Pramāṇa – J. Phys., Vol. 27, Nos 1 & 2, July & August 1986, pp. 105–116.

Boson-fermion relations in several dimensions

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Abstract. Zero and positive temperature fermion field operators in several dimensions are constructed as stochastic integrals of certain reflection valued processes with respect to the corresponding boson field operator processes.

Keywords. Boson and fermion Fock spaces; boson stochastic calculus; quantum Ito's formula; canonical anticommutation relation; canonical commutation relations.

PACS No. 05·50; 05·30; 03·65

1. Introduction

Zero temperature fermion field operators in one space dimension were constructed by Hudson and Parthasarathy (1986) as quantum-stochastic integrals of a certain reflection valued process with respect to the boson field operator processes using the boson stochastic calculus developed by the same authors Hudson and Parthasarathy (1984). This leads to a canonical unitary isomorphism between the boson and Fermion Fock spaces over L_2 (**R**). Parthasarathy and Sinha (1986) showed that the stochastic integral representation of the fermion field with respect to the boson field over **R** is unique subject to the requirement of irreducibility, martingale property and existence of a vacuum. Here we extend this construction and some of the results to the case of arbitrary dimension and arbitrary temperature. As a consequence we obtain a new reducible, cyclic, non-Fock (nonzero temperature) fermion representation in terms of a reducible, cyclic, non-Fock boson representation in a boson Fock space. There have been other constructions of fermion operators as functionals of boson operators in the literature (Dell' Antonio *et al* 1972; Coleman 1975; Garbaczewski 1975; Carey and Hurst 1985).

It was observed by Dell' Antonio *et al* and Coleman that in some models in $1+1$ dimension (for example, massless Thirring model) certain formal expressions of boson fields can be formed having the vacuum expectation values and statistics of fermion fields. In Carey *et al*, this process is made rigorous for the canonical anticommutation relation (CAR) algebra over L_2 (S^1, C). However, the constructions employed by both Carey *et al* and Garbaczewski are complicated and the fermion operators so obtained could not be expressed in terms of an operator martingale process. The quantum-stochastic calculus used here as well as in Hudson and Parthasarathy (1986) makes the construction of fermion operator martingales in terms of the boson operator

The authors felicitate Prof. D S Kothari on his eightieth birthday and dedicate this paper to him on this occasion.

martingales not only transparent but also keeps the relationship entirely kinematic and hence totally independent of any model.

Section 2 is devoted to the construction of an abstract fermion representation without reference to any dimension. The uniqueness of such a representation upto unitary equivalence is expected to be true but still remains an open question. In § 3 we extend the construction to the positive temperature case in several dimensions. This leads to a direct relation between the canonical commutation relations (CCR) representations of Araki and Woods (1963) and the CAR representations of Araki and Wyss (1964) and Dell' Antonio (1968).

2. Zero temperature boson-fermion relations in several dimensions

Let $\mathscr{k}$ be any complex separable Hilbert space and let P be a continuous spectral measure on $\mathbf{R}$ whose values are orthogonal projection operators in $\mathscr{k}$. In the boson Fock space $\Gamma(\mathscr{k}) = \mathbf{C} \oplus \mathscr{k} \oplus \ldots \oplus \mathscr{k}^{ⓢ^n} \oplus \ldots$ over $\mathscr{k}$ were ⓢ^{n} denotes n-fold symmetric tensor product, we consider the annihilation and creation operators $a(u)$, $a^\dagger(u)$, $u \in \mathscr{k}$ and define for every $t \in \mathbf{R}$

$$A_{P,u}(t) = a(P(-\infty, t]u), \quad A^\dagger_{P,u}(t) = a(P(-\infty, t]u). \tag{1}$$

Writing $\psi(u)$ for the coherent vector $1 \oplus u \oplus \ldots \oplus (n!)^{-1/2} u^{\otimes^n} \oplus \ldots$ define the second-quantized reflection operators $J_P(t)$ by the relations

$$\begin{aligned} &J_P(t)\,\psi(u) = \psi(R_t u),\\ &R_t u = -P(-\infty, t]u + P(t, \infty)u. \end{aligned} \tag{2}$$

Then, in the language of Hudson and Parthasarathy (1984) $A_{P,u}$, $A^\dagger_{P,u}$ and J_P can be interpreted as *adapted processes.* Furthermore

$$\begin{aligned} &J_P(s)\,J_P(t) = J_P(t)\,J_P(s) \quad \text{for all } s, t,\\ &J_P(t)^2 = 1, \quad J^\dagger_P(t) = J_P(t). \end{aligned} \tag{3}$$

The stochastic calculus and the quantum Ito's formula, developed by Hudson and Parthasarathy (1984) for the case where the spectral measure P is absolutely continuous with respect to the Lebesgue measure, can easily be extended to the present more general case. Therefore following the central idea in Hudson and Parthasarathy (1986), we define:

$$F_{P,u}(t) = \int_{-\infty}^{t} J_P(s)\,\mathrm{d}A_{P,u}(s), \quad F^\dagger_{P,u}(t) = \int_{-\infty}^{t} J_P(s)\,\mathrm{d}A^\dagger_{P,u}(s); \tag{4}$$

$$F_P(u) = F_{P,u}(\infty), \quad F^\dagger_P(u) = F^\dagger_{P,u}(\infty). \tag{5}$$

Our aim is to show that $\{F_P(u), F^\dagger_P(u), u \in \mathscr{k}\}$ is a representation of CAR and establish some of its basic properties. We start with the observation that (4) and (5) define the operators in the domain $\mathscr{E}$ which is the linear manifold generated by all coherent vectors.

