Ketu* and the second invariant of a quadratic space

M. OJANGUREN
Section de Mathématiques, UNIL, CH-1015 Lausanne

R. PARIMALA and R. SRIDHARAN
School of Mathematics, TIFR, Bombay 400005

Abstract. *Using the Chern classes defined by Grothendieck, we map a K-theoretic invariant of invertible symmetric matrices defined by Giffen to the Clifford invariant.*

Key words. *Chern classes, quadratic forms, Clifford invariant.*

1. Introduction

An invariant for quadratic forms over a commutative ring A with values in a quotient of $\mathrm{K}_2(A)$ has been defined by Giffen [**6**]. He also verified that if A is a field, his invariant maps to the Clifford invariant of the quadratic form under the well-known symbol map to the Brauer group.

Using Grothendieck's theory of equivariant Chern classes [**7**], Shekhtman [**13**] and Soulé [**14**] defined homomorphisms $c_{ji} : \mathrm{K}_j(A) \longrightarrow \mathrm{H}^{2i-j}_{\text{ét}}(\mathrm{Spec}A, \boldsymbol{\mu}_m^{\otimes i})$ for all possible i, j and m. We show that Giffen's invariant maps under c_{22} to the Clifford invariant in $\mathrm{H}^2_{\text{ét}}(\mathrm{Spec}A, \boldsymbol{\mu}_2)$, defined by the connecting map associated to the sequence

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathbf{Spin} \longrightarrow \mathbf{SO} \longrightarrow 1 \ .$$

The method of proof involves basically two steps. The first is a general reduction to rank 2 quadratic spaces, via some calculations of étale cohomology groups of affine quadrics. The second is an explicit calculation of the invariants for rank 2 quadratic spaces over smooth affine curves. In this calculation we use Suslin's results on K-cohomology of smooth affine varieties. Since no explicit description of c_{22} is known, except on elements generated by symbols, our results can be interpreted as a computation of c_{22} on that part of K_2 arising from invariants of quadratic forms.

In what follows A will denote a commutative ring in which 2 is invertible.

2. The Giffen invariant

Let A be a commutative ring in which 2 is invertible, $\mathrm{WG}(A)$ the Witt-Grothendieck group of A (see for instance [**9**]) and $\mathrm{W}_0(A)$ its quotient by the subgroup generated by the hyperbolic plane $H(A)$. We denote by $\mathrm{W}_1(A)$ the kernel of the forgetful map

$$s_0 : \mathrm{W}_0(A) \longrightarrow \mathrm{K}_0(A)/2\mathbb{Z} \ .$$

---

* Ketu may not be connected with K_2 but, according to Herrmann Graßman's *Wörterbuch zum Rig-veda*, *"es bezeichnet das, was sich sichtbar oder kenntlich macht ... "*

Every class of $\mathrm{W}_1(A)$ is represented by a symmetric matrix of even rank. We define a homomorphism (called discriminant)

$$s_1 : \mathrm{W}_1(A) \longrightarrow \mathrm{K}_1(A)/\mathrm{Tr}(\mathrm{K}_1(A)) \ ,$$

where $\mathrm{Tr}(\alpha) = \alpha + \alpha^t$. Let S be a symmetric $2n \times 2n$ matrix representing an element of $\mathrm{W}_1(A)$. We set $s_1(S) = (-1)^n$(class of S). Let $\mathrm{W}_2(A)$ be the kernel of s_1. Every class of $\mathrm{W}_2(A)$ can be represented by an elementary symmetric matrix whose rank is a multiple of 4. Let $\mathrm{ES}(A)$ be the subset of $\mathrm{E}(A)$ consisting of symmetric matrices of rank divisible by 4 and trivial discriminant.

We say that α and β in $\mathrm{ES}(A)$ are *equivalent* if there exists an elementary matrix ϵ and integers m, n, such that

$$\epsilon^t \begin{pmatrix} \alpha & 0 \\ 0 & J_{2m} \end{pmatrix} \epsilon = \begin{pmatrix} \beta & 0 \\ 0 & J_{2n} \end{pmatrix} \ ,$$

where J_k is the matrix with k diagonal blocks equal to $J = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

These equivalence classes of elementary matrices form an abelian group $\mathrm{EW}_2(A)$ in which the class of J_{2m} is the identity and the class of $-\alpha$ is the inverse of the class of α. Clearly, W_2 is a quotient of $\mathrm{EW}_2(A)$. We define a map

$$s_2 : \mathrm{EW}_2(A) \longrightarrow \mathrm{K}_2(A)/\mathrm{Tr}(\mathrm{K}_2(A)) \ .$$

This time, Tr is the map $\alpha \mapsto \alpha - \alpha^t$, where $\alpha \mapsto \alpha^t$ is the involution induced on $\mathrm{K}_2(A)$ by the unique involution of the Steinberg group $\mathrm{St}(A)$ that maps each generator $x_{ij}(\lambda)$ to $x_{ji}(\lambda)$.

Let $\alpha \in \mathrm{ES}(A) \cap \mathrm{GL}_{4n}(A)$ represent an element of $\mathrm{EW}_2(A)$ and $\widetilde{\alpha}$ denote a lift of α in $\mathrm{St}(A)$. We set

$$s_2(\alpha) = n(-1,-1) + (\text{class of } \widetilde{\alpha}^{-1}\widetilde{\alpha}^t) \ .$$

Clearly this respects the equivalence in $\mathrm{ES}(A)$ and passes down to a homomorphism s_2.

Remark. The Steinberg group is written multiplicatively, but K_2 additively.

3. Catch 22 and transposition

We recall a few facts about equivariant Chern classes, from [**7**], [**13**] and [**14**].

Let X be a G-scheme, i.e. a scheme X with a discrete group G operating on it. A sheaf $\mathcal{F}$ over X is a G-sheaf if for every $g \in G$ there is a morphism of sheaves $\varphi_g : g^*\mathcal{F} \to \mathcal{F}$ such that $\varphi_{gh} = \varphi_h \circ h^*(\varphi_g)$ for all $g, h \in G$ and $\varphi_e = \mathrm{Id}_{\mathcal{F}}$. A global section s of $\mathcal{F}$ is invariant if $\varphi_g(s) = s$ for every g. This condition makes sense because $g^*\mathcal{F}(X) = \mathcal{F}(g^{-1}(X)) = \mathcal{F}(X)$. Associating to every abelian G-sheaf the group of its global invariant sections we get a functor whose n-th derived functor is, by definition, the n-th equivariant cohomology group $\mathrm{H}^n(X, G, \mathcal{F})$.

