(t+m)}{p-1}} w^m(t) \right) - |\widetilde{w}|^{p-1}\widetilde{w} \right] dt. \tag{10}$$

Finally, (9) and (10) imply that $\widetilde{w}$ is a solution to (8). Moreover, using Proposition 5 and the arguments above, one easily proves that

$$E^\infty[\widetilde{w}](t) = \frac{|\widetilde{w}'(t)|^2}{2} + \frac{c\,|\widetilde{w}(t)|^2}{2} + \frac{\lambda|\widetilde{w}(t)|^{p+1}}{p+1} \equiv \mathcal{E}(\gamma,\lambda) - L_{p,d} \int_0^{+\infty} |w'(s)|^2 ds \tag{11}$$

for all $t \in \mathbb{R}$, which immediately implies that $\widetilde{w}$ is periodic. Moreover, up to translation, there is a unique solution of (8) for every given positive value of $E^\infty[\widetilde{w}]$. Then the statement of Proposition 6 easily follows. $\square$

When the asymptotics of the solutions to (6) is given by $\widetilde{w} \equiv 0$, we can describe more precisely the behavior at infinity. In order to do that, we make a new assumption on f : assume *the existence of a continuous function* $g : (0, +\infty) \to \mathbb{R}^+$ *such that* $g(s) \to 0$ *as* $s \to +\infty$, $g(s)s^{-1} \in L^1(M, +\infty)$ *for some* $M > 0$, $g(s)s^{p-1}$ *is nondecreasing in* $s > 0$ *and for all* $u \in \mathbb{R}$,

$$|f(u) - |u|^{p-1}u| \leq g(|u|)|u|^p. \tag{$H3$}$$

Note once again that this assumption is satisfied for instance by $g(s) = \alpha|s|^{-\epsilon}$, $\alpha \in \mathbb{R}^+$, $0 < \epsilon \leq p-1$ and by $g(s) = |\log(\beta + s)|^a$, with $\beta > 1$, $a < -1$, and either $a + (p-1)(1 + \log\beta) \geq 0$ or $a + (p-1)(1 + \log\beta) < 0$ and $a + (p-1)(1 + \log(|a|\beta/(p-1))) \geq 0$.

Remark 8 *It can be easily seen that if f satisfies $(H3)$, then $(H1)$ and $(H2)$ are also satisfied.*

We can now prove:

Proposition 9 *Let $d/(d-2) < p \leq (d+2)/(d-2)$ and assume that f satisfies $(H3)$. If w is a solution to (6) which converges asymptotically to 0, there exists a constant $C \neq 0$ such that as s goes to $+\infty$, $w(s) \sim C e^{-\frac{2s}{p-1}}$.*

The proof of the above result can be done by following the same arguments as in Lemma 2.11 of [5], which derive from classical results on the asymptotic behaviour of linear O.D.E.'s (more precisely, Theorem 8.1 in [14]). Important elements of the proof are the boundedness of w (see Proposition 5) and the fact that whenever a solution to (6), w, converges to 0 as $t \to +\infty$, neither $w(t)$ nor $w'(t)$ change sign for t large.

Remark 10 *If u is a solution of (4) and the solution of (6), w, defined by (5) is asymptotic to 0 at infinity, then $u \in L^\infty(B_1)$.*

Corollary 11 *Assume $p = (d+2)/(d-2)$ and assumption $(H3)$. If for some $\gamma > 0$, $\mathcal{E}(\gamma, \lambda) = 0$, then,*

$$\lim_{t \to +\infty} e^{\frac{d-2}{2}t} w_\gamma(t) \in \mathbb{R},$$

and therefore, the corresponding solution of (2), u_γ, is bounded.

