Probability and Geometry on some Noncommutative Manifolds

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Abstract : In a noncommutative torus, effect of perturbation by inner derivation on the associated quantum stochastic process and geometric parameters like volume and scalar curvature have been studied. Cohomological calculations show that the above perturbation produces new spectral triples. Also for the Weyl C^*-algebra, the Laplacian associated with a natural stochastic process is obtained and associated volume form is calculated.
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1 Introduction

For a fixed θ, an irrational number in $[0,1]$, consider the C^*-algebra $\mathcal{A}_\theta$ generated by a pair of unitary symbols subject to the relation :

$$UV = \exp(2\pi i\theta)VU \equiv \lambda VU. \tag{1.1}$$

For details of the properties of such a C^*-algebra, the reader is referred to [2], [17]. The algebra has many interesting representations :
(i) $\mathcal{H} = L^2(\mathbb{T}^1)$, $\mathbb{T}^1$ is the circle, and for $f \in \mathcal{H}$, $(\pi_1(U)f)(z) = f(\lambda z)$, $(\pi_1(V)f)(z) = zf(z),\ z \in \mathbb{T}^1$.
(ii) In the same $\mathcal{H}$, with the roles of U and V reversed :
for $f \in \mathcal{H}$, $(\pi_2(V)f)(z) = f(\bar{\lambda}z),\ (\pi_2(U)f)(z) = zf(z),\ z \in \mathbb{T}^1$.
(iii) In $\mathcal{H} = L^2(\mathbb{R})$,
$(\pi_3(U)f)(x) = f(x+1),\ (\pi_3(V)f)(x) = \lambda^x f(x)$.
While the first two were inequivalent irreducible representations, the ultraweak closure of the third one is a factor of type II_1.

There is a natural action of the abelian compact group $\mathbb{T}^2$ (2-torus) on

$\mathcal{A}_\theta$ given by,

$$\alpha_{(z_1,z_2)}(\sum a_{mn}U^mV^n) = \sum a_{mn}z_1^m z_2^n U^mV^n,$$

where the sum is over finitely many terms and $\|z_1\| = \|z_2\| = 1$. α extends as a $*$-automorphism on $\mathcal{A}_\theta$ and has two commuting generators d_1 and d_2 which are skew-*-derivations obtained by extending linearly the rule:

$$\begin{aligned} d_1(U) = U,\ d_1(V) = 0 \\ d_2(U) = 0, d_2(V) = V. \end{aligned} \tag{1.2}$$

Both d_1 and d_2 are clearly well defined on ${\mathcal{A}_\theta}^\infty \equiv \{a \in \mathcal{A}_\theta \mid z \mapsto \alpha_z(a)$ is $C^\infty\} \equiv \{\sum_{m,n\in Z} a_{mn}U_mV^n \mid sup_{m,n}|m^kn^la_{mn}| \leq c$ for all $k,l \in N\}$. Since the action is norm continuous $\mathcal{A}_\theta^\infty$ is a dense $*$-subalgebra of $\mathcal{A}_\theta$. A theorem of Bratteli, Elliot, Jorgensen [1] describes all the derivaions of $\mathcal{A}_\theta$ which maps $\mathcal{A}_\theta^\infty$ to itself : for almost all θ (Lebesgue), a derivation $\delta : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$ is of the form $\delta = c_1d_1 + c_2d_2 + [r, .]$, with $r \in \mathcal{A}_\theta^\infty,\ c_1, c_2 \in \mathbb{C}$. Another important fact about $\mathcal{A}_\theta$ is the existence of a unique faithful trace τ on $\mathcal{A}_\theta$ defined as follows:

$$\tau(\sum a_{mn}U^mV^n) = a_{00}. \tag{1.3}$$

Then one can consider the Hilbert space $\mathcal{H} = L^2(\mathcal{A}_\theta, \tau)$ (see [13] for an account on noncommutative L^p spaces.) and study the derivations there. It is easy to see that the family $\{U^mV^n\}_{m,n\in Z}$ constitute a complete orthonormal basis in $\mathcal{H}$. The next simple theorem is stated without proof.

Theorem 1.1 *The canonical derivations d_1, d_2 are self adjoint on their natural domains: $Dom(d_1) = \{\sum a_{mn}U^mV^n \mid \sum(1+m^2)|a_{mn}|^2 < \infty\}$ $Dom(d_2) = \{\sum a_{mn}U^mV^n \mid \sum(1+n^2)|a_{mn}|^2 < \infty\}$. Furthermore if we denote by $d_r = [r, .]$ with $r \in \mathcal{A}_\theta \subset L^\infty(\mathcal{A}_\theta, \tau)$ acting as left multiplication in $\mathcal{H}$, then ${d_r}^* = d_{r^*} \in \mathcal{B}(\mathcal{H})$*

2 Diffusion on $\mathcal{A}_\theta$ and a noncommutative Laplacian

There is a canonical construction of a quantum stochastic flow or diffusion on a von Neumann [8] or a C^*-algebra $\mathcal{A}$ [7] associated with a completely

positive semigroup on $\mathcal{A}$. The question about which of these semigroups have 'local' generators $\mathcal{L}$ remains open , though Sauvageot studied these in [19] . Following these studies , we know that $\mathcal{L}$ is characterized by :
(i) $\mathcal{D} \subseteq Dom(\mathcal{L}) \subseteq \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, dense in $\mathcal{A}$ such that $\mathcal{D}$ itself is a $*$-algebra,
(ii) a $*$-representation π in some Hilbert space $\mathcal{K}$ and an associated π derivation δ such that $\delta(x) \in \mathcal{B}(\mathcal{H},\mathcal{K})$ and $\delta(xy) = \delta(x)y + \pi(x)\delta(y)$,
(iii) a second order cocycle relation : $\mathcal{L}(x^*y) - \mathcal{L}(x)^*y - x^*\mathcal{L}(y) = \delta(x)^*\delta(y)$, for $x, y \in \mathcal{D}$. In analogy with the heat semigroup in the case of classical diffusion, we shall call $\mathcal{L}$ the non-commutative Laplacian or Lindbladian.
Hudson and Robinson [10] studied the above question for $\mathcal{A}_\theta$ in the case where the representation π is the identity representation in $\mathcal{H}$ itself and concluded that while there exist classical stochastic dilations for the Lindbladians $\mathcal{L}(x) = -\frac{1}{2}{d_1}^2(x)$ or $-\frac{1}{2}{d_2}^2(x)$, there does not exist any $\mathcal{L}$ corresponding to $\delta = d_1 + id_2$ so that there is no quantum stochastic dilation corresponding to this case. We claim that if we choose $\pi(x) = x \otimes I$ in $\mathcal{K} = \mathcal{H} \otimes C^2 \cong \mathcal{H} \oplus \mathcal{H}$, and $\delta_0 = d_1 \oplus d_2$, then $\mathcal{L}_0 = -\frac{1}{2}({d_1}^2 + {d_2}^2), \mathcal{D} = {\mathcal{A}_\theta}^\infty$ satisfies all the properties (i) - (iii) and one can construct a quantum stochastic flow driven by $(\pi, \delta_0, \mathcal{L}_0)$. In analogy, one can have the perturbed triple $(\pi, \delta, \mathcal{L})$ where $\delta = \delta_1 \oplus \delta_2$ with $\delta_1 = d_1 + d_{r_1}$ and $\delta_2 = d_2 + d_{r_2}$ and $\mathcal{L} = -\frac{1}{2}({\delta_1}^2 + {\delta_2}^2), \mathcal{D} = {\mathcal{A}_\theta}^\infty$.

Thus we have two triples $(\pi, \delta_0, \mathcal{L}_0)$ and $(\pi, \delta, \mathcal{L})$ both satisfying (i)-(iii). Hence they should give rise to two quantum stochastic processes and that they indeed do so is the content of theorem 2.1. Therefore from the quantum stochastic point of view also, the two "Laplacians" $\mathcal{L}_0$ and $\mathcal{L}$ are equally good candidates for driving the processes. Then the question arises: can we associate the same geometric features with these two Laplacians or are there geometrically discernible changes as we go from the Laplacian $\mathcal{L}_0$ to the perturbed one $\mathcal{L}$? This will be addressed in the following section.