Suppose now that $\mathcal{E}$ is a G-vector bundle of rank r over X. The projective bundle $\mathbb{P}(\mathcal{E})$ associated to E is a G-scheme. For every integer m invertible in $\mathcal{O}_X$ let $\boldsymbol{\mu}_m$ be the sheaf of m-th roots of unity on X or $\mathbb{P}(\mathcal{E})$. The sheaf $\mathcal{E}$ determines a canonical element $[\mathcal{O}_{\mathcal{E}}(1)] \in \mathrm{H}^1(\mathbb{P}(\mathcal{E}), G, \mathbf{G}_m)$ which maps, under the connecting map ∂ associated to the Kummer sequence

$$1 \longrightarrow \boldsymbol{\mu}_2 \longrightarrow \mathbf{G}_m \xrightarrow{\ 2\ } \mathbf{G}_m \longrightarrow 1 \ ,$$

to an element $\xi(\mathcal{E}) = \partial[\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)] \in \mathrm{H}^2(\mathbb{P}(\mathcal{E}), G, \boldsymbol{\mu}_m)$. By [**7**],

$$\mathrm{H}^*(\mathbb{P}(\mathcal{E}), G, \boldsymbol{\mu}_m^{\otimes r}) = \bigoplus_{0 \leq j \leq r-1} \mathrm{H}^*(X, G, \boldsymbol{\mu}_m^{\otimes(r-j)})\xi^j$$

where ξ^j is the j-th cup-product power of $\xi = \xi(\mathcal{E})$. Thus, ξ^r is a linear combination of its lower powers and there is a relation

$$\xi^r + c_1(\mathcal{E})\xi^{r-1} + \cdots + c_r(\mathcal{E}) = 0 \ ,$$

where $c_i(\mathcal{E}) \in \mathrm{H}^{2i}(X, G, \boldsymbol{\mu}_m^{\otimes i})$.
If G operates trivially on X, the Künneth formula gives, for any j, a natural homomorphism

$$\mathrm{H}^{2j}(X, G, \boldsymbol{\mu}_m^{\otimes j}) \longrightarrow \bigoplus_{0 \leq i \leq 2j} \mathrm{Hom}\left(\mathrm{H}_i(G, \mathbb{Z}), \mathrm{H}^{2j-i}(X, \boldsymbol{\mu}_m^{\otimes j})\right)$$

which, for $i = j = 2$, maps $c_2(\mathcal{E})$ to a homomorphism

$$\varphi_{22}(\mathcal{E}) : \mathrm{H}_2(G, \mathbb{Z}) \longrightarrow \mathrm{H}^2(X, \boldsymbol{\mu}_m^{\otimes 2}) \ .$$

Taking now $X = \mathrm{Spec}A$, $\mathcal{E} = A^r$, $m = 2$ and $G = \mathbf{GL}_r(A)$ with the trivial action on X and the natural action on $\mathcal{E}$, we get, for any r, homomorphisms

$$\varphi_{22}^{(r)} : \mathrm{H}_2(\mathbf{GL}_r(A), \mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{Spec}A, \boldsymbol{\mu}_2)$$

which are compatible with the inclusions $\mathbf{GL}_r \subset \mathbf{GL}_{r+1}$. We therefore have a homomorphism

$$c_{22} : \mathrm{K}_2(A) = \mathrm{H}_2(\mathrm{E}(A), \mathbb{Z}) \longrightarrow \mathrm{H}^2(\mathrm{Spec}A, \boldsymbol{\mu}_2) \ .$$

Proposition 3.1. *For any $\alpha \in \mathrm{K}_2(A)$, $c_{22}(\alpha) = c_{22}(\alpha^t)$.*

Proof. We denote by # the inverse of the transpose of matrices and of elements of K_1, K_2, etc. On $\mathrm{K}_2(A) = \mathrm{H}_2(\mathrm{E}(A), \mathbb{Z})$ the involution # is induced by # on $\mathrm{E}(A)$, hence we only have to show that $\varphi_{22}^{(r)}(\alpha^\#) = \varphi_{22}^{(r)}(\alpha)$ for any $\alpha \in \mathrm{H}_2(\mathbf{GL}_r(A), \mathbb{Z})$. Let $G = \mathbf{GL}_r(A)$. The isomorphism $\# : G \longrightarrow G$ induces $\# : \mathrm{H}^{2i}(X, G, \boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^{2i}(X, G, \boldsymbol{\mu}_2)$ with the property that, for any G-bundle $\mathcal{E}$ on X,

$$\#c_i(\mathcal{E}) = c_i(\mathcal{E}^\#) \ , \tag{1}$$

where $\mathcal{E}^{\#}$ is $\mathcal{E}$ with the action of G twisted through $\#$ (see [**7**], §2.3).
For $\mathcal{E} = A^r$, $\mathcal{E}^{\#}$ is the dual $\check{\mathcal{E}}$ of $\mathcal{E}$, in the category of G-bundles. For any scheme Y with a G-action and any G-bundle $\mathcal{F}$ on Y, $c_i(\mathcal{F}) = (-1)^i c_i(\check{\mathcal{F}}) = c_i(\check{\mathcal{F}})$ in $\mathrm{H}^{2i}(Y, G, \boldsymbol{\mu}_2)$. (This can be proved by the splitting principle as in [**7**], page 247, noting that the statement is clear for line bundles with G-action.) From (1) we get $\# c_i(\mathcal{E}) = c_i(\mathcal{E})$. Since $\# c_i(\mathcal{E})$ induces the map $\varphi_{ij}^{(r)} \circ \#$ from $\mathrm{H}_j(\mathbf{GL}_r(A), \mathbb{Z})$ to $\mathrm{H}^{2i-j}(\mathrm{Spec} A, \boldsymbol{\mu}_2)$, it follows that $c_{22}(\alpha) = c_{22}(\alpha^{\#}) = -c_{22}(\alpha^t) = c_{22}(\alpha^t)$ and the proposition is proved.

4. An interlude

Theorem 4.1. *Let X be a noetherian reduced scheme and U an open subscheme containing all the singular points and all generic points of X. Then the restriction map*

$$\mathrm{H}^2(X, \mathbf{G}_m) \longrightarrow \mathrm{H}^2(U, \mathbf{G}_m)$$

is injective.

Proof. Let $i : U \longrightarrow X$ be the inclusion. Since U contains the generic points of X, the sheaf $\mathbf{G}_m$ on X injects into $i_*\mathbf{G}_m$. Let the sheaf D be defined by the exact sequence

$$1 \longrightarrow \mathbf{G}_m \longrightarrow i_*\mathbf{G}_m \longrightarrow D \longrightarrow 0 .$$

Associating to a rational function its order at a codimension 1 regular point induces a map of (étale) sheaves

$$i_*\mathbf{G}_m \longrightarrow \bigoplus_{x \in X^{(1)} \setminus U} i_{x_*}\mathbb{Z} ,$$

where $i_x : x \longrightarrow X$ is the inclusion of the point $x = \mathrm{Spec}\ k(x)$ into X. This map vanishes on $\mathbf{G}_m$ and induces a surjective map

$$d : D \longrightarrow \bigoplus_{x \in X^{(1)} \setminus U} i_{x_*}\mathbb{Z} .$$