Proof. By Proposition 6 and (11), if $\mathcal{E}(\gamma, \lambda) = 0$, then the asymptotic behavior of w_γ is described by $\widetilde{w} \equiv 0$. Then, we just apply Proposition 9 to conclude. $\square$

Proposition 12 *If $p = (d+2)/(d-2)$ and assumptions $(H1)$, $(H2)$ hold, then*

$$\lim_{\gamma \to +\infty} \mathcal{E}(\gamma, \lambda) = +\infty\ .$$

Proof. Suppose by contradiction the existence of a sequence $\gamma_m \underset{m \to +\infty}{\longrightarrow} +\infty$ such that $|\mathcal{E}(\gamma_m, \lambda)| \leq C, \quad C > 0$, for all m. If so, the sequence $\{w_{\gamma_m}\}$ is bounded in $L^\infty(\mathbb{R}^+)$. Indeed, $\widetilde{w}_m$, the solutions to (8) which describe the asymptotic behaviour of w_{γ_m}, are uniformly bounded in $\mathbb{R}^+$, which can be easily seen by the analysis of the solution set to (8) and the boundedness of $\mathcal{E}(\gamma_m, \lambda)$. Hence, if the sequence $\{w_{\gamma_m}\}$ is not bounded in $L^\infty(\mathbb{R}^+)$, up to the extraction of a subsequence there is some $b_m \in \mathbb{R}^+$ such that

$$\|w_{\gamma_m}\|_{L^\infty(\mathbb{R}^+)} = |w_{\gamma_m}(b_m)| \quad \underset{m \to +\infty}{\longrightarrow} \quad +\infty\ .$$

Now, by assumption $(H2)$ and Remark 4, we have

$$|E_{\gamma_m}(b_m) - \mathcal{E}(\gamma_m, \lambda)| \leq C + \lambda \|w_{\gamma_m}\|_\infty^{\frac{2d}{d-2}} \phi(w_{\gamma_m}(b_m)) \tag{12}$$

for some $C > 0$, and by $(H1)$,

$$E_{\gamma_m}(b_m) \geq \lambda \, \frac{d-2}{4d} \, |w_{\gamma_m}(b_m)|^{\frac{2d}{d-2}}$$

for m large enough. By the boundedness of $\mathcal{E}(\gamma_m, \lambda)$, this contradicts (12) and shows that $\|w_{\gamma_m}\|_{L^\infty(\mathbb{R}^+)}$ is bounded.

Let $r_m := \inf\{t > 0 \ ; \quad w'_{\gamma_m}(t) = \frac{\gamma_m}{2}\}$. Clearly, $0 < r_m < +\infty$ for all m. Moreover, for some $\theta_m \in [0, 1]$,

$$\frac{\gamma_m}{2} = \Big| w'_{\gamma_m}(r_m) - w'_{\gamma_m}(0) \Big| = \Big| w''_{\gamma_m}(\theta_m r_m) \Big| r_m \leq C \, r_m, \tag{13}$$

for some $C > 0$ independent of m, by equation (6). Finally, for all m,

$$\|w_{\gamma_m}\|_{L^\infty(\mathbb{R}^+)} \geq |w_{\gamma_m}(r_m)| \geq \frac{\gamma_m \, r_m}{2} \geq \frac{\gamma_m^2}{4C}$$

by (13). But the L^∞-norm of w_{γ_m} was shown to be bounded, while γ_m was assumed to be unbounded. This contradiction proves the proposition.

□

Remark 13 *If we make a further assumption on f (or on H), namely if*

$$H(u) \leq 0 \quad \text{for all } \ u, \tag{$H4$}$$

then by (7), $E_\gamma(\cdot)$ is a non-increasing function for all $\gamma > 0$. Therefore, for all $t > 0$, for all $\gamma > 0$, $\quad E_\gamma(t) \leq \frac{\gamma^2}{2}$. Hence,

$$\limsup_{\gamma \to 0^+} \ \mathcal{E}(\gamma, \lambda) \leq 0 \, .$$

An example of function f such that $(H4)$ holds is given by

$$f(u) = |u|^{\frac{4}{d-2}} u + \sum_{i=1}^{l} C_i |u|^{q_i} u,$$

with $C_i > 0, \quad 0 \leq q_i < \frac{4}{d-2}$.