Theorem 2.1 *(i) The quantum stochastic differential equation (q.s.d.e)* [14] *for* $x \in {\mathcal{A}_\theta}^\infty$

$$dj_t^0(x) = j_t^0(id_1(x))dw_1(t) + j_t^0(id_2(x))dw_2(t) + j_t^0(\mathcal{L}_0(x))dt; \\ j_0^0(x) = x \otimes I \qquad (2.1)$$

has unique solution j_t^0 *which is a* $*$*-homomorphism from* $\mathcal{A}_\theta$ *to* $\mathcal{A}_\theta \otimes \mathcal{B}(\Gamma(L^2(R_+) \otimes C^2))$. *In fact* $j_t^0(x) = \alpha_{(exp2\pi iw_1(t), exp2\pi iw_2(t))}(x)$, *where* $(w_1, w_2)(t)$ *is the standard two dimensional Brownian motion. Also* $Ej_t^0(x) = e^{t\mathcal{L}_0}(x)$, *where* E *is the vacuum expectation in the Fock space*

$\Gamma(L^2(R_+) \otimes C^2)$.
(ii) The q.s.d.e in $\mathcal{H} \otimes \Gamma$*:*

$$dU_t = \sum_{l=1}^{2} U_t\{ij_t{}^0(r_l)dA_l{}^\dagger + ij_t{}^0(r_l{}^*)dA_l - \frac{1}{2}j_t{}^0(r_l{}^*r_l)dt\},$$

$$U_0 = I \tag{2.2}$$

has a unique unitary solution [3]. Setting $j_t(x) = U_t j_t^0(x) U_t^*$*, one has the q.s.d.e :*

$$dj_t(x) = \sum_{l=1}^{2}\{j_t(i\delta_l(x))dA_l{}^\dagger + j_t(i\delta_l{}^\dagger(x))dA_l\} + j_t(\mathcal{L}(x))dt, \tag{2.3}$$

and $Ej_t(x) = e^{t\mathcal{L}}(x)$*.*

We do not give the proof here since most of it is available in the references cited above.

3 Weyl Asymptotics for $\mathcal{A}_\theta$

For classical compact Riemannian manifold (M,g) of dimension d with metric g, one has the natural heat semigroup $\mathcal{T}_t$ as the expectation semigroup of the Brownian motion on the manifold [18] so that the Laplace-Beltrami operator Δ is the generator of $\mathcal{T}_t$. It is known [18] that $\mathcal{T}_t$ is an integral operator on $L^2(M, dvol)$ with a smooth integral kernel $\mathcal{T}_t(x, y)$, which admits an asymptotic expansion as $t \to 0+$:

$$\mathcal{T}_t(x, y) = \sum_{j=0}^{\infty} \mathcal{T}^{(j)}(x, y)t^{-d/2+j}, \tag{3.1}$$

and that

$$vol(M) = \int_M \mathcal{T}^0(x, x)dvol(x)$$

$$= \lim_{t\to 0+} t^{d/2}\int_M \mathcal{T}_t(x, x)dvol(x) = \lim_{t\to 0+} t^{d/2}(Tr\mathcal{T}_t),$$

where we have taken the trace in $L^2(M, dvol)$. Similarly the scalar curvature s at $x \in M$ is given as $s(x) = \frac{1}{6}\mathcal{T}^{(1)}(x, x)$. This gives the integrated scalar curvature

$$s = \int_M s(x)dvol(x) = \frac{1}{6}\int_M \mathcal{T}^{(1)}(x, x)dvol(x)$$

$$= \frac{1}{6} \lim_{t \to 0+} t^{d/2-1} \int [\mathcal{T}_t(x,x) - t^{-d/2} \mathcal{T}^0(x,x)] dvol(x)$$

$$= \frac{1}{6} \lim_{t \to 0+} t^{d/2-1} [Tr \mathcal{T}_t - t^{-d/2} vol(M)]$$

For the non-commutative d-torus (with d even) one possibility is to define its volume V and integrated scalar curvature s by analogy from their classical counterparts as :

$$V(\mathcal{A}_\theta) \equiv V \equiv \lim_{t \to 0+} t^{d/2} Tr \mathcal{T}_t, \tag{3.2}$$

$$s(\mathcal{A}_\theta) \equiv s \equiv \frac{1}{6} \lim_{t \to 0+} t^{d/2-1} [Tr \mathcal{T}_t - t^{-d/2} V] \tag{3.3}$$

where the heat semigroup $\mathcal{T}_t$ in the classical case is replaced by the expectation semigroups of the last section: $\mathcal{T}_t^0 = e^{t\mathcal{L}_0}$ and the perturbed one $\mathcal{T}_t = e^{t\mathcal{L}}$ respectvely acting on $L^2(\mathcal{A}_\theta, \tau)$. Before we can compute these numbers, we need to study the operators $\mathcal{L}_0$ and $\mathcal{L}$ in $L^2(\tau)$ more carefully . The next theorem summarizes their properties for $d = 2$ and we have denoted by $\mathcal{B}_p$ the Schatten ideals in $\mathcal{B}(\mathcal{H})$ with the respective norms.

Theorem 3.1 *(i) $\mathcal{L}_0$ is a negative selfadjoint operator in $L^2(\tau)$ with compact resolvent. In fact $\mathcal{L}_0(U^m V^n) = -\frac{1}{2}(m^2 + n^2) U^m V^n; m, n \in Z$ so that $(\mathcal{L}_0 - z)^{-1} \in \mathcal{B}_p(L^2(\tau))$ for $p > 1$ and $z \in \rho(\mathcal{L}_0)$*

(ii) If $r_1, r_2 \in \mathcal{A}_\theta^\infty$ and are selfadjoint, then $\mathcal{L} = \mathcal{L}_0 + B + A$, where $B = -\frac{1}{2}(d_{r_1}^2 + d_{r_2}^2 + d_{d_1(r_1)} + d_{d_2(r_2)})$ and $A = -d_{r_1} d_1 - d_{r_2} d_2$, so that A is compact relative to $\mathcal{L}_0$ and $\mathcal{L}$ is selfadjoint on $\mathcal{D}(\mathcal{L}_0)$ with compact resolvent.

If $r_1, r_2 \in \mathcal{A}_\theta$, then $-\mathcal{L} = -\mathcal{L}_0 - B - A$ as quadratic form on $D((-\mathcal{L}_0)^{\frac{1}{2}})$ and

$$(-\mathcal{L} + n^2)^{-1} = (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} (I + Z_n)^{-1} (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \tag{3.4}$$

where
$Z_n = (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} (B + A)(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}$, is compact for each n with $B = -\frac{1}{2}(d_{r_1}^2 + d_{r_2}^2), A = \frac{1}{2}(d_1 d_{r_1} + d_{r_1} d_1 + d_2 d_{r_2} + d_{r_2} d_2)$. This defines $\mathcal{L}$ as a selfadjoint operator in $L^2(\tau)$ with compact resolvent. Furthermore, in both cases of (ii) , the difference of resolvents $(\mathcal{L} - z)^{-1} - (\mathcal{L}_0 - z)^{-1}$ is trace class for $z \in \rho(\mathcal{L}) \cap \rho(\mathcal{L}_0)$.