We show that d is injective; let V be an étale neighborhood of a point $x \in X$. Suppose that $f \in i_*\mathbf{G}_m(V)$ maps to zero in $i_{x_*}\mathbb{Z}$ for every $x \in X^{(1)} \setminus U$. By definition f is a unit at every point of $V \times_X U$. Every point $y \in V$ outside $V \times_X U$ is regular and f has order zero at every codimension 1 generization of y belonging to $V \setminus V \times_X U$ because it maps to zero under d. On the other hand, f has order zero at the generizations of y belonging to $V \times_X U$ because f is a unit on $V \times_X U$. Hence f is a unit at y and d is an isomorphism.
By Lemma 1.9 of [**8**] we have $\mathrm{H}^1(X, i_*\mathbb{Z}) = 0$ and hence $\mathrm{H}^1(X, D) = 0$. We therefore have an injection

$$\mathrm{H}^2(X, \mathbf{G}_m) \longrightarrow \mathrm{H}^2(X, i_*\mathbf{G}_m) .$$

There is a spectral sequence

$$\mathrm{H}^p(X, R^q i_* \mathbf{G}_m) \Longrightarrow \mathrm{H}^n(U, \mathbf{G}_m) \ .$$

Since the étale sheaves $R^1 i_* \mathbf{G}_m$ and $R^2 i_* \mathbf{G}_m$ are zero we have an isomorphism

$$\mathrm{H}^2(X, i_* \mathbf{G}_m) \xrightarrow{\sim} \mathrm{H}^2(U, \mathbf{G}_m) \ .$$

Remark. It follows easily from Theorem 4.1 that for any commutative noetherian ring A such that SpecA has finitely many singularities the Giffen invariant and the Clifford invariant have the same image in the Brauer group of A. To see this, it suffices to replace A by its semilocalization at the singular primes and use the fact that, for a semilocal ring, $\mathrm{K}_2(A)$ is generated by symbols. This was our first approach to the comparison of the two invariants. For the result of the last section Theorem 4.1 is only needed in the well-known special case of a regular scheme, but since it does not seem to appear anywhere in print, we decided to record it here.

5. Quadrics

Let $\alpha = (a_{ij}) \in \mathbf{GL}_n(A)$ be a symmetric matrix, $u \in \mathbf{G}_m(A)$, $q(X_1, \ldots, X_n) = \sum_{i \leq j} a_{ij} X_i X_j$,

$$C = \frac{A[X_1, \ldots, X_n]}{(q(X_1, \ldots, X_n) - u)}$$

and $B = C_{X_1}$. For any A-algebra A', we write $B_{A'}$, $C_{A'}$ for $B \otimes_A A'$, $C \otimes_A A'$. Let $\pi : \mathrm{Spec}B \longrightarrow \mathrm{Spec}A$ be the structure map.

Proposition 5.1. *Let A be reduced and $n \geq 3$. The homomorphisms $\mathbb{Z} \longrightarrow \pi_* \mathbf{G}_m$ given by $n \mapsto X_1^n$ and $\mathbf{G}_m \longrightarrow \pi_* \mathbf{G}_m$ induce an isomorphism of étale sheaves*

$$\mathbf{G}_m \times \mathbb{Z} \xrightarrow{\sim} \pi_* \mathbf{G}_m \ .$$

Proof. Since $\mathbf{G}_m$ commutes with direct limits we may assume that A is noetherian. Supppose first that A is a separably closed field. If $n \geq 4$, X_1 is a prime in C and $\mathbf{G}_m(C) = \mathbf{G}_m(A)$, hence $\mathbf{G}_m(B) = \mathbf{G}_m(A) \times \mathbb{Z}$. If $n = 3$, by Proposition 5.3, either X_1 is a prime or $B \simeq (A[X_1, X_2, X_3]/(X_1 X_2 - X_3^2 - 1))_{X_1} \simeq A[X_1, X_1^{-1}, X_3]$, so that $\mathbf{G}_m(B)$ is again as above. If A is any field and $\overline{K}$ its separable closure, since $\overline{K} \cap B = A$, we have again $\mathbf{G}_m(B) = \mathbf{G}_m(A) \times \mathbb{Z}$.
Suppose A is any domain and K its field of fractions. Since B is faithfully flat over A, $\mathbf{G}_m(B) \cap K = \mathbf{G}_m(A)$, hence

$$\mathbf{G}_m(A) \times \mathbb{Z} \subseteq \mathbf{G}_m(B) \subseteq \mathbf{G}_m(B) \cap \mathbf{G}_m(B_K) = \mathbf{G}_m(B) \cap (\mathbf{G}_m(K) \times \mathbb{Z}) = \mathbf{G}_m(A) \times \mathbb{Z} \ .$$

Suppose now that A is strictly henselian with maximal ideal $\mathfrak{m}$ and minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$. Any unit u of B can be written, in $B_{A/\mathfrak{p}_i}$ as $v_i X_1^{n_i}$, with $v_i \in \mathbf{G}_m(A/\mathfrak{p}_i)$. If $\overline{u} = wX_1^m$ in $B_{A/\mathfrak{m}}$ with $w \in \mathbf{G}_m(A/\mathfrak{m})$, then $n_i = m$ for all i and thus ux_1^{-m} maps into $\mathbf{G}_m(A/\mathfrak{p}_i)$ for each i. Since A is strictly henselian, there is a retraction $\eta : B \longrightarrow A$. The element $uX_1^{-m} - \eta\left(uX_1^{-m}\right)$ maps to zero in $B_{A/\mathfrak{p}_1} \times \ldots \times B_{A/\mathfrak{p}_s}$ and, A being reduced, the map $B \longrightarrow B_{A/\mathfrak{p}_1} \times \ldots \times B_{A/\mathfrak{p}_s}$ is injective. Thus $uX_1^{-m} \in \mathbf{G}_m(A)$.
The map $\mathbf{G}_m \times \mathbb{Z} \longrightarrow \pi_* \mathbf{G}_m$ is an isomorphism on stalks because the strict henselization of a reduced local ring is reduced. Hence this map is an isomorphism .

Corollary 5.2. *With the same notation as above, $\boldsymbol{\mu}_2 \longrightarrow \pi_* \boldsymbol{\mu}_2$ is an isomorphism.*

Proof. Restrict the isomorphism of the above proposition to the 2-torsion subsheaves.

Proposition 5.3. *Let A be a local domain and $C = A[X, Y, Z]/(XY - Z^2 - u)$, u a unit of A. Let $\ell = aX + bY + cZ$ with $Aa + Ab + Ac = A$. If ℓ is not a prime in C, $C_\ell \simeq A[X_1, X_1^{-1}, Z]$.*

Proof. If a is invertible in A, we may assume that $a = 1$ and take $\ell = X + bY + cZ = X_1$ as a variable, so that

$$C = \frac{A[X_1, Y, Z_1]}{(X_1 Y - b_1 Y^2 - Z_1^2 - u)}$$

with $Z_1 = Z + \frac{1}{2}cY$ and $b_1 = b - \frac{1}{2}c$.
Since A is a domain and X_1 is not a prime, $b_1 = 0$ and $B \simeq A[X_1, X_1^{-1}, Z]$.
Suppose a and b are both non units. Then c is a unit and we may assume $c = 1$. Let $Z_1 = Z + aX + bY$. Then

$$\begin{aligned} C/(Z_1) &= A[X, Y, Z_1]/(XY - (aX + bY)^2 - u, Z_1) \\ &= A[X, Y]/(XY - (aX + bY)^2 - u) \end{aligned}$$

is a domain since $4ab \neq 1$, contradicting the assumption that ℓ is not a prime.