4 Proof of the main results

We are now ready to prove the main results of this paper. First we may notice that (H) is equivalent to $(H3)$ if f is locally Lipschitz. We start with the subcritical case, and next, we will deal with the critical one.

Proof of Theorem 1: By Propositions 6 and Remark 7, for all $\gamma > 0$, $w_\gamma(t)$ converges either to 0 or to one of the constants $\pm(2\frac{(d-1)(p-1)-p-1}{(p-1)^2\lambda})^{1/(p-1)}$ as t goes to $+\infty$. Indeed, those three constants are the only critical points of the function $w \to V^\infty(t,w)$. Hence, by Proposition 9, either u_γ is bounded or it behaves at the origin like $C\,r^{-2/(p-1)}$. Now, assumption $(H1)$ ensures that for a given $\lambda \in \mathbb{R}$, there is an $L^\infty(B_1)$ *a priori* bound on all bounded solutions of (1) (see the arguments in the proof of Lemma 6 in [17] for the L^∞ estimate) and the C^1 bound trivially follows. Hence, for γ large, u_γ is unbounded.

That all solutions of (2) are distributional solutions of (1) follows from Proposition 6 and a simple computation (for a similar argument, see Lemma 2.1 in [5]). $\square$

For any given $k \in \mathbb{N}$, if we define by Λ_b^k and Λ_u^k the sets of parameters $\lambda \in \mathbb{R}$ for which respectively bounded and unbounded radial solutions of (2) with k zeros in $(0,1)$ exist, we may rephrase the main statements of Theorem 2 as follows:

Theorem 14 *Assume that* $p = (d+2)/(d-2)$ *and that* f *is a locally Lipschitz function such that* $(H3)$ *and* $(H4)$ *are satisfied. Assume moreover that for any* $r > 0$, $sup_{|s|>r}\,|s|^{-4/(d-2)}\,|f'(s)| < +\infty$. *Then* Λ_u^k *is open in* $(0,+\infty)$ *and*

$$Int(\Lambda_b^k) \subset \Lambda_u^k \subset \bigcup_{j\geq 0} \Lambda_b^j\ .$$

Here $Int(\Lambda_b^k)$ denotes the interior of Λ_b^k.

Proof of Theorem 2: The assertion concerning the oscillating solutions is a straightforward consequence of Propositions 6 and 12. That all solutions of (2) are distributional solutions of (1) follows from Proposition 6 and a simple computation (for a similar argument, see Lemma 2.1 in [5]). Theorem 2 is then easy to deduce from Theorem 14. $\square$

Proof of Theorem 14:

Let $\bar{\lambda} \neq 0$ be such that there exists a bounded radial solution $\bar{u}$ of (2) with $\lambda = \bar{\lambda}$, which has k zeros in the interval $(0,1)$. Without loss of generality we may assume that $\bar{c} = \bar{u}(0) > 0$ and define $\bar{\gamma} = -\bar{u}'(1)$. Let also $\bar{w}$ denote the corresponding solution of (6) given by (5) with $u = \bar{u}$. Obviously, for t large, $|\bar{w}(t)| + |\bar{w}'(t)|$ is close to 0,

$$\bar{w}''(t) \sim \left(\tfrac{(d-2)^2}{4} - \lambda \frac{f(\bar{c})}{\bar{c}} e^{-2t}\right) \bar{w}(t) \sim \tfrac{(d-2)^2}{4}\, \bar{w}(t) \quad \text{and} \quad \bar{w}(t) \sim \bar{c}\, e^{-\frac{d-2}{2}t}\,.$$