Proof:–

The proof of (i) is obvious and hence is omitted. (ii) It is easy to verify that $\mathcal{L} = \mathcal{L}_0 + B + A$ on $\mathcal{A}_\theta^\infty$ and that $A(-\mathcal{L}_0 + n^2)^{-1}$ is compact for every $n = 1, 2, \ldots$. Therefore $(\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + n^2)^{-1} = (\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + 1)^{-1}(\mathcal{L}_0 + 1)(-\mathcal{L}_0 + n^2)^{-1} \to 0$ in operator norm as $n \to \infty$. By the Kato-Rellich theorem [15], $\mathcal{L}$ is selfadjoint and since $(-\mathcal{L} + n^2)^{-1} = (-\mathcal{L}_0 + n^2)^{-1}[1 + (\mathcal{L}_0 - \mathcal{L})(-\mathcal{L}_0 + n^2)^{-1}]^{-1}$ for sufficiently large n, one also concludes that $\mathcal{L}$ has compact resolvent. Furthermore for $z \in \rho(\mathcal{L}) \cap \rho(\mathcal{L}_0)$,

$$(\mathcal{L} - z)^{-1} - (\mathcal{L}_0 - z)^{-1} = (\mathcal{L}_0 - z)^{-1}[1 + (\mathcal{L} - \mathcal{L}_0)(\mathcal{L}_0 - z)^{-1}]^{-1}(\mathcal{L}_0 - \mathcal{L})(\mathcal{L}_0 - z)^{-1}$$

Since $(\mathcal{L} - \mathcal{L}_0)(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}$ is bounded , $(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3(L^2(\tau))$ and since $(-\mathcal{L}_0 + z)^{-1} \in \mathcal{B}_{3/2}(L^2(\tau))$, It follows that $(\mathcal{L} - n^2)^{-1} - (\mathcal{L}_0 - n^2)^{-1}$ is trace class for $n = 1, 2, \ldots$ by the Holder inequality.

When $r_1, r_2 \in \mathcal{A}_\theta$, we cannot write the expression for $\mathcal{L}$ as above on $\mathcal{A}_\theta^\infty$, since r_1, r_2 may not be in the domain of the derivations d_1, d_2. For this reason, we need to define $-\mathcal{L}$ as the sum of quadratic forms and standard results as in [15] can be applied here . From the structure of B and A it is clear that Z_n is compact for each n and hence an identical reasoning as above would yield that $\|Z_n\| \to 0$ as $n \to \infty$ and therefore $(I + Z_n)^{-1} \in \mathcal{B}$ for sufficiently large n and the right hand side of (3.4) defines the operator $-\mathcal{L}$ associated with the quadratic form with $D((-\mathcal{L})^{\frac{1}{2}}) = D((-\mathcal{L}_0)^{\frac{1}{2}})$. Clearly

$$\begin{aligned}(-\mathcal{L} + n^2)^{-1} - (-\mathcal{L}_0 + n^2)^{-1} &= -(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(I + Z_n)^{-1}Z_n(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \\ &= -(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(I + Z_n)^{-1}(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}(B + A)(-\mathcal{L}_0 + n^2)^{-1}\end{aligned}$$

for sufficiently large n and since

$$(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3, (-\mathcal{L}_0 + n^2)^{-\frac{1}{2}}A(-\mathcal{L}_0 + n^2)^{-\frac{1}{2}} \in \mathcal{B}_3,$$

it is clear that $(\mathcal{L} - n^2)^{-1} - (\mathcal{L}_0 - n^2)^{-1}$ is trace class. □

The next theorem studies the effect of the perturbation from $\mathcal{L}_0$ to $\mathcal{L}$ on the volume and the integrated sectional curvature for $\mathcal{A}_\theta$.

Theorem 3.2 *(i) The volume V of $\mathcal{A}_\theta(d = 2)$ as defined in (3.2) is invariant under the perturbation from $\mathcal{L}_0$ to $\mathcal{L}$.*
(ii) The integrated scalar curvature for $r \in \mathcal{A}_\theta^\infty$, in general is not invariant under the above perturbation.

Proof :– We need to compute $Tr(e^{t\mathcal{L}} - e^{t\mathcal{L}_0})$. Note that if $r_1, r_2 \in \mathcal{A}_\theta^\infty$, then $e^{t\mathcal{L}} - e^{t\mathcal{L}_0} = -\int_0^t e^{(t-s)\mathcal{L}}(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}ds$ which on two iterations yields:

$$e^{t\mathcal{L}} - e^{t\mathcal{L}_0} = -\int_0^t e^{(t-s)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}ds + \int_0^t dt_1 e^{(t-t_1)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)\times$$

$$\int_0^{t_1} dt_2 e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L}-\mathcal{L}_0)e^{t_2\mathcal{L}_0} - \int_0^t dt_1 e^{(t-t_1)\mathcal{L}}(\mathcal{L}-\mathcal{L}_0)\int_0^{t_1} dt_2 e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L}-\mathcal{L}_0)\times$$

$$\int_0^{t_2} dt_3 e^{(t_2-t_3)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)e^{t_3\mathcal{L}_0} \equiv I_1(t) + I_2(t) + I_3(t). \qquad (3.5)$$

For estimating the trace norms of these terms , we note that the $\mathcal{B}_p$-norm of $(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}$ is estimated as

$$\|(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}\|_p = \|(B + A)e^{s\mathcal{L}_0}\|_p \leq \|B\|\|e^{s\mathcal{L}_0}\|_p +$$

$$c_1(\|d_1e^{s\mathcal{L}_0}\|_p + \|d_2e^{s\mathcal{L}_0}\|_p) \leq c''(\|e^{s\mathcal{L}_0}\|_p + \|d_2e^{s\mathcal{L}_0}\|_p)$$

$$\leq c'(s^{-p^{-1}} + s^{-p^{-1}-\frac{1}{2}}) \leq c\ s^{-p^{-1}-\frac{1}{2}}$$

for constants c, c_1, c', c'' since we are interested only for the region $0 < s \leq t \leq 1$. Using Holder inequality for Schatten norms and the fact that

$$\|(\mathcal{L} - n^2)^{-1}\| \leq \|(\mathcal{L}_0 - n^2)^{-1}[1 + (\mathcal{L} - \mathcal{L}_0)(\mathcal{L}_0 - n^2)^{-1}]^{-1}\| \leq \frac{2}{n^2}$$

for sufficiently large n. We get for the third term in 3.5

$$\|I_3(t)\|_1 \leq 2\int_0^t dt_1 \int_0^{t_1} dt_2 \|(\mathcal{L} - \mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}\|_{p_1}\times$$

$$\int_0^{t_2} dt_3 \|(\mathcal{L} - \mathcal{L}_0)e^{(t_2-t_3)\mathcal{L}_0}\|_{p_2}\|(\mathcal{L} - \mathcal{L}_0)e^{t_3\mathcal{L}_0}\|_{p_3}$$

$$\leq c(p_1, p_2, p_3)\int_0^t t_1^{-\frac{1}{2}}dt_1 \to 0$$

as $t \to 0$ where $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$. A very similar estimate shows that

$$\|I_1(t)\|_1 \leq \int_0^t ds\|e^{(t-s)\mathcal{L}_0}\|_{p_1}\|(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}\|_{p_2} \leq ct^{-\frac{1}{2}}$$

(with $p_2 > 2$ *and* $p_1^{-1} + p_2^{-1} = 1$) and

$$\|I_2(t)\|_1 \leq \int_0^t dt_1 \|e^{(t-t_1)\mathcal{L}_0}\|_{p_1} \int_0^{t_1} dt_2 \|(\mathcal{L}-\mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}\|_{p_2} \|(\mathcal{L}-\mathcal{L}_0)e^{t_2\mathcal{L}_0}\|_{p_3} \leq c',$$

(with $p_1^{-1} + p_2^{-1} + p_3^{-1} = 1$, in particular the choice $p_1 = p_2 = p_3 = 3$ will do) a constant independent of t . From this it follows that $\lim_{t\to 0+} t\, Tr(e^{t\mathcal{L}} - e^{t\mathcal{L}_0}) = 0$ and thus the invariance of volume under perturbation.

In the case when $r_1, r_2 \in \mathcal{A}_\theta$ only , then $\mathcal{L} - \mathcal{L}_0 = B + d_1B_1 + d_2B_2 + B_1'd_1 + B_2'd_2$ where B, B_1, B_1', B_2, B_2' are bounded. Therefore the term like $e^{(t-s)\mathcal{L}_0} d_1 B_1 e^{s\mathcal{L}_0} = [e^{s\mathcal{L}_0} B_1^* d_1 e^{(t-s)\mathcal{L}_0}]^*$ admits similar estimates as above and the same result follows.