Proposition 5.4. *Let $q = (a_{ij})$ be a symmetric stably elementary matrix of size $n \geq 3$. Let $Q = \mathrm{Spec} B$ where $B = C_{X_1}$ and $C = A[X_1, \ldots, X_n]/\left(\sum a_{ij} X_i X_j \pm 1\right)$. Let $\pi : Q \longrightarrow \mathrm{Spec} A$ be the structure map. The induced homomorphism*

$$\mathrm{H}^2(A, \boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(B, \boldsymbol{\mu}_2)$$

is injective.

We postpone the proof of this proposition to the end of the section.

Lemma 5.5. *Suppose that A is local and reduced and that q is split. For $n \geq 2$, $\mathrm{Pic} B$ has no 2-torsion.*

Proof. Suppose that A is a normal domain. If $n \geq 4$, X_1 generates a prime ideal in C and $\mathrm{Pic} B$ injects into $\mathrm{Cl}(B)/\mathrm{Cl}(A)$ which in turn injects into $\mathrm{Cl}(B_K) = \mathrm{Cl}(C_K) = \mathrm{Pic} C_K = 0$, K being the field of fractions of A.

If $n = 3$ and X_1 generates a prime ideal, arguing as above we get that $\mathrm{Pic} B$ injects into $\mathrm{Pic} C_K = \mathbb{Z}$. If X_1 is not a prime, by Proposition 5.3, $B \simeq A[X_1, X_1^{-1}, X_2]$, hence $\mathrm{Pic} B = \mathrm{Pic} A = 0$. If $n = 2$, $B \simeq A[X_1, X_1^{-1}, \ell^{-1}]$, where $\ell = aX_1 + b$, with $a, b \in A$ and $Aa + Ab = A$. In particular, B is faithfully flat over A. An invertible B-module is equivalent to the image of a divisorial ideal J of A. Since B is faithfully flat over A, J must be projective over A, hence free. Thus $\mathrm{Pic} B = 0$.

To deal with the general case, we may assume that A is essentially of finite type over $\mathbb{Z}$ and proceed by induction on its dimension. The zero-dimensional case is already proved. Since A is essentially of finite type over $\mathbb{Z}$, its integral closure $\overline{A}$ is a finite A-module. Let $\mathfrak{c}$ be the conductor of $\overline{A}$ in A. The cartesian square

$$\begin{array}{ccc} B & \longrightarrow & \overline{B} \\ \downarrow & & \downarrow \\ B/\mathfrak{c}B & \longrightarrow & \overline{B}/\mathfrak{c}\overline{B} \end{array}$$

gives an exact sequence ([**2**], IX, 5.4)

$$\mathbf{G}_m(B/\mathfrak{c}B) \times \mathbf{G}_m(\overline{B}) \longrightarrow \mathbf{G}_m(\overline{B}/\mathfrak{c}\overline{B}) \xrightarrow{\alpha} \mathrm{Pic} B \longrightarrow \mathrm{Pic}(B/\mathfrak{c}B) \times \mathrm{Pic}\overline{B} \ ,$$

where $\overline{B} = B \otimes_A \overline{A}$. Since $\dim A/\mathfrak{c} < \dim A$, by induction, $\mathrm{Pic}\,(B/\mathfrak{c}B)$ has no 2-torsion. Since $\overline{A}$ is a product of normal domains, the argument above shows that $\mathrm{Pic}\overline{B}$ has no 2-torsion. It remains to show that $\mathrm{Coker}\,\alpha$ has no 2-torsion. Let $A_0 = (A/\mathfrak{c})_{red}$, $B_0 = B \otimes_A A_0$, $\overline{A}_0 = \left(\overline{A}/\mathfrak{c}\right)_{red}$, $\overline{B}_0 = B \otimes \overline{A}_0$.

Let u be a unit in $\mathbf{G}_m(\overline{B}/\mathfrak{c}\overline{B})$ which represents a 2-torsion element in $\mathrm{Coker}\,\alpha$. Its image v in $\mathbf{G}_m(\overline{B}_0)$ can be lifted to an element of $\mathbf{G}_m(B_0) \times \mathbf{G}_m(\overline{B})$. In fact, by 5.1, on each connected component of $\mathrm{Spec}\overline{B}_0$, v is of the form $v_0 X_1^m$, $v_0 \in \mathbf{G}_m(\overline{A}_0)$. The map $\mathbf{G}_m(\overline{A}/\mathfrak{c}) \longrightarrow \mathbf{G}_m(\overline{A}_0)$ is surjective and, $A/\mathfrak{c}$ being local, the map $\mathbf{G}_m(A/\mathfrak{c}) \times \mathbf{G}_m(\overline{A}) \longrightarrow \mathbf{G}_m(\overline{A}/\mathfrak{c})$ is also surjective. On each component, X_1^m lifts to a unit of $B/\mathfrak{c}B$, thus, modifying u by an element coming from $\mathbf{G}_m(B/\mathfrak{c}B) \times \mathbf{G}_m(\overline{B})$, we may assume that u maps to 1 in $\mathbf{G}_m(\overline{B}_0)$, so that $u \in 1 + \mathrm{nil}(\overline{B}/\mathfrak{c}\overline{B})$. Since this group has a finite filtration whose quotients, being isomorphic to $\left(\mathrm{nil}\left(\overline{B}/\mathfrak{c}\overline{B}\right)\right)^i / \left(\mathrm{nil}\left(\overline{B}/\mathfrak{c}\overline{B}\right)\right)^{i+1}$, are uniquely 2-divisible, it follows that the image of u in $\mathrm{Coker}\,\alpha$ is zero. Thus the lemma is proved.

Corollary 5.6. *Let $n \geq 3$. The sheaf $R^1\pi_*\boldsymbol{\mu}_2$ is constant, isomorphic to $\mathbb{Z}/2$, generated by the image of X_1.*

Proof. Let s be a global section of $R^1\pi_*\boldsymbol{\mu}_2$, represented on an étale open set $\mathrm{Spec}A'$ by an element $\zeta \in \mathrm{H}^1(B_{A'}, \boldsymbol{\mu}_2)$. In view of 5.5 and 5.1 we may assume, by shrinking $\mathrm{Spec}A'$, that $\zeta = X_1^{\varepsilon}$.

Corollary 5.7. *Let $n \geq 3$. If A is connected, the group $\mathrm{H}^0(A, R^1\pi_*\boldsymbol{\mu}_2)$ is generated by X_1.*

Proof of Proposition 5.4. Since $\boldsymbol{\mu}_2(R) = \boldsymbol{\mu}_2(R_{red})$ for every ring R in which 2 is invertible, we may assume that A is reduced. The Leray spectral sequence for $\pi : Q \longrightarrow \mathrm{Spec}A$ gives an exact sequence

$$\mathrm{H}^1(B, \boldsymbol{\mu}_2) \xrightarrow{\alpha} \mathrm{H}^0(A, R^1\pi_*\boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(A, \pi_*\boldsymbol{\mu}_2) \xrightarrow{\beta} \mathrm{H}^2(B, \boldsymbol{\mu}_2) \ .$$

By Corollary 5.7, α is surjective and hence β is injective. Since A is reduced, it follows from Corollary 5.2 that $\boldsymbol{\mu}_2 \longrightarrow \pi_*\boldsymbol{\mu}_2$ is an isomorphism and this completes the proof of the proposition.