Proposition 1 For every $t \in \mathbf{R}$, $u \in \mathscr{k}$

$$J_P(t)\,F_{P,u}(t) + F_{P,u}(t)\,J_P(t) = 0 \text{ on the domain } \mathscr{E}. \tag{6}$$

Proof For any $v, w \in \mathscr{k}$, (2), (3) and (4) imply

$$\langle \psi(v), J_P(t) F_{P,u}(t)\psi(w) \rangle = \left\langle \psi(R_t v), \int_{-\infty}^{t} J_P(s)\, \mathrm{d}A_{P,u}(s)\psi(w) \right\rangle$$

$$= \int_{-\infty}^{t} \langle \psi(R_t v), \psi(R_s w) \rangle \langle P(\mathrm{d}s)\, u, w \rangle \tag{7}$$

and on the other hand

$$\langle \psi(v), F_{P,u}(t)\, J_P(t)\psi(w) \rangle = \left\langle \psi(v), \int_{-\infty}^{t} J_P(s)\, \mathrm{d}A_{P,u}(s)\, \psi(R_t w) \right\rangle$$

$$= \int_{-\infty}^{t} \langle \psi(v), \psi(R_s\, R_t\, w) \rangle \langle P(\mathrm{d}s) u, R_t w \rangle$$

$$= \int_{-\infty}^{t} \langle \psi(R_t v), \psi(R_s w) \rangle \langle P(\mathrm{d}s)\, u, w \rangle. \tag{8}$$

Adding (7) and (8) we conclude (6).

Proposition 2 The operators $F_{P,u}(t)$ are bounded for all $t \leqslant \infty$, $u \in \mathscr{k}$.

Proof We have the stochastic differential equations

$$\mathrm{d}F_{P,u} = J_P\, \mathrm{d}A_{P,u}, \quad \mathrm{d}F_{P,u}^{\dagger} = J_P\, \mathrm{d}A_{P,u}^{\dagger}.$$

By the quantum Ito's formula

$$\langle F_{P,u}^{\dagger}(t)\psi(w_1), F_{P,v}^{\dagger}(t)\psi(w_2) \rangle$$

$$= \int_{-\infty}^{t} \langle F_{P,u}^{\dagger}(s)\psi(w_1), J_P(s)\, \psi(w_2) \rangle\, \overline{\langle P(\mathrm{d}s)v, w_1 \rangle}$$

$$+ \int_{-\infty}^{t} \langle J_P(s)\psi(w_1), F_{P,v}^{\dagger}(s)\psi(w_2) \rangle \langle P(\mathrm{d}s)\, u, w_2 \rangle$$

$$+ \langle \psi(w_1), \psi(w_2) \rangle \langle u, P(-\infty, t]v \rangle. \tag{9}$$

Similarly,

$$\left\langle F_{P,v}(t)\, \psi(w_1), F_{P,u}(t)\, \psi(w_2) \right\rangle$$

$$= \int_{-\infty}^{t} \langle F_{P,v}(s)\, \psi(w_1), J_P(s)\, \psi(w_2) \rangle \langle P(\mathrm{d}s)u, w_2 \rangle$$

$$+ \int_{-\infty}^{t} \langle J_P(s)\, \psi(w_1), F_{P,u}(s)\, \psi(w_1) \rangle\, \overline{\langle P(\mathrm{d}s)\, v, w_1 \rangle}. \tag{10}$$

Adding (9) and (10) we obtain

$$\langle F_{P,u}^{\dagger}(t)\psi(w_1), F_{P,v}^{\dagger}(t)\, \psi(w_2) \rangle + \langle F_{P,v}(t)\, \psi(w_1), F_{P,u}(t)\, \psi(w_2) \rangle$$

$$= \langle \psi(w_1), \psi(w_2) \rangle \langle u, P(-\infty, t]v \rangle \text{ for all } t. \tag{11}$$

Then for any $\xi \in \mathscr{E}$, we obtain by putting $u = v$ in (11)

$$\|F^{\dagger}_{P,u}(t)\xi\|^2 + \|F_{P,u}(t)\xi\|^2 = \|\xi\|^2 \langle u, P(-\infty, t]u\rangle. \tag{12}$$

This completes the proof.

Corollary The operators $F_{P,u}(t)$ and $F^{\dagger}_{P,u}(t)$ can be extended uniquely to the whole space $\Gamma(\mathscr{h})$. If these extensions are denoted by the same symbols then $F^{\dagger}_{P,u}(t)$ is the adjoint of $F_{P,u}(t)$.