(ii) From the expression (3.3) for the integrated scalar curvature s, we see that for $d = 2$

$$s(\mathcal{L}) - s(\mathcal{L}_0) = \frac{1}{6} \lim_{t\to 0+} Tr(e^{t\mathcal{L}} - e^{t\mathcal{L}_0}) \tag{3.6}$$

if it exists, and conclude that the contribution to (3.6) from the term $I_3(t)$ vanishes as we have seen in (i).We claim that though $\|I_2(t)\|_1 \leq$ constant, $TrI_2(t) \to 0$ as $t \to 0+$. In fact since the integrals in $I_2(t)$ converges in trace norm

$$TrI_2(t) = \int_0^t dt_1 \int_0^{t_1} dt_2 Tr((\mathcal{L} - \mathcal{L}_0)e^{(t_1-t_2)\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)e^{(t-t_1+t_2)\mathcal{L}_0})$$

and by a change of variable we have that $|TrI_2(t)| \leq t \int_0^t \|(\mathcal{L} - \mathcal{L}_0)e^{s\mathcal{L}_0}(\mathcal{L} - \mathcal{L}_0)e^{(t-s)\mathcal{L}_0}\|_1 ds$ For $r \in \mathcal{A}_\theta^\infty$, the perturbation $(\mathcal{L} - \mathcal{L}_0)$ is of the form $b_0 + b_1d_1 + b_2d_2$ with $b_i \in \mathcal{B}(\mathcal{H})$ for $i = 0, 1, 2$ and the Hilbert-Schmidt norm estimates are as follows :

$$\|(\mathcal{L}-\mathcal{L}_0)e^{s\mathcal{L}_0}\|_2 \leq \|b_0\|\|e^{s\mathcal{L}_0}\|_2 + \sqrt{2}(\|b_1\|+\|b_2\|)\|(-\mathcal{L}_0)^{\frac{1}{2}}e^{s\mathcal{L}_0}\|_2 \leq c(s^{-\frac{1}{2}}+s^{-\frac{3}{4}}).$$

Therefore

$$|TrI_2(t)| \leq ct \int_0^t (s^{-\frac{1}{2}} + s^{-\frac{3}{4}})((t-s)^{-\frac{1}{2}} + (t-s)^{-\frac{3}{4}})$$

and this clearly converges to zero as $t \to 0+$.

This leaves only $I_1(t)$ contribution so that

$$6(s(\mathcal{L}) - s(\mathcal{L}_0)) = -\lim_{t\to 0+} t\, Tr((\mathcal{L} - \mathcal{L}_0)e^{t\mathcal{L}_0}).$$

As before we note that $(\mathcal{L}-\mathcal{L}_0)$ contains two kinds of terms :
$B=-\frac{1}{2}(d_{r_1}^2+d_{r_2}^2), A=-\frac{1}{2}(d_{r_1}d_1+d_1d_{r_1}+d_{r_2}d_2+d_2d_{r_2})$
We show that the term $Tr(Ae^{t\mathcal{L}_0})=0$ for all $t>0$. It suffices to show that $Tr(d_rd_1e^{t\mathcal{L}_0})=0$ for $r\in\mathcal{A}_\theta^\infty$ and for this we note that

$$Tr(d_rd_1e^{t\mathcal{L}_0})=\sum_{m,n}<U^mV^n,d_rd_1e^{t\mathcal{L}_0}(U^mV^n)>$$

$$=\sum_{m,n}me^{-t/2(m^2+n^2)}\tau(V^{-n}U^{-m}d_r(U^mV^n))=\sum_{m,n}me^{-t/2(m^2+n^2)}\tau(V^{-n}U^{-m}rU^mV^n-r)=0$$

identically. This leaves only the contribution due to B. Thus

$$s(\mathcal{L})-s(\mathcal{L}_0)=\frac{1}{12}\lim_{t\to 0+}t\ Tr((d_{r_1}^2+d_{r_2}^2)e^{t\mathcal{L}_0}), \tag{3.7}$$

if it exists. However since $\{tTr(({d_{r_1}}^2+{d_{r_2}}^2)e^{t\mathcal{L}_0})\}$ is bounded as $t\to 0+$, we can and will interpret the above limit as a special kind of Banach limit as in Connes [2,p.563]

$$s(\mathcal{L})-s(\mathcal{L}_0)=\frac{1}{12}Lim_{t^{-1}\to\omega}tTr(({d_{r_1}}^2+{d_{r_2}}^2)e^{t\mathcal{L}_0}) \tag{3.8}$$

$$=\frac{1}{12}Tr_\omega(({d_{r_1}}^2+{d_{r_2}}^2)\hat{\mathcal{L}_0}^{-1} \tag{3.9}$$

The notation $\hat{\mathcal{L}_0}$ will be explained in the next section. In the following we show that in general the right hand side of (3.8) is strictly positive.

For example set $r_1=(U+U^{-1})$ and $r_2=0$, then $r_1,r_2\in\mathcal{A}_\theta^\infty$, and

$$6(s(\mathcal{L})-s(\mathcal{L}_0))=\frac{1}{2}Lim_{t^{-1}\to\omega}t\sum_{m,n}e^{-t/2(m^2+n^2)}<U^mV^n,d_{r_1}^2(U^mV^n)>$$

$$=2^{-1}Lim_{t^{-1}\to\omega}t\sum_{m,n}e^{-t/2(m^2+n^2)}\tau((1-\lambda^{-n})^2\lambda^{2n}U^2+(1-\lambda^n)^2\lambda^{-2n}U^{-2}+(2-\lambda^n-\lambda^{-n}))$$

$$=2^{-1}Lim_{t^{-1}\to\omega}t\left(2\sum_{m=1}^\infty e^{-m^2t/2}+1\right)\left(8\sum_{n=1}^\infty sin^2(\pi\theta n)e^{-n^2t/2}\right)$$

Next note that for $0<t<2$

$$\sqrt{t}\sum_{n=1}^\infty sin^2(\pi\theta n)e^{-n^2t/2}\geq\sqrt{t}\sum_{n=1}^{[\sqrt{2/t}]}sin^2(\pi\theta n)e^{-n^2t/2}$$

$$\geq e^{-1}(\sqrt{2}-\sqrt{t})\sum_{n=1}^{[\sqrt{2/t}]}[\sqrt{2/t}]^{-1}sin^2\pi(n\theta-[n\theta]) = e^{-1}(\sqrt{2}-\sqrt{t})E(sin^2\pi X_t),$$

where for each $0 < t \leq 2$, X_t is a $[0,1]$-valued random variable with Probability$(X_t = k\theta - [k\theta]) = [\sqrt{2/t}]^{-1}$ for $k = 1, 2, \ldots, [\sqrt{\frac{2}{t}}]$ and E is the associated expectation. Since θ is irrational, it is known that ([9]) as $t \to 0+$, the random variable X_t converges weakly to one with uniform distribution on $[0,1]$ and therefore

$$\liminf_{t\to 0+}\sqrt{t}\sum_{n=1}^{\infty}sin^2(\pi\theta n)e^{-n^2t/2} \geq \lim_{t\to 0+}\sqrt{t}\sum_{n=1}^{[\sqrt{2/t}]}sin^2(\pi\theta n)e^{-n^2t/2}$$

$$\geq \sqrt{2}e^{-1}\int_0^1 sin^2\pi x dx = (\sqrt{2}e)^{-1}.$$

We also have by Connes (page 563) [2] $\lim_{t\to 0+}\sqrt{t}\sum_{m=1}^{\infty}e^{-m^2t/2} = \frac{\sqrt{\pi}}{\sqrt{2}}$ Now by the general properties of the limiting procedure as expounded in [2]

$$s(\mathcal{L}) - s(\mathcal{L}_0) \geq \frac{2\sqrt{\pi}}{3e}$$

□

Remark:– From the expression for $s(\mathcal{L}_0)$, we see that for $d = 2$, $s(\mathcal{L}_0) = \lim_{t\to 0+}(Tre^{t\mathcal{L}_0} - \frac{V}{t})$. Since the expression for $Tre^{t\mathcal{L}_0}$and the volume V are exactly the same as in the case of classical two-torus with its heat semigroup, the integrated scalar curvature for $\mathcal{L}_0$ is the same as in the classical case, which is clearly zero. Therefore $s(\mathcal{L})$ is strictly positive for the case considered here.