6. The generalized Rees ring of an invertible module

Let I be an invertible A-module and $L = \bigoplus_{n\in\mathbb{Z}} I^n$ where $I^0 = A$, I^n is the n-fold tensor product $I \otimes_A \ldots \otimes_A I$ if n is positive and $(I^{-n})^{-1}$ if n is negative. The canonical isomorphism $I^m \otimes_A I^n \longrightarrow I^{m+n}$ defines on L the structure of a graded A-algebra, usually called *the generalized Rees ring of I*.

Proposition 6.1. *Let A be reduced. The A-algebra L is faithfully flat and the kernel of the map* $\mathrm{Pic}A \longrightarrow \mathrm{Pic}L$ *is the cyclic group generated by the class of I.*

Proof. Since locally I is free and L isomorphic to $A[t, t^{-1}]$, L is faithfully flat over A. Clearly, $L \otimes_A I \simeq L$ as L-modules. Let P be an invertible A-module and $\varphi : L \longrightarrow L \otimes_A P$ an isomorphism of L-modules. The fact that $\varphi(1)$ belongs to some homogeneous component $I^n \otimes_A P$ and that φ induces an isomorphism $A \xrightarrow{\sim} I^n \otimes_A P$ may be verified locally on A, using the fact that A is reduced (compare with [**4**], III.3). Thus $P \simeq I^{-n}$ for some n.

Proposition 6.2. *Let A be reduced. The kernel of* $\mathrm{H}^2(A, \boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(L, \boldsymbol{\mu}_2)$ *is generated by $\partial[I]$, where $[I]$ is the class of I in* $\mathrm{H}^1(A, \mathbf{G}_m)$ *and ∂ is the connecting homomorphism of the Kummer exact sequence.*

Proof. Let $\pi : \mathrm{Spec}L \longrightarrow \mathrm{Spec}A$ be the structure map. The natural map $\boldsymbol{\mu}_2 \longrightarrow \pi_*\boldsymbol{\mu}_2$ is an isomorphism since, for a local ring A, the ring L is just $A[t, t^{-1}]$. Hence the Leray spectral sequence gives an exact sequence

$$\mathrm{H}^1(A, \boldsymbol{\mu}_2) \xrightarrow{\mathrm{H}^1(\pi)} \mathrm{H}^1(L, \boldsymbol{\mu}_2) \xrightarrow{\delta} \mathrm{H}^0(A, R^1\pi_*\boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(A, \boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(L, \boldsymbol{\mu}_2) \ .$$

For any open set $\mathrm{Spec}A_f$ over which I is free there is a split exact sequence

$$0 \longrightarrow \mathrm{H}^1(A_f, \boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^1(L_f, \boldsymbol{\mu}_2) \xrightarrow{\delta_f} \mathrm{H}^0(A_f, R^1\pi_*\boldsymbol{\mu}_2) \longrightarrow 0$$

with $L_f \simeq A_f[t, t^{-1}]$. We claim that $\mathrm{H}^0(A_f, R^1\pi_*\boldsymbol{\mu}_2) = \mathbb{Z}/2$. Let s be a global section of $R^1\pi_*\boldsymbol{\mu}_2$, $x \in \mathrm{Spec}A_f$ and A_x the strict henselisation of A at x. On $\mathrm{Spec}A_x$, s

is represented by an element $\xi \in \mathrm{H}^1(L_{A_x}, \boldsymbol{\mu}_2) \simeq L^*_{A_x}/(L^*_{A_x})^2 \times {}_2\mathrm{Pic}L_{A_x}$. Since $L_{A_x} \simeq A_x[t,t^{-1}]$ and A_x is reduced, $L^*_{A_x} \simeq A^*_x \times \mathbb{Z}$. Further, by 5.5, ${}_2\mathrm{Pic}L_{A_x} = 0$, so that $\xi = t^\epsilon$, $\epsilon = 0$ or 1. Thus $\mathrm{H}^0(A_f, R^1\pi_*\boldsymbol{\mu}_2) = \mathbb{Z}/2$, generated by t and hence $\mathrm{H}^0(A, R^1\pi_*\boldsymbol{\mu}_2)$ is either $\mathbb{Z}/2$ or trivial.

Suppose I is not a square in $\mathrm{Pic}A$. Then, under the connecting map ∂ of the Kummer sequence, $\partial[I]$ is a nontrivial element in $\ker(\mathrm{H}^2(A,\boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(L,\boldsymbol{\mu}_2))$ by Proposition 6.1 and hence it generates the whole kernel.

Suppose $I = J^2$ for some $J \in \mathrm{Pic}A$. If $\{\mathrm{Spec}A_f, f \in S\}$ is a covering which trivializes J, we can choose local generators t_f for I such that $t_f = u^2_{fg}t_g$, $u_{fg} \in A^*_{fg}$ denoting the cocycle associated to J. The set $\{t_f, f \in S\}$ represents a nonzero section of $R^1\pi_*\boldsymbol{\mu}_2$ since it is nonzero even locally on A. This section is the image under δ of the discriminant module $JL \otimes_L JL \simeq J^2L = IL \simeq L$ over L. In this case $\mathrm{H}^2(A,\boldsymbol{\mu}_2) \longrightarrow \mathrm{H}^2(L,\boldsymbol{\mu}_2)$ is injective.

7. Rank 2 forms over an affine curve

Assume that A is a smooth affine algebra of dimension 1 over a field of characteristic not equal to 2 and let $H(I)$ be the hyperbolic plane over an invertible A-module I. We can represent $H(I)$ by a symmetric matrix

$$\alpha = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

with $b^2 - ac = 1$. We assume that $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right)$ is stably elementary, so that it represents a class in $\mathrm{EW}_2(A)$. In this case, since $\dim A = 1$, this matrix is in fact elementary.

Proposition 7.1. *$c_{22} \circ s_2 \left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right) = 0$ if and only if the class of I is a square in* $\mathrm{Pic}A$.