Let us now consider $T > 0$ large, $\delta > 0$ small and define the set

$$\mathcal{V}_{T,\delta}(T) := \{(a,b) \in \mathbb{R}^2\,;\ \bar{w}(T) < a < \bar{w}(T) + \delta,\ \bar{w}'(T) < b < \bar{w}'(T) + \delta\}\,.$$

For every $(a,b) \in \mathcal{V}_{T,\delta}(T)$, we solve the O.D.E. problem

$$z'' + \lambda\, e^{-\frac{d+2}{2}t} f\left(e^{\frac{d-2}{2}t} z\right) - \tfrac{(d-2)^2}{4} z = 0 \ (t \in \mathbb{R}),\ z(T) = a,\ z'(T) = b\,. \tag{14}$$

and we denote by $z_{a,b}$ its unique solution (f is assumed to be locally Lipschitz).

Our strategy is to prove that any trajectory corresponding to (14) with an initial datum in $\mathcal{V}_{T,\delta}(T)$ at $t = T$ either converges to $(0,0)$ or has an asymptotically negative energy as $t \to +\infty$, and has the same number of zeros as $\bar{u}$. This is achieved thanks to an order preserving property (step 1).

Using the flow (14), we pull back these solutions at time $t = 0$ to $\mathcal{V}_{T,\delta}(0)$, which is included in a neighborhood of $(0, \bar{\gamma})$. One shows (step 2) that in this neighborhood, the solutions which are asymptotically converging to 0 as $t \to +\infty$ belong to a one-dimensional manifold of class C^1, by means of a convenient parametrization of the bounded solutions of (2).

The solutions of (14) which at time $t = 0$ are in $\mathcal{V}_{T,\delta}(0)$ cross the axis $w = 0$ of the phase space for t close to 0, or actually at $t = 0$ after a shift and a slight change of λ. Among these solutions, which are parametrized by a two-dimensional manifold of class C^1, there are certainly solutions with an asymptotically negative energy as $t \to +\infty$.

The conclusion (step 3) then holds due to elementary considerations on the topological properties of the sets of solutions.

First step. An order preserving property

By the mean value theorem,

$$\frac{w''-z''_{a,b}}{w-z_{a,b}}-\frac{(d-2)^2}{4}=-\lambda\,\frac{f'(e^{\frac{d-2}{2}t}\,y)}{(e^{\frac{d-2}{2}t}\,y)^{\frac{4}{d-2}}}\,y^{\frac{4}{d-2}}$$

for some function y such that $y(t)\in[w(t),z_{a,b}(t)]$, at least as long as $w(t)\leq z_{a,b}(t)$ for $t>T$. Thus $e^{\frac{d-2}{2}t}\,y(t)\geq e^{\frac{d-2}{2}t}\,w(t)$ (which tends to $\bar{c}>0$ as $t\to+\infty$) and with the assumption

$$\sup_{|s|>\frac{1}{2}\bar{c}}|s|^{-4/(d-2)}\,|f'(s)|<+\infty\ ,$$

which proves that for $e^{\frac{d-2}{2}T}\,w(T)>\frac{1}{2}\bar{c}$, (*i.e.* for T large enough),

$$\frac{w''-z''_{a,b}}{w-z_{a,b}}=\frac{(d-2)^2}{4}+O\left(|w(T)+\delta|^{\frac{4}{d-2}}\right)>0\ ,$$

at least for T large and δ small enough. Hence the sign of $(w''-z''_{a,b})$ and that of $(w-z_{a,b})$ are the same for $t\geq T$, at least as long as $z_{a,b}(t)\leq z_{a,b}(T)\leq w(T)+\delta$, which is certainly true as long as $z'_{a,b}(t)<0$.