4 Spectral Triple on $\mathcal{A}_\theta^\infty$, its perturbation and cohomology

Following Connes [2] we consider the even spectral triple $(\mathcal{A} = \mathcal{A}_\theta^\infty, \mathcal{H} = L^2(\tau)\oplus L^2(\tau), D_0, \Gamma)$ where D_0 , the unperturbed Dirac operator= $\begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix} \equiv i\gamma_1 d_1(a) + i\gamma_2 d_2(a)$ in $\mathcal{H}$. Here γ_1, γ_2 are the

2×2 clifford matrices. The grading operator is given by $\Gamma = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$. One easily verifies that $a\Gamma = \Gamma a, \Gamma^* = \Gamma = \Gamma^{-1}, \Gamma D_0 = -D_0\Gamma$. Note also D_0 has compact resolvent since $D_0^2 = -2 \begin{pmatrix} \mathcal{L}_0 & 0 \\ 0 & \mathcal{L}_0 \end{pmatrix}$ and $ker D_0 = ker\mathcal{L}_0 \otimes C^2$ is two dimensional. The perturbed spectral triple is taken to be $(\mathcal{A}, \mathcal{H}, D, \Gamma)$ where $D = D_0 + \begin{pmatrix} 0 & d_r \\ d_{r^*} & 0 \end{pmatrix}$ for some $r \in \mathcal{A}_\theta^\infty$. It is not difficult to see that D_0 and D are both essentially selfadjoint on $\mathcal{A} \subseteq \text{Ł}^2(\tau)$ and that the perturbed triple is also an even one. Here, as in Connes [2], by the volume form $v(a)$ on $\mathcal{A}$ we mean the linear functional $v(a) = \frac{1}{2}Tr_w(a|\hat{D}|^{-2}P)$ where Tr_w is the Dixmier trace [2], and we have used the notation that for a self-adjoint operator T with compact resolvent $\hat{T} = T|_{N(T)^\perp} \equiv TP$, where P is the projection on $N(T)^\perp$. Next we prove that the volume form is invariant under the above perturbation. For this we need a lemma.

Lemma 4.1 *Let T be a selfadjoint operator with compact resolvent such that $\hat{T}^{-1}$ is Dixmier trace-able. Then for $a \in \mathcal{A}$ and every $z \in \rho(T)$,* $Tr_w(a\hat{T}^{-1}P) = Tr_w(a(T-z)^{-1})$.

Proof:–
Note that $(T-z)^{-1} = (\hat{T}-z)^{-1}P \oplus -z^{-1}P^\perp$ and $P^\perp$ is finite dimensional. Therefore $Tr_w(a(T-z)^{-1}) = Tr_w(PaP(\hat{T}-z)^{-1}P)$. On the other hand $Tr_w(PaP\hat{T}^{-1}P - PaP(\hat{T}-z)^{-1}P) = -zTr_w(PaP\hat{T}^{-1}(\hat{T}-z)^{-1}P) = 0$, since $\hat{T}^{-1}$ is Dixmier trace-able and $(\hat{T}-z)^{-1}$ is compact [2] □

Theorem 4.2 *If we set $v_0(a) = \frac{1}{2}Tr_w(a|\hat{D}_0|^{-2})$ and $v(a) = \frac{1}{2}Tr_w(a|\hat{D}|^{-2})$ for $a \in \mathcal{A}$, then $v_0(a) = v(a)$*

Proof :–
Note that $D^2 = -2 \begin{pmatrix} \mathcal{L}_1 & 0 \\ 0 & \mathcal{L}_2 \end{pmatrix}$, where
$\mathcal{L}_1 = \mathcal{L}_0 + d_r d_{r^*} + (d_1 d_{r^*} + d_r d_1) + i(d_2 d_{r^*} - d_r d_2)$ and
$\mathcal{L}_2 = \mathcal{L}_0 + d_{r^*} d_r + (d_1 d_r + d_{r^*} d_1) + i(d_2 d_{r^*} - d_r d_2)$, and that by theorem (3.1) of section 3, both $\mathcal{L}_1$ and $\mathcal{L}_2$ have compact resolvents with P_1, P_2 projections on $\mathcal{N}(\mathcal{L}_1)^\perp$ and $\mathcal{N}(\mathcal{L}_2)^\perp$ respectively. Therefore by the previous lemma for $Imz \neq 0$

$$v(a) = Tr_w(a(-\hat{\mathcal{L}}_1)^{-1}P_1) + Tr_w(a(-\hat{\mathcal{L}}_2)^{-1}P_2)$$

$$= Tr_w(a(-\mathcal{L}_1 - z)^{-1} + a(-\mathcal{L}_2 - z)^{-1})$$
$$= Tr_w(a(-\mathcal{L}_0 - z)^{-1} + a(-\mathcal{L}_0 - z)^{-1}) + Tr_w(a(-\mathcal{L}_1 - z)^{-1}$$
$$-a(-\mathcal{L}_0 - z)^{-1}) + Tr_w(a(-\mathcal{L}_2 - z)^{-1} - a(-\mathcal{L}_0 - z)^{-1}) = v_0(a)$$

since $(-\mathcal{L}_i - z)^{-1} - (-\mathcal{L}_0 - z)^{-1}$ is trace class for $i = 1, 2$ □

We say that two spectral triples $(\mathcal{A}_1, \mathcal{H}_1, D_1)$ and $(\mathcal{A}_2, \mathcal{H}_2, D_2)$ are unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $D_2 = UD_1U^*$ and $\pi_2(.) = U\pi_1(.)U^*$, where $\pi_j,\ j = 1, 2$ are the representation of $\mathcal{A}_j$ in $\mathcal{H}_j$ respectively. Now, we want to prove that in general the perturbed spectral triple is not unitarily equivalent to the unperturbed one. Let $\Omega^1(\mathcal{A}_\theta^\infty)$ be the universal space of 1-forms ([2]) and π be the representation of $\Omega^1 \equiv \Omega^1(\mathcal{A}_\theta^\infty)$ in $\mathcal{H}$ given by

$$\pi(a) = a, \pi(\delta(a)) = [D, a],$$

where δ is the universal derivation.
Note that $[D, a] = i[\delta_1(a)\gamma_1 + \delta_2(a)\gamma_2]$, where $r_1 = \text{Re } r,\ r_2 = \text{Im } r\ \delta_1 = d_1 + d_{r_1}, \delta_2 = d_2 + d_{r_2}$.

Theorem 4.3 *(i) Let $r = U^m$, then $\Omega^1_D(\mathcal{A}_\theta^\infty) := \pi(\Omega^1) = \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$.*
(iii) $\Omega^2(\mathcal{A}_\theta^\infty) = 0$ for $r = U^m$.

Proof :-
(i) Clearly $\pi(\Omega^1) \subseteq \mathcal{A}_\theta^\infty\gamma_1 + \mathcal{A}_\theta^\infty\gamma_2$. The other inclusion follows from the facts that $\delta_2(U^k) = 0, \delta_1(U^k)$ is invertible, and that $\delta_2(V^l)$ is invertible for sufficiently large l.