Proof. By 4.2.1 of [**15**] the map $c_{22} : K_2(A) \longrightarrow \mathrm{H}^2(A,\boldsymbol{\mu}_2)$ factors through $\mathrm{H}^0(X, \mathcal{K}_2)$, where $\mathcal{K}_2$ is the Zariski sheaf associated to the presheaf $U \mapsto K_2(U)$. Since the transposition on K_2 coincides locally with the inverse, this induces a factoring of c_{22} as a composite

$$\mathrm{K}_2(A)/\mathrm{Tr}(\mathrm{K}_2(A)) \longrightarrow \mathrm{H}^0(X,\mathcal{K}_2)/2 \longrightarrow \mathrm{H}^2(A,\boldsymbol{\mu}_2)\ .$$

Since A is smooth, by 4.2.1 and Theorem 4.3 of [**15**] the second map is injective. Thus, $c_{22} \circ s_2(q) = 0$ if and only if the image of $s_2(q)$ in $\mathrm{H}^0(X,\mathcal{K}_2)/2$ is zero, q denoting the class of $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right)$ in $\mathrm{EW}_2(A)$. Let $\mathcal{U} = \{U_i,\ 1 \le i \le N\}$ be an affine open covering of $X = \mathrm{Spec}A$ which trivializes I and let $\{u_{ij} \in \mathbf{G}_m(\mathcal{O}_X(U_i \cap U_j)),\ 1 \le i,j \le N\}$ be a cocycle representing I. Let $\widetilde{\alpha}$ be a lift $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right)$ in $\mathrm{St}(A)$. Then

$$s_2(q) = \text{class of } (-1,-1) + \widetilde{\alpha}^{-1}\widetilde{\alpha}^t \text{ in } \mathrm{K}_2(A)/\mathrm{Tr}(\mathrm{K}_2(A)).$$

By shrinking $\mathcal{U}$, we may assume that

$$\begin{pmatrix} \alpha & 0 \\ 0 & J \end{pmatrix} = \beta_i J_2 \beta_i^t$$

on U_i, with $\beta_i \in \mathrm{E}_4(\mathcal{O}_X(U_i))$. Let $\widetilde{\beta}_i$ be a lift of β_i in $\mathrm{St}(\mathcal{O}_X(U_i))$. Let

$$\widetilde{J} = w_{21}(-1)w_{43}(-1)h_{13}(-1) \ ,$$

where $w_{ij}(\lambda) = x_{ij}(\lambda)x_{ji}(-\lambda^{-1})x_{ij}(\lambda)$, $h_{ij}(\lambda) = w_{ij}(\lambda)w_{ij}(-1)$ (cf [**10**] page 71). Then $\widetilde{J}$ is a lift of J_2 and $\widetilde{J}^{-1}\widetilde{J}^t = (-1,-1)$. On U_i, $\widetilde{\alpha} = c_i\widetilde{\beta}_i\widetilde{J}\widetilde{\beta}_i^t$ with $c_i \in \mathrm{K}_2(\mathcal{O}_X(U_i))$. By further shrinking $\mathcal{U}$, we may assume that each c_i is a sum of symbols, so that $-c_i = c_i^t$. Then

$$(-1,-1) + \widetilde{\alpha}^{-1}\widetilde{\alpha}^t = -2c_i$$

on U_i so that $2c_i = 2c_j$ on $U_i \cap U_j$. The image of $s_2(q)$ in $\mathrm{H}^0(X,\mathcal{K}_2)$ is represented by the cocycle $\{-2c_i,\ 1 \le i \le N\}$. Since $2(c_i - c_j) = 0$ on $U_i \cap U_j$, $\{c_i - c_j,\ 1 \le i,\ j \le N\}$ defines a 1-cocycle on $\mathrm{Spec}A$ with values in ${}_2\mathcal{K}_2$.

Lemma 7.2. *The class of the 1-cocycle* $\{c_i - c_j,\ 1 \le i,\ j \le N\}$ *in* $\mathrm{H}^1(X, {}_2\mathcal{K}_2)$ *is equal to the class of the 1-cocycle* $\{(-1,u_{ij}),\ 1 \le i,\ j \le N\}$.

We grant this lemma. Suppose the image of $s_2(q)$ in $\mathrm{H}^0(X,\mathcal{K}_2)/2$ is zero. After possibly shrinking $\mathcal{U}$, we may assume that there exists a cocycle $\{c_i',\ 1 \le i \le N\}$ in $\mathrm{H}^0(X,\mathcal{K}_2)$ such that $2c_i = 2c_i'$ on U_i. Then $2(c_i - c_i') = 0$ and $c_i - c_j = (c_i - c_i') - (c_j - c_j')$, which shows that the 1-cocycle $\{c_i - c_j\}$ with values in ${}_2\mathcal{K}_2$ is a coboundary. By Lemma 7.2, the cocycle $\{(-1,u_{ij})\}$ is trivial in $\mathrm{H}^1(X, {}_2\mathcal{K}_2)$.

Lemma 7.3. *Let X be a smooth curve over a field k of characteristic different from 2. There is an isomorphism* $\mathrm{Pic}X/2 \longrightarrow \mathrm{H}^1(X, {}_2\mathcal{K}_2)$, *which, in terms of Čech cocycles, is given by mapping the class of* $\{f_{ij}\}$ *to that of* $\{(-1,f_{ij})\}$.

By Lemma 7.3, $[I]$ is trivial in $\mathrm{Pic}A/2$. Conversely, if the class of I in $\mathrm{Pic}A/2$ is trivial, reversing the above arguments, we see that the image of $s_2(q)$ in $\mathrm{H}^0(X,\mathcal{K}_2)/2$ is zero. This proves Proposition 7.1, provided we prove the two lemmas.

Proof of Lemma 7.2. We can choose the trivializations β_i such that

$$\beta_j^{-1}\beta_i = \begin{pmatrix} u_{ij}^{-1} & 0 \\ 0 & u_{ij} \end{pmatrix} \ .$$

This has a lift $h_{12}(u_{ij})$ in $\mathrm{St}(U_i \cap U_j)$. Putting $u_{ij} = u$ we get, using the identities

$$w_{ij}(u)^t = w_{ij}(u)$$
$$h_{ij}(u)^t = h_{ij}(-u)h_{ij}(-1)^{-1}$$

and Corollary 9.4 of [**10**],

$$\begin{aligned}
c_j - c_i &= \widetilde{J}^{-1} h_{12}(u) \widetilde{J} h_{12}(u)^t = \\
&= h_{13}(-1)^{-1} w_{43}(-1)^{-1} w_{21}(-1)^{-1} h_{12}(u) w_{21}(-1) w_{43}(-1) h_{13}(-1) h_{12}(-u) h_{13}(-1)^{-1} \\
&= h_{13}(-1)^{-1} h_{21}(-u) h_{21}(-1)^{-1} h_{13}(-1) h_{12}(-u) h_{12}(-1)^{-1} \\
&= h_{21}(u) h_{12}(-u) h_{12}(-1)^{-1} \\
&= h_{12}(u)^{-1} h_{12}(-u) h_{12}(-1)^{-1} \\
&= h_{12}(u) h_{12}(-1)^{-1} h_{12}(u)^{-1} = (-1, u) \ .
\end{aligned}$$