Since at $t=T$, $0<\bar{w}(T)<z_{a,b}(T)$, $\bar{w}'(T)<z'_{a,b}(T)<0$, if we define

$$T_{a,b}:=\sup\{t>T\,;\ 0<\bar{w}(t)<z_{a,b}(t),\ \bar{w}'(t)<z'_{a,b}(t)<0\}\ ,$$

there are two possibilities:

(i) either $T_{a,b}=+\infty$ and $|z_{a,b}(t)|+|z'_{a,b}(t)|\ \underset{t\to+\infty}{\longrightarrow}\ 0\,,$

(ii) or $T_{a,b}<+\infty$, $z_{a,b}(T_{a,b})>0$, $z'_{a,b}(T_{a,b})=0$.

Let $E^z(t)=\frac{1}{2}z'^2+V(t,z(t))$. In the first case, $\lim_{t\to+\infty}E^{z_{a,b}}(t)=0$. In the latter, by $(H1)$,

$$E^{z_{a,b}}(T_{a,b})\leq C\,|z_{a,b}(T_{a,b})|^{\frac{2d}{d-2}}-\tfrac{(d-2)^2}{8}\,|z_{a,b}(T_{a,b})|^2<0\,,$$

if T is chosen large and δ small enough. Therefore, by $(H4)$ and (7), if (ii) holds, then

$$\frac{d}{dt}\,(E^{z_{a,b}}(t))=\lambda\,\tfrac{d-2}{2}\,e^{-\frac{d+2}{2}t}H(e^{\frac{d-2}{2}t}z_{a,b})\leq 0\quad\forall\,t\geq T\ ,$$

and therefore,

$$\lim_{t\to+\infty}E^{z_{a,b}}(t)<0\ .$$

However, in both cases the number of zeros is the same:

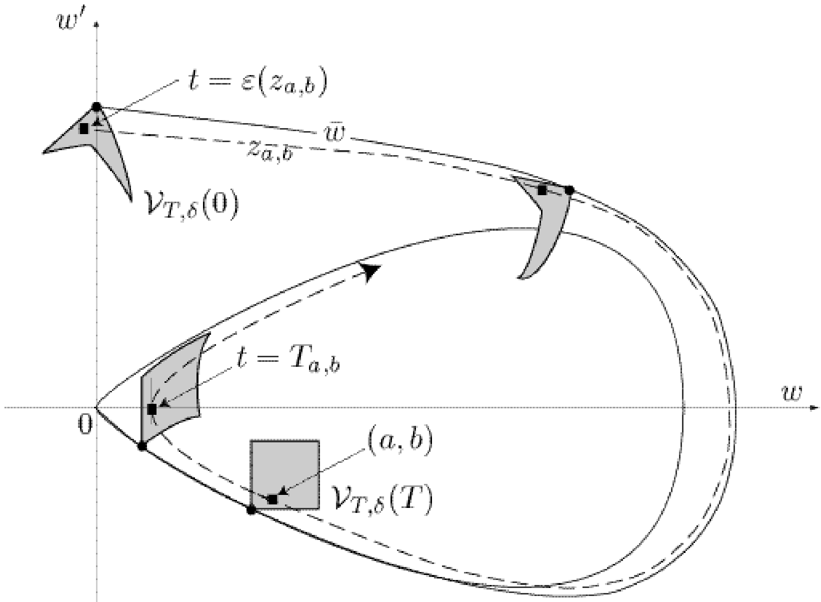
Proposition 15 *With the above notations, for T large, if $\delta > 0$ is small enough, for any $(a,b) \in \mathcal{V}_{T,\delta}(T)$, $z_{a,b}$ has the same number of zeros in $(0,+\infty)$ as $\bar{u}$.*

The proof easily follows by the continuity properties of the solutions of O.D.E.s with Lipschitz coefficients and the above discussion.

Second step. The phase space at $t = 0$.

Let $\mathcal{V}_{T,\delta}(0)$ be the image of $\mathcal{V}_{T,\delta}(T)$ by the flow (14) at $t = 0$:

$$(x,y) \in \mathcal{V}_{T,\delta}(0) \iff \exists (a,b) \in \mathcal{V}_{T,\delta}(T) \text{ such that } z_{a,b}(0) = x \text{ and } z'_{a,b}(0) = y.$$



Motion of the set $\mathcal{V}_{T,\delta}(\cdot)$ under the flow of equation (14).