(iii) Let $J_1 = \text{Ker}\pi|_{\Omega^1},\ J_2 = \text{Ker}\pi|_{\Omega^2}$. Then $J_2 + \delta J_1$ is an ideal, implying that $\pi(\delta J_1) = \pi(J_2 + \delta J_1)$ is a nonzero submodule of $\pi(\Omega^2) \subseteq \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. Since $\mathcal{A}_\theta^\infty$ is simple there are two possibilities, namely either $\pi(\delta J_1) \cong \mathcal{A}_\theta^\infty$, or $\pi(\delta J_1) = \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. To rule out the first possibility we take a closer look at J_1 and $\pi(\delta J_1)$. $J_1 = \{\sum_i a_i\delta(b_i) | \sum_i a_i\delta_1(b_i) = 0, \sum_i a_i\delta_2(b_i) = 0\}$. Using the fact that δ_1, δ_2 are derivations we get

$$\sum_i \delta_1(a_i)\delta_2(b_i) = -\sum_i a_i\delta_1(\delta_2(b_i)) \quad (4.1)$$
$$\sum_i \delta_2(a_i)\delta_1(b_i) = -\sum_i a_i\delta_2(\delta_1(b_i)) \quad (4.2)$$

for $\sum_i a_i\delta(b_i) \in J_1$

$$\pi(\sum_i \delta(a_i)\delta(b_i)) = \sum_i (\delta_1(a_i)\gamma_1 + \delta_2(a_i)\gamma_2)(\delta_1(b_i)\gamma_1 + \delta_2(b_i)\gamma_2)$$

$$= \sum_i (\delta_1(a_i)\delta_1(b_i) + \delta_2(a_i)\delta_2(b_i)) + \sum (\delta_1(a_i)\delta_2(b_i) - \delta_2(a_i)\delta_1(b_i))\gamma_{12},$$

where $\gamma_{12} = \gamma_1\gamma_2 = -\gamma_2\gamma_1$. Taking $x = U^{-1}\delta(U) + U\delta(U^{-1}) \in \Omega^1$ it is easy to verify that $x \in J_1$ and $\pi(\delta x) = -2$. This proves $\mathcal{A}_\theta^\infty \oplus 0 \subseteq \pi(\delta J_1)$. We show that the inclusion is proper by showing the nontriviality of coefficient of γ_{12}. Using 4.1, 4.2 we get coefficient of γ_{12} to be $\sum a_i[\delta_1, \delta_2](b_i) = \sum -ima_i[r_1, b_i]$. As before we can find n_0 such that for $l \geq n_0$, $\delta_2(V^l)$ is invertible. If we now choose $a_1 = I, b_1 = V^{n_0}, a_2 = -\delta_2(V^{n_0})\delta_2(V^l)^{-1}, b_2 = V^l, a_3 = (-a_1\delta_1(b_1) - a_2\delta_2(b_2))U^{-1}, b_3 = U$, then the vanishing of the coefficient of γ_{12} will imply that $[r_1, V^{n_0}] = \delta_2(V^{n_0})\delta_2(V^l)^{-1}[r_1, V^l]$ for all $l \geq n_0$ and we note that while the left hand side is nonzero and independent of l, the right hand side converges to 0 as $l \to \infty$ leading to a contradiction. Therefore $\mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty = \pi(\delta J_1) \subseteq \pi(\Omega^2) \subseteq \mathcal{A}_\theta^\infty \oplus \mathcal{A}_\theta^\infty$. Hence $\Omega_D^2(\mathcal{A}_\theta^\infty) = \frac{\pi(\Omega^2)}{\pi(\delta J_1)} = 0$. □

Thus we have the following :

Theorem 4.4 *The spectral triples $(\mathcal{A}_\theta^\infty, \mathcal{H}, D_0)$ and $(\mathcal{A}_\theta^\infty, \mathcal{H}, D)$ are not unitarily equivalent for $r = U^m$.*

The proof is clear since $\Omega_{D_0}^2(\mathcal{A}_\theta^\infty) = \mathcal{A}_\theta^\infty \neq 0 = \Omega_D^2(\mathcal{A}_\theta^\infty)$.

Classically there is a correspondence between connection form and covariant differentiation. This correspondence comes from the duality between the module of derivations and the module of sections in the cotangent bundle. Unfortunately there is no such duality in the non-commutative context. Here for defining the connection form we visualize it more as the connection form arising from covariant differentiation. We need to do so because if we take the existing definition [5] then the curvature form becomes trivial.

Let $\mathcal{K}$ be the vector space of all derivations $d : \mathcal{A}_\theta^\infty \to \mathcal{A}_\theta^\infty$. This space is same as $\{c_1d_1 + c_2d_2 + [r, .] : r \in \mathcal{A}_\theta^\infty\}$ for almost all θ (lebesgue) [1] for the rest of this section we will be using those $\theta's$ only. Let δ_{mn} be the element of $\mathcal{K}$ given by $\delta_{mn}(a) = [U^mV^n, a]$. We turn $\mathcal{K}$ into an inner product space by requiring that $\{d_1, d_2, \delta_{mn}\}$ to be orthonormal, for example as in [11]. Let $\mathcal{E}$ be any normed $\mathcal{A}_\theta^\infty$-module. For $\delta \in \mathcal{K}$, let $c_\delta : \mathcal{E} \otimes \mathcal{K} \to \mathcal{E}$, be the contraction with respect to δ. Topologize $\mathcal{E} \otimes \mathcal{K}$ with the weak topology inherited from

$c_\delta, \delta \in \mathcal{K}$. Then a *connection* is a complex-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes \mathcal{K}$ such that $c_\delta \nabla(\xi a) = c_\delta \nabla(\xi) a + \xi \delta(a), \ \forall \delta \in \mathcal{K}$.

Theorem 4.5 *Suppose that* ∇_1, ∇_2 *are maps from* $\mathcal{E}$ *to* $\mathcal{E}$ *satisfying*

$$\nabla_i(\xi a) = \nabla_i(\xi) a + \xi d_i(a), \ i = 1, 2.$$

Then the map ∇ *given by*

$$\nabla(\xi) = \nabla_1 \otimes d_1 + \nabla_2 \otimes d_2 - \sum \xi U^m V^n \otimes \delta_{mn}$$

is well-defined and is a connection.

Proof :-
Let $\delta \in \mathcal{K}$, such that $\delta = c_1 d_1 + c_2 d_2 + \sum c_{mn} \delta_{mn}$, where $\{c_{mn}\} \in \mathcal{S}(\mathbb{Z}^2) \subseteq l_1(\mathbb{Z}^2)$. Therefore the sum in the right hand side of the definition of ∇ converges in the topology referred above. The rest is straightforward. □

It is clear from the definition of ∇ in the above theorem that $\nabla_j = c_{d_j} \nabla$ for $(j = 1, 2)$. We also set $\nabla_r = c_{d_r} \nabla$ for $r \in \mathcal{A}_\theta^\infty$

Definition 4.6 *Let* $R : \mathcal{K} \otimes \mathcal{K} \to \mathcal{L}(\mathcal{E})$ *be the map given by* $R(\delta_1, \delta_2) = c_{[\delta_1, \delta_2]} \nabla - [c_{\delta_1} \nabla, c_{\delta_2} \nabla]$. *We call* R *the curvature 2-form associated with the connection* ∇.

Theorem 4.7 *We have*

$$R(d_1, d_2) = R(d_1 + d_{r_1}, d_2 + d_{r_2}).$$

Proof :-
$[d_1 + d_{r_1}, d_2 + d_{r_2}] = [d_1(r_2), .] - [d_2(r_1), .] + [[r_1, r_2], .]$. So we have

$$\begin{aligned}
&R(d_1 + d_{r_1}, d_2 + d_{r_2})(\xi) \\
&= -\xi d_1(r_2) + \xi d_2(r_1) - \xi [r_1, r_2] - (\nabla_1 + \nabla_{r_1})(\nabla_2 \xi - \xi r_2) + (\nabla_2 + \nabla_{r_2})(\nabla_1 \xi - \xi r_1) \\
&= -[\nabla_1, \nabla_2]\xi + \nabla_1(\xi r_2) + (\nabla_2 \xi) r_1 - \xi r_2 r_1 - \nabla_2(\xi r_1) \\
&- (\nabla_1 \xi) r_2 + \xi r_1 r_2 - \xi d_1(r_2) + \xi d_2(r_1) - \xi [r_1, r_2] \\
&= -[\nabla_1, \nabla_2]\xi = R(d_1, d_2)(\xi) \ \ (\text{since} \ \ [d_1, d_2] = 0).
\end{aligned}$$

□

Remark:– In section 3, we have seen that the integrated scalar curvature under the perturbed Lindbladian is different from zero, whereas in section 4,the curvature 2-form has been shown to be invariant under the same perturbation.