Proof of Lemma 7.3. We denote by $\mathcal{H}^n$ the Zariski sheaf on X, associated to the presheaf $U \mapsto \mathrm{H}^n(U, \boldsymbol{\mu}_2)$. By Theorem 6.1 of [**3**], $\mathrm{Pic} X/2 \simeq \mathrm{H}^1(X, \mathcal{H}^1)$ is the cokernel of the map

$$\mathrm{H}^1(k(X), \boldsymbol{\mu}_2) = k(X)/(k(X)^*)^2 \xrightarrow{\partial} \bigoplus_{x \in X^{(1)}} \mathrm{H}^0(k(x), \boldsymbol{\mu}_2) = \bigoplus_{x \in X^{(1)}} \mathbb{Z}/2 \ ,$$

where the map is given, at x, by $\partial(f) = (-1)^{v_x(f)}$. By [**11**], 8.7.8.(b), $\mathrm{H}^1(X, {}_2\mathcal{K}_2)$ is the cokernel of the tame symbol map

$${}_2\mathrm{K}_2(k(X)) \longrightarrow \bigoplus_{x \in X^{(1)}} {}_2\mathrm{K}_1(k(x)) = \bigoplus_{x \in X^{(1)}} \mathbb{Z}/2 \ .$$

By Theorem 1.8 of [**15**], mapping $f \in k(X)$ to the symbol $(-1, f)$ yields a surjection

$$k(X)/(k(X)^*)^2 \longrightarrow {}_2\mathrm{K}_2(k(X)) \ ,$$

such that the diagram

$$\begin{array}{ccc}
k(X)/(k(X)^*)^2 & \longrightarrow & \bigoplus \mathbb{Z}/2 \\
\downarrow & & \downarrow = \\
{}_2\mathrm{K}_2(k(X)) & \longrightarrow & \bigoplus \mathbb{Z}/2
\end{array}$$

is commutative. This yields an isomorphism $\mathrm{Pic} X/2 \longrightarrow \mathrm{H}^1(X, {}_2\mathcal{K}_2)$ on the cokernels. Since all the maps in the resolutions are explicit, it is easy to verify that, in terms of cocycles, the isomorphism is as claimed.

Theorem 7.4. *The class of I in $\mathrm{Pic} A/2 \hookrightarrow \mathrm{H}^2(A, \boldsymbol{\mu}_2)$ is precisely $c_{22} \circ s_2 \left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right)$. In particular, $c_{22} \circ s_2 \left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right) = e_2 \left(\begin{smallmatrix} \alpha & 0 \\ 0 & J \end{smallmatrix}\right)$.*

Proof. Let $L = \bigoplus_{n \in \mathbb{Z}} I^n$. Since IL is trivial in $\mathrm{Pic} L$, by Proposition 7.1, $c_{22} \circ s_2(q \otimes L) = 0$. In view of (5.3), $c_{22} \circ s_2(q)$ is either zero or the class of I in $\mathrm{Pic} A/2 \subset \mathrm{H}^2/A, \boldsymbol{\mu}_2)$. If I is not a square in $\mathrm{Pic} A$, by Proposition 7.1, $c_{22} \circ s_2(q) \neq 0$ so that $c_{22} \circ s_2(q) = [I]$. If I is is a square in $\mathrm{Pic} A$, by Proposition 7.1, $c_{22} \circ s_2(q) = [I] = 0$. On the other hand, by [**12**], Theorem 20, $e_2(q) = [I]$ and this completes the proof of the theorem.

8. The comparison

Lemma 8.1 (Casanova). *Let $\alpha \in \mathbf{GL}_{4n}(A)$ represent an element of $\mathrm{EW}_2(A)$ and $\theta \in \mathbf{GL}_{4n}(A)$ a matrix such that $\theta^t\theta \in \mathrm{E}(A)$. Then*

$$s_2(\theta^t\alpha\theta) = s_2(\alpha) + s_2(\theta^t\theta) + n(-1,-1) \ .$$

Proof. Let $\widetilde{\alpha}$ be a lift of α and $\widetilde{\theta^t\theta}$ a lift of $\theta^t\theta$ in $\mathrm{St}(A)$. For any $\varphi \in \mathbf{GL}(A)$, let $x \mapsto x^\varphi$ be the unique isomorphism of $\mathrm{St}(A)$ lifting the conjugation by φ in $\mathrm{E}(A)$. Choosing $\widetilde{\theta^t\theta}\ \widetilde{\alpha}^\theta$ as a lift of $\theta^t\alpha\theta$, we get

$$\begin{aligned} s_2(\theta^t\alpha\theta) &= n(-1,-1) + \left(\widetilde{\theta^t\theta}\cdot\widetilde{\alpha}^\theta\right)^t\left(\widetilde{\theta^t\theta}\cdot\widetilde{\alpha}^\theta\right)^{-1} \\ &= s_2(\theta^t\theta) + (\widetilde{\alpha}^\theta)^t(\widetilde{\alpha}^{-1})^{(\theta^t)^{-1}} \\ &= s_2(\theta^t\theta) + \left(\widetilde{\alpha}^t\widetilde{\alpha}^{-1}\right)^{(\theta^t)^{-1}} \end{aligned}$$

since, for any $x \in \mathrm{St}(A)$, $(x^\theta)^t = (x^t)^{(\theta^t)^{-1}}$. To show that the second term coincides with $\widetilde{\alpha}^t\widetilde{\alpha}^{-1}$ and thus complete the proof, it suffices to observe that the action of any $\varphi \in \mathbf{GL}(A)$ on $\mathrm{K}_2(A)$ is trivial: indeed, by [**1**], $\mathrm{K}_2(A) = \mathrm{H}_2(\mathrm{E}(A),\mathbb{Z})$ is a direct factor of $\mathrm{H}_2(\mathbf{GL}(A),\mathbb{Z})$ and an inner homomorphism of any group G induces the identity map on $\mathrm{H}_*(G,\mathbb{Z})$.

Lemma 8.2. *Let $\theta \in \mathrm{SL}_2(A)$. Then $\theta^t\theta$ is stably elementary and e_2 coincides with $c_{22}\circ s_2$ on*

$$\begin{pmatrix} 1_2 & 0 \\ 0 & \theta^t\theta \end{pmatrix}$$

They both are $(-1)\smile(-1)$ in $\mathrm{H}^2(A,\boldsymbol{\mu}_2)$.

Proof. Since $\theta^t = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\theta^{-1}\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{-1}$, $\theta^t\theta$ is a commutator and hence stably elementary. To prove the second assertion, it suffices to consider the generic case of the matrix $\theta = \left(\begin{smallmatrix} x & y \\ z & t \end{smallmatrix}\right)$ over the ring $A = \mathbb{Z}[\frac{1}{2},x,y,z,t]/(xt-yz-1)$. Since x is a prime in A and A is regular, $\mathrm{Pic}\,A \simeq \mathrm{Pic}\,A_x = 0$, so that $\mathrm{H}^2(A,\boldsymbol{\mu}_2) \simeq {}_2\mathrm{Br}(A)$. Since $\mathrm{Br}(A)$ injects (§4!) into $\mathrm{Br}(K)$, K the field of fractions of A, it suffices to compare the two invariants in $\mathrm{Br}(K)$, which has been done in [**6**]. Clearly

$$e_2\begin{pmatrix} 1_2 & 0 \\ 0 & \theta^t\theta \end{pmatrix} = e_2(1_4) = (-1)\smile(-1) \ .$$

Theorem 8.3. *The Giffen invariant maps, under c_{22}, to the Clifford invariant.*

Proof. Let $\alpha \in \mathbf{GL}_{4n}(A)$ represent an element of $\mathrm{EW}_2(A)$ and let q be the corresponding quadratic form.