Consider now the set S_η of the solutions of

$$w'' + \lambda\, e^{-\frac{d+2}{2}t} f\left(e^{\frac{d-2}{2}t} w\right) - \tfrac{(d-2)^2}{4} w = 0\,,\ t \in \mathbb{R}\,, \tag{15}$$

such that $|w(0)|^2 + |w'(0) - \bar{\gamma}|^2 < \eta^2$. In what follows, we will identify S_η and the corresponding set of initial data: $B((0,\bar{\gamma}),\eta) \subset \mathbb{R}^2$. For $\delta > 0$ small enough, it is clear that $\mathcal{V}_{T,\delta}(0)$ is contained in S_η.

Proposition 16 *For $\eta > 0$ small enough,*

$$\forall\, w \in S_\eta\ , \quad \exists \varepsilon =: \varepsilon(w) \quad \textit{such that} \quad w(\varepsilon) = 0\ ,$$

and

$$\lim_{\substack{\eta\to 0\\ \eta>0}} \sup_{w\in S_\eta} \varepsilon(w) = 0\ .$$

Proof. Since by assumption, $\bar{u}(0) > 0$ and by the uniqueness of the solution of (15) due to the Lipschitz regularity of f, certainly $\bar{w}'(0) \neq 0$. A straightforward analysis of the phase space then gives the result. □

For each $w \in S_\eta$, we may build a solution $\widetilde{w}$ of (6) (with $\widetilde{w}(0) = 0$) up to a small change of λ. Let indeed $\widetilde{w}$ be such that

$$w(t) = e^{-\frac{d-2}{2}\varepsilon}\,\widetilde{w}(t-\varepsilon)\ ,$$

with $\varepsilon = \varepsilon(w)$ as above. The function $\widetilde{w}$ is such that

$$\widetilde{w}(0) = 0\ , \quad \widetilde{w}'(0) = e^{\frac{d-2}{2}\varepsilon} w'(\varepsilon)\ ,$$

$$-\widetilde{w}'' = \widetilde{\lambda}\, e^{-\frac{d+2}{2}t} f\left(e^{\frac{d-2}{2}t}\,\widetilde{w}\right) - \tfrac{(d-2)^2}{4}\widetilde{w} = 0$$

with $\widetilde{\lambda} = \lambda e^{-2\varepsilon}$. In other terms, $\widetilde{w}$ corresponds to a radial solution of equation (2) with λ replaced by $\widetilde{\lambda}$. As a consequence of Proposition 16, we have the

Corollary 17 *With the above notations,*

$$\lim_{\substack{\eta\to 0\\ \eta>0}} \sup_{w\in S_\eta} \left|\widetilde{\lambda}(w) - \bar{\lambda}\right| = 0\ .$$

Moreover, on $(0, +\infty)$, $\widetilde{w}$ and w have the same number of zeros.

Now, we are going to parametrize the solutions in S_η which converge to 0 as $t \to +\infty$. For any $a \in \mathbb{R}$, consider $v = v_a$ on $[0, +\infty)$ such that

$$-v'' - \frac{d-1}{r} v' = f(v) \quad \forall\, r \in (0, +\infty)\ , \quad v(0) = a\ , \quad v'(0) = 0\ , \qquad (16)$$

and denote by $r_k(a)$ $(k = 1, 2, ...)$ its zeros in $\mathbb{R}^+$. The rescaling

$$u_{a,k}(r) = v_a(r_{k+1}(a) \cdot r) \quad \forall\, r \in (0, 1)$$

gives a complete parametrization of the branches of bounded solutions with k zeros. To be precise, a function u is a bounded radial solution of (2) with k zeros if and only if there exists an $a \in \mathbb{R}$ such that $u = u_{a,k}$, and the set of such solutions is therefore given by $\{(\lambda, a) \; : \; a \in \mathbb{R}, \; \lambda = \frac{1}{(r_{k+1}(a))^2}\}$ in $(0, +\infty) \times L^\infty(B)$.