5 Non-commutative $2d$-dimensional space

In this section we shall discuss the geometry of the simplest kind of non-compact manifolds, namely the Euclidean $2d$-dimensional space and its non-commutative counterpart. Let $d \geq 1$ be an integer and let $\mathcal{A}_c \equiv C_0(\mathbb{R}^{2d})$, the (nonunital) C^*-algebra of all complex-valued continuous functions on $\mathbb{R}^{2d}$ which vanish at infinity. Then $\partial_j (j = 1, 2, \ldots, 2d)$, the partial derivative in the j-th direction, can be viewed as a densely defined derivation on $\mathcal{A}_c$, with the domain $\mathcal{A}_c^\infty \equiv C_c^\infty(\mathbb{R}^{2d})$, the set of smooth complex valued functions on $\mathbb{R}^{2d}$ having compact support. We consider the Hilbert space $L^2(\mathbb{R}^{2d})$ and naturally imbed $\mathcal{A}_c^\infty$ in it as a dense subspace. Then $i\partial_j$ is a densely defined symmetric linear map on $L^2(\mathbb{R}^{2d})$ with domain $\mathcal{A}_c^\infty$, and we denote its self-adjoint extension by the same symbol. Also, let $\mathcal{F}$ be the Fourier transform on $L^2(\mathbb{R}^{2d})$ given by

$$\hat{f}(k) \equiv (\mathcal{F}f)(k) = (2\pi)^{-d} \int e^{-ik.x} f(x) dx,$$

and M_φ be the operator of multiplication by the function φ. We set $\widetilde{M_\varphi} = \mathcal{F}^{-1} M_\varphi \mathcal{F}$, thus $i\partial_j = \widetilde{M_{x_j}}$. $\Delta \equiv \widetilde{M}_{-\sum x_j^2}$ is the self-adjoint negative operator, called the $2d$-dimensional Laplacian. Clearly, the restriction of Δ on $\mathcal{A}_c^\infty$ is the differential operator $\sum_{j=1}^{2d} \partial_j^2$. Let $h = L^2(\mathbb{R}^d)$ and U_α, V_β be two strongly continuous groups of unitaries in h, given by the following :

$$(U_\alpha f)(t) = f(t + \alpha), \quad (V_\beta f)(t) = e^{it.\beta} f(t), \quad \alpha, \beta, t \in \mathbb{R}^d, \; f \in C_c^\infty(\mathbb{R}^d).$$

Here $t.\beta$ is the usual Euclidean inner product of $\mathbb{R}^d$. It is clear that

$$\begin{aligned} U_\alpha U_{\alpha'} &= U_{\alpha+\alpha'}, \\ V_\beta V_{\beta'} &= V_{\beta+\beta'}, \\ U_\alpha V_\beta &= e^{i\alpha.\beta} V_\beta U_\alpha. \end{aligned} \tag{5.1}$$

For convenience, we define a unitary operator W_x for $x = (\alpha, \beta) \in \mathbb{R}^{2d}$ by

$$W_x = U_\alpha V_\beta e^{-\frac{i}{2}\alpha.\beta},$$

so that the Weyl relation (5.1) is now replaced by $W_x W_y = W_{x+y} e^{\frac{i}{2} p(x,y)}$, where $p(x, y) = x_1.y_2 - x_2.y_1,$ for $x = (x_1, x_2), y = (y_1, y_2)$. This is exactly the Segal form of the Weyl relation ([4]). For f such that $\hat{f} \in L^1(\mathbb{R}^{2d})$, we set

$$b(f) = \int_{\mathbb{R}^{2d}} \hat{f}(x) W_x dx \in \mathcal{B}(h).$$

Let $\mathcal{A}^\infty$ be the $*$-algebra generated by $\{b(f) | f \in C_c^\infty(\mathbb{R}^{2d})\}$ and let $\mathcal{A}$ be the C^*-algebra generated by $\mathcal{A}^\infty$ with the norm inherited from $\mathcal{B}(h)$. It is easy to verify using the commutation relation (5.1) that $b(f)b(g) = b(f \odot g)$ and $b(f)^* = b(f^\natural)$, where

$$\widehat{(f \odot g)}(x) = \int \hat{f}(x - x')\hat{g}(x') e^{\frac{i}{2} p(x,x')} dx'; \quad f^\natural(x) = \bar{f}(-x).$$

We define a linear functional τ on $\mathcal{A}^\infty$ by setting $\tau((b(f)) = \hat{f}(0)$ $(= (2\pi)^{-d} \int f(x)dx)$, and easily verify ([4], page 36) that it is a well-defined faithful trace on $\mathcal{A}^\infty$. It is natural to consider $\mathcal{H} = L^2(\mathcal{A}^\infty, \tau)$ and represent $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$ by left multiplication. From the definition of τ, it is clear that the map $C_c^\infty(\mathbb{R}^{2d}) \ni f \mapsto b(f) \in \mathcal{A}^\infty \subseteq \mathcal{H}$ extends to a unitary isomorphism from $L^2(\mathbb{R}^{2d})$ onto $\mathcal{H}$ and in the sequel we shall often identify the two.

There is a canonical $2d$-paramater group of automorphism of $\mathcal{A}$ given by $\varphi_\alpha(b(f)) = b(f_\alpha)$, where $\hat{f}_\alpha(x) = e^{i\alpha.x}\hat{f}(x),\ f \in C_c^\infty(\mathbb{R}^{2d}),\ \alpha \in \mathbb{R}^{2d}$. Clearly, for any fixed $b(f) \in \mathcal{A}^\infty$, $\alpha \mapsto \varphi_\alpha(b(f))$ is smooth, and on differentiating this map at $\alpha = 0$, we get the canonical derivations $\delta_j, j = 1, 2, \ldots, 2d$ as $\delta_j(b(f)) = b(\partial_j(f))$ for $f \in C_c^\infty(\mathbb{R}^{2d})$. We shall not notationally distinguish between the derivation δ_j on $\mathcal{A}^\infty$ and its extension to $\mathcal{H}$, and continue to denote by $i\delta_j$ both the derivation on $*$-algebra $\mathcal{A}^\infty$ and the associated self-adjoint operator in $\mathcal{H}$.

Let us now go back to the classical case. As a Riemannian manifold, $\mathbb{R}^{2d}$ does not posses too many interesting features; it is a flat manifold and thus there is no nontrivial curvature form. Instead, we shall be interested in obtaining the volume form from the operator-theoretic data associated with the $2d$-dimensional Laplacian Δ. Let $\mathcal{T}_t = e^{\frac{t}{2}\Delta}$ be the contractive C_0-semigroup generated by Δ, called the heat semigroup on $\mathbb{R}^{2d}$. Unlike compact manifolds, Δ has only absolutely continuous spectrum. But for any $f \in C_c^\infty(\mathbb{R}^{2d})$ and $\epsilon > 0$, $M_f(-\Delta + \epsilon)^{-d}$ has discrete spectrum. Furthermore, we have the following :

Theorem 5.1 *$M_f\mathcal{T}_t$ is trace-class and $Tr(M_f\mathcal{T}_t) = t^{-d}\int f(x)dx$. Thus, in particular, $v(f) \equiv \int f(x)dx = t^d Tr(M_f\mathcal{T}_t)$.*

Proof :-
We have $Tr(M_f \mathcal{T}_t) = Tr(\mathcal{F} M_f \mathcal{F}^{-1} M_{e^{-\frac{t}{2}\sum x_j^2}})$, and $\mathcal{F} M_f \mathcal{F}^{-1} M_{e^{-\frac{t}{2}\sum x_j^2}}$ is an integral operator with the kernel $k_t(x,y) = \hat{f}(x-y)e^{-\frac{t}{2}\sum y_j^2}$. It is continuous in both arguments and $\int |k_t(x,x)|dx < \infty$, we obtain by using a result in [6], (p. 114, ch.3) that $M_f \mathcal{T}_t$ is trace class and $Tr(M_f \mathcal{T}_t) = \int k_t(x,x)dx = (2\pi)^d t^{-d} \hat{f}(0) = t^{-d} v(f)$. □

As in section 4, we get an alternative expression for the volume form v in terms of the Dixmier trace.