Replacing A by the A-algebra B of Proposition 5.4, we can make the matrix α elementarily equivalent to $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \beta \end{smallmatrix}\right)$. The comparison of the invariants in $\mathrm{H}^2(A, \boldsymbol{\mu}_2)$ reduces, by Proposition 5.4, to their comparison in $\mathrm{H}^2(B, \boldsymbol{\mu}_2)$ for the form $\left(\begin{smallmatrix} \pm 1 & 0 \\ 0 & \beta \end{smallmatrix}\right)$. Repeating this process $4n-3$ times we are reduced to compare the invariants of the matrix

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & b & c \end{pmatrix} .$$

Since $ac - b^2 = -1$, the quadratic space $\left(A^2, \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)\right)$ is isometric to $H(I)$ for some invertible module I. By Theorem 20 of [**12**], $e_2(H(I))$ is the class of I in $\mathrm{Pic}\,A/2 \subset \mathrm{H}^2(A, \boldsymbol{\mu}_2)$. If we extend the scalars to $L = \bigoplus_{n \in \mathbb{Z}} I^n$, I becomes principal and

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} = \gamma^t J \gamma$$

for some $\gamma \in \mathrm{SL}_2(L)$. By Lemma 8.1,

$$s_2(\beta) = s_2 \begin{pmatrix} 1 & & \\ & -1 & \\ & & J \end{pmatrix} + s_2 \begin{pmatrix} 1_2 & \\ & \gamma^t \gamma \end{pmatrix} + (-1, -1)$$

and by Lemma 8.2,

$$c_{22} \circ s_2 \begin{pmatrix} 1_2 & \\ & \gamma^t \gamma \end{pmatrix} = (-1) \smile (-1) .$$

For a matrix of the form

$$\alpha = \begin{pmatrix} 1 & & & \\ & -a & & \\ & & -b & \\ & & & ab \end{pmatrix} ,$$

$s_2(\alpha) = (a, b)$ modulo $\mathrm{Tr}(\mathrm{K}_2(A))$, and $c_{22} \circ s_2(\alpha) = (a) \smile (b)$ (see [**6**]). Since

$$\begin{pmatrix} 1 & & \\ & -1 & \\ & & J \end{pmatrix}$$

is elementarily equivalent to

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix} ,$$

it follows that over L, $c_{22} \circ s_2(\beta) = 0$. Therefore $c_{22} \circ s_2(\beta)$, by Proposition 6.1, is either the class of I or zero. Suppose

$$\begin{pmatrix} 1_{4n} & \\ & \beta \end{pmatrix}$$

is elementary. Then this holds for the matrix

$$\beta = \begin{pmatrix} 1_{4n} & & & & \\ & 1 & 0 & & \\ & 0 & -1 & & \\ & & & x & y \\ & & & y & z \end{pmatrix}$$

over the ring

$$A = \frac{\mathbb{Z}[\frac{1}{2}, x, y, z, X_{ij}, Y_{ij}, Z_{ij}, T_{ij}][(\mathrm{det} X)^{-1}, \ldots, (\mathrm{det} T)^{-1}]}{(xz - y^2 + 1, XYX^{-1}Y^{-1}ZTZ^{-1}T^{-1} = \beta)} ,$$

where $X = (X_{ij}), \ldots, T = (T_{ij})$ are generic matrices and $1 \leq i, j \leq 4n+4$. We note that if

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & a & b \\ & & b & c \end{pmatrix} = \gamma$$

is such that $ac - b^2 = -1$ and

$$\begin{pmatrix} 1_{4n} & \\ & \gamma \end{pmatrix}$$

is elementary over a ring B, there is a specialization from A to B, specializing β to γ, since over any commutative ring, any elementary matrix is a product of two commutators ([**5**], Remark 21). If $c_{22} \circ s_2(\beta) = 0$ this invariant would be zero for any matrix of the form β over any ring. However the computations in §7 show the existence of matrices β over Dedekind domains C for which it is not zero: we may take $C = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ and

$$\beta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & x^2 - y^2 & 2xy \\ 0 & 0 & 2xy & y^2 - x^2 \end{pmatrix} .$$

The corresponding ideal I is the generator of $\mathrm{Pic} C = \mathbb{Z}/2$. This shows that the two invariants coincide.

Acknowledgements

Financial support from IFCPAR/CEFIPRA and FNRS is gratefully acknowledged. The first author also thanks Sandro di Pippozzo for moral support during the monsoon.

Bibliography

1. D. Arlettaz: A splitting result for the second homology group of the general linear group. In *Adams Memorial Symposium on Algebraic Topology, volume 1*, London Math. Soc. Lecture Notes Series **175** (1992), 83–88.

2. H. Bass: *Algebraic K–theory.* Benjamin, New-York, 1968.

3. S. Bloch and A. Ogus: Gersten's conjecture and the homology of schemes. Ann. scient. Éc. Norm. Sup. **7** (1974), 181–201.

4. S. Caenepeel and F. Van Oystaeyen: *Brauer groups and the cohomology of graded rings*, Monographs and Textbooks in Pure and Appl. Math. **121**. Marcel Dekker, New York 1988.

5 . R.K. Dennis and L.N. Vaserstein. On a question of M. Newman on the number of commutators. J. of Algebra **118** (1988), 150–161.

6. C.H. Giffen: Hasse-Witt Invariants for (α, u)-Reflexive Forms and Automorphisms. I: Algebraic K_2-Valued Hasse-Witt Invariants. J. Algebra **44** (1977), 434–456.

7. A. Grothendieck: Classes de Chern et représentations linéaires des groupes discrets. In *Dix exposés sur la cohomologie des schémas*, Masson and North-Holland, Amsterdam, 1968.

8. A. Grothendieck: Le groupe de Brauer II. In *Dix exposés sur la cohomologie des schémas*, Masson and North-Holland, Amsterdam, 1968.

9. M. Knebusch: Symmetric bilinear forms over algebraic varieties. In *Conference on Quadratic Forms - 1976*, Queen's Papers Pure Appl. Math. **46** (1977), 103–283.

10. J. Milnor: Introduction to algebraic K-theory. *Annals of Math. Studies* **72**, Princeton Univ. Press, 1971.

11. A.S. Merkur'ev and A.A. Suslin: K-cohomology of Severi-Brauer varieties and norm residue homomorphism. Izv. Akad. Nauk SSSR **46** (1982), 1011–1146 = Math. USSR Izv. **21** (1983), 307–341.

12. M. Ojanguren: A splitting theorem for quadratic forms. Comment. Math. Helv. **57** (1982), 145–157.

13. V.V. Shekhtman: Chern classes in algebraic K-theory. Trans. Moscow Math. Soc. **1** (1984), 243–271.

14. C. Soulé: K-théorie des anneaux de corps de nombres et cohomologie étale. Invent. Math. **55** (1979), 251–295.

15. A.A. Suslin: Torsion in K_2 of fields. K-Theory **1** (1987), 5–29.