Remark 18 *Note that any bounded solution u is certainly of class C^2 as soon as we assume that f is continuous, and then $a = u(0)$ is uniquely defined. The parametrization of all bounded radial solutions does not require further regularity assumption provided we consider all possible solutions of (16). Reciprocally, the solution of (16) is unique if f is assumed to be locally Lipschitz. Such a regularity has further consequences:*

(i) There exists at most one branch of bounded radial solutions which changes sign exactly k times for any given $k \in \mathbb{N}$.

(ii) If $uf(u) > 0$ for any $u \in \mathbb{R}$, such a branch exists for any $k \in \mathbb{N}$.

(iii) For $\eta > 0$ small enough, the solutions w in S_η such that $\lim_{t \to +\infty} w(t) = 0$ is a C^1 connected manifold.

Let us denote by R_η the set of the solutions $w \in S_\eta$ such that $\lim_{t \to +\infty} w(t) = 0$. By the above remark (iii), for η small enough, there exists an interval $I \subset \mathbb{R}$ such that R_η is parametrized by $a \in I$. Since f is assumed to be locally Lipschitz, this parametrization is one-to-one and continuous. Assume that for $\delta > 0$ small enough,

$$\mathcal{V}_{T,\delta}(0) \subset R_\eta \; .$$

Since f is assumed to be locally Lipschitz, the flow which maps $\mathcal{V}_{T,\delta}(T)$ into $\mathcal{V}_{T,\delta}(0)$ is also one-to-one and continuous, which clearly contradicts the *Theorem of Invariance of Domain*. Thus, for any $\delta > 0$ arbitrarily small, there certainly exists one solution $z_{a,b}$ of Equation (14) with $(a, b) \in \mathcal{V}_{T,\delta}(T)$ such that $T_{a,b} < +\infty$ (*i.e.* such that $\lim_{t \to +\infty} E^{z_{a,b}}(t) < 0$: see case (ii) of the first step). Clearly, such a $z_{a,b}$ is also in S_η for some $\eta > 0$ arbitrarily small if $\delta > 0$ is small enough, and to $\widetilde{z_{a,b}}$ corresponds an unbounded radial solution $\tilde{u}(r) = r^{\frac{d-2}{2}} \widetilde{z_{a,b}}(-\log r)$, $r \in (0, 1)$, with k zeros, of Equation (1) with $\lambda = \widetilde{\lambda}(z_{a,b}) = \bar{\lambda} e^{-2\varepsilon(z_{a,b})}$ using the notations of Proposition 16 and Corollary 17 (λ can be taken arbitrarily close to $\bar{\lambda}$ in the limit $\delta \to 0$).

Third step. Topological properties of the sets of solutions

The set Λ_u^k is an open set because for any unbounded radial solution u of (1) with a finite number of zeros, $\mathcal{E}(-u'(1), \lambda) < 0$ and by continuity of the map $\lambda \mapsto \mathcal{E}(\gamma, \lambda)$ (see Proposition 5). As seen in the above step,

$$\Lambda_b^k \subset \overline{\Lambda_u^k} \, .$$

Since Λ_b^k is an interval, then certainly

$$Int(\Lambda_b^k) \subset \Lambda_u^k \, .$$

From the continuity of $\gamma \mapsto \mathcal{E}(\gamma, \lambda)$ (see Proposition 5) and $\lim_{\gamma \to +\infty} \mathcal{E}(\gamma, \lambda) = +\infty$ (see Proposition 12), we also get

$$\Lambda_u^k \subset \bigcup_{j \geq 0} \Lambda_b^j \, ,$$

which ends the proof. □

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