Theorem 5.2 *For $\epsilon > 0$, $M_f(-\Delta + \epsilon)^{-d}$ is of Dixmier trace class and its Dixmier trace is equal to $\pi^d v(f)$.*

For convenience, we shall give the proof only in the case $d = 1$. We need following two lemmas.

Lemma 5.3 *If $f, g \in L^p(\mathbb{R}^2)$ for some p with $2 \leq p < \infty$, then $M_f \widetilde{M_g}$ is a compact operator in $L^2(\mathbb{R}^2)$.*

Proof :-
It is a consequence of the Holder and Hausdorff-Young inequalities. We refer the reader to [16], volume III for a proof. □

Lemma 5.4 *Let S be a square in $\mathbb{R}^2$ and f be a smooth function with* $\mathrm{Supp}(f) \subseteq \mathrm{int}(S)$. *Let Δ_S denote the Laplacian on S with the periodic boundary condition. Then $Tr_\omega(M_f(-\Delta_S + \epsilon)^{-1}) = \pi \int f(x)dx$.*

Proof :-
This follows from [12] by identifying S with the two-dimensional torus in the natural manner. □

Proof of the theorem :-
Note that for $g \in \mathcal{D}(\Delta) \subseteq L^2(\mathbb{R}^2)$, we have $fg \in \mathcal{D}(\Delta_S)$ and $(\Delta_S M_f - M_f \Delta)(g) = (\Delta M_f - M_f \Delta)(g) = Bg$, where $B = -M_{\Delta f} + 2i \sum_{j=1}^2 M_{\partial_j(f)} \circ \partial_j$. From this follows the identity

$$\begin{aligned} & M_f(-\Delta + \epsilon)^{-1} - (-\Delta_S + \epsilon)^{-1} M_f \\ = & \ (-\Delta_S + \epsilon)^{-1} B (-\Delta + \epsilon)^{-1}. \end{aligned} \tag{5.2}$$

Now, from the Lemma 5.3, it follows that $B(-\Delta + \epsilon)^{-1}$ is compact, and since $(-\Delta_S + \epsilon)^{-1}$ is of Dixmier trace class (by the Lemma 5.4), we have that the right hand side of (5.2) is of Dixmier trace class with the Dixmier trace = 0. The theorem follows from the genaral fact that $Tr_\omega(xy) = Tr_\omega(yx)$, if y is of Dixmier trace class and x is bounded (see [2]). □

Similar computation can be done for the non-commutative case. The Lindbladian $\mathcal{L}_0$ generated by the canonical derivation δ_j on $\mathcal{A}$ is given by

$$\mathcal{L}_0(a(f)) = \frac{1}{2}a(\Delta f),\ f \in C_c^\infty(R^{2d}) \tag{5.3}$$

Since in $L^2(R^{2d})$ $\frac{1}{2}\Delta$ has a natural selfadjoint extension (which we continue to express by the same symbol), $\mathcal{L}_0$ also has an extension as a negative selfadjoint operator in $\mathcal{H} \cong L^2(R^{2d})$, and we define the heat semigroup for this case as $\mathcal{T}_t = e^{t\mathcal{L}_0}$. By analogy we can define the volume form on $\mathcal{A}^\infty$ by setting $v(a(f)) = \lim_{t\to 0+} t^d Tr(a(f)\mathcal{T}_t)$. Then we have

Theorem 5.5 $v(a(f)) = \int f dx$

Proof:–
The kernel $\tilde{K}_t$ of the integral operator $a(f)\mathcal{T}_t$ in $\mathcal{H}$ is given as $\tilde{K}_t(x,y) = \hat{f}(\underline{x}-\underline{y})e^{-t|y|^2/2}e^{ip(x,y)/2}$. As before we note that K_t is continuous in R^{2d} and $\tilde{K}_t(x,x) = k_t(x,x) = \hat{f}(0)e^{-t|x|^2/2}$. Using [6] we get the required result. □

Remark:– (i) Note that in the theorem 5.2, $Tr_\omega(M_f(-\Delta+\epsilon)^{-d}) = \pi^d v(f)$ which is independent of $\epsilon > 0$. This could also have been arrived at directly as in section 4 for the algebra $\mathcal{A}_\theta$ once we have observed in the proof of the theorem that $Tr_\omega M_f(\Delta-\epsilon)^{-1} = Tr_\omega M_f(\Delta_S-\epsilon)^{-1}$.

We want to end this section with a brief discussion on the stochastic dilation of the heat semigroups on the spaces considered. For the classical (or commutative) C^*-algebra of $C_0(R^{2d})$ the stochastic process associated with the heat semigroup is the well known standard Brownian motion. For the non-commutative C^*-algebra $\mathcal{A}$ we first realize it in $\mathcal{B}(L^2(R^d))$ by the Stone-von-Neumann theorem on the representation of the Weyl relations [4]

$$\begin{aligned}(U_\alpha f)(x) &= f(x+\beta)\\ (V_\beta f)(x) &= e^{i\alpha.x}f(x).\end{aligned} \tag{5.4}$$

Let $q_j, p_j (j = 1,2\ldots d)$ be the generators of V_β and U_α respectively, in fact they are the position and momentum operators in the above Schrodinger representation. For simplicity of writing we shall restrict ourselves to the case $d = 1$, and consider the q.s.d.e in $L^2(R) \otimes \Gamma(L^2(R_+, C^2))$:

$$dX_t = X_t[-ip\ dw_1(t) - \frac{1}{2}p^2dt - iq\ dw_1(t) - \frac{1}{2}q^2dt], X_0 = I \tag{5.5}$$

where w_1, w_2 are independent standard Brownian motions as in section 2. The following theorem summarizes the results.

Theorem 5.6 *(i) The q.s.d.e (5.5) has a unique unitary solution.*
(ii) If we set $j_t(x) = X_t(x \otimes I_t)X_t^*$ then j_t satisfies the q.s.d.e :

$$dj_t(x) = j_t(-i[p, x])dw_1(t) + j_t(-i[q, x])dw_2(t) + j_t(\mathcal{L}(x))dt$$

for all $x \in \mathcal{A}^\infty$ and $Ej_t(x) = e^{t\mathcal{L}}(x)$ for all $x \in \mathcal{A}$

Proof:–
Consider the q.s.d.e in $\Gamma(L^2(R_+))$ for each $\lambda \in R$ for a.a w_1,

$$dW_t^{(\lambda)} = W_t^{(\lambda)}(-i(\lambda + w_1(t))dw_2(t) - \frac{1}{2}(\lambda + w_1(t))^2dt), W_0^{(\lambda)} = I.$$

It is clear from [14] that $W_t^{(\lambda)} = exp(-i\int_0^t(\lambda+w_1(s))dw_2(s))$ which is unitary in $\Gamma(L^2(R_+))$ for fixed λ and w_1. Next we set $W_t = \int_R E^q(d\lambda) \otimes W_t^{(\lambda)}$ which can be easily seen to be unitary in $L^2(R) \otimes \Gamma(L^2(R_+))$ for fixed w_1, where E^q is the spectral measure of the self adjoint operator q in $L^2(R)$. Writing $X_t = W_t e^{-ipw_1(t)}$ it is clear that X_t is unitary in $L^2(R) \otimes \Gamma(L^2(R_+, C^2))$. A simple calculation using Ito calculus shows that X_t indeed satisfies equation 5.5.
The part two follows from the observation that for fixed w_1 and w_2, X_t^* and $b(f) \otimes I_\Gamma$ with $f \in C_c^\infty(R^2)$ maps $\mathcal{S}(R) \otimes \Gamma(L^2(R_+, C^2))$ into itself. It is also easy to see that $j_t(x) = X_t x X_t^* = e^{-iqw_2(t)}e^{-ipw_1(t)}xe^{ipw_1(t)}e^{iqw_2(t)} = \phi_{(-w_1(t),-w_2(t))}$. □

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