Proof This is immediate from the definition of stochastic integrals (4) and the density of $\mathscr{E}$.

Hereafter we define the operators $F_{P,u}(t)$ and $F^{\dagger}_{P,u}(t)$ to be the extensions of (4) to the whole boson Fock space $\Gamma(\mathscr{h})$ and put $F_{P,u}(-\infty) = F^{\dagger}_{P,u}(-\infty) = 0$.

Proposition 3 The operators $\{F_{P,u}(t), F^{\dagger}_{P,u}(t), u \in \mathscr{h}\}$ obey the CAR

$$[F_{P,u}(t), F_{P,v}(t)]_+ \equiv F_{P,u}(t)F_{P,v}(t) + F_{P,v}(t)F_{P,u}(t) = 0,$$

$$[F_{P,u}(t), F^{\dagger}_{P,v}(t)]_+ = \langle u, P(-\infty, t]v\rangle \text{ for all } u, v \in \mathscr{h}, -\infty \leqslant t \leqslant \infty.$$

Proof The second relation is immediate from (11). In order to prove the first relation we deduce from Proposition 1 and the quantum Ito's formula the identity

$$\begin{aligned} \mathrm{d}(F_{P,u}F_{P,v} + F_{P,v}F_{P,u}) &= J_P F_{P,v}\,\mathrm{d}A_{P,u} + F_{P,u}J_P\,\mathrm{d}A_{P,v} \\ &\quad + J_P F_{P,u}\,\mathrm{d}A_{P,v} + F_{P,v}J_P\,\mathrm{d}A_{P,u} = 0. \end{aligned}$$

Since $F_{P,u}$ and $F_{P,v}$ vanish at $-\infty$ the proof is complete.

Proposition 4 Let U be a unitary operator on $\mathscr{h}$ and let $\Gamma(U)$ be its second quantization defined by

$$\Gamma(U)\psi(u) = \psi(Uu) \text{ for all } u \in \mathscr{h}.$$

Then

$$\Gamma(U)F_{P,u}(t)\Gamma(U)^{-1} = F_{UPU^{-1},Uu}(t) \quad \text{for all } -\infty \leqslant t \leqslant \infty,\ u \in \mathscr{h},$$

where UPU^{-1} is the spectral measure defined by $(UPU^{-1})(E) = UP(E)U^{-1}$ for any Borel set $E \subset \mathbf{R}$.

Proof We have from definitions

$$\begin{aligned} \langle\psi(v), \Gamma(U)F_{P,u}(t)\Gamma(U^{-1})\psi(w)\rangle &= \langle\psi(U^{-1}v), F_{P,u}(t)\psi(U^{-1}w)\rangle \\ &= \int_{-\infty}^{t} \langle\psi(U^{-1}v), J_P(s)\psi(U^{-1}w)\rangle\langle P(\mathrm{d}s)u, U^{-1}w\rangle \\ &= \int_{-\infty}^{t} \langle\psi(v), J_{UPU^{-1}}(s)\psi(w)\rangle\langle UP(\mathrm{d}s)U^{-1}Uu, w\rangle \\ &= \langle\psi(v), F_{UPU^{-1},Uu}(t)\psi(w)\rangle. \end{aligned}$$

From now on we fix the spectral measure P in $\mathscr{h}$ and drop the suffix P from $A_{P,u}$, J_P, $F_{P,u}$ etc.

Proposition 5 Let $\Omega = \psi(0)$ be the vacuum vector in $\Gamma(\mathscr{h})$.

Then $F_u(t)\Omega = 0$,

$$F^{\dagger}_{u_1}(t)\dots F^{\dagger}_{u_n}(t)\Omega = \sum_{\sigma\in S_n}\varepsilon(\sigma)\int_{-\infty<s_1<\cdots<s_n<t} dA^{\dagger}_{u_{\sigma(1)}}(s_1)\cdots dA^{\dagger}_{u_{\sigma(n)}}(s_n)\Omega, \tag{13}$$

$$A^{\dagger}_{u_1}(t)\cdots A^{\dagger}_{u_n}(t)\Omega = \sum_{\sigma\in S_n}\int_{-\infty<s_1<\cdots<s_n<t} dF^{\dagger}_{u_{\sigma(1)}}(s_1)\cdots dF^{\dagger}_{u_{\sigma(n)}}(s_n)\Omega \tag{14}$$

for every positive integer n, $u_1, u_2, \dots, u_n \in \mathscr{k}$, $-\infty \leqslant t \leqslant \infty$, where S_n is the permutation group acting on $\{1, 2, \dots, n\}$ and $\varepsilon(\sigma)$ denotes the parity of the permutation σ.

Proof This is proved exactly along the same lines as in Hudson and Parthasarathy (1985) by using induction, quantum Ito's formula and the relation (6) on the whole Fock space.

Proposition 6 Let $\mathscr{k}_t$ denote the range of the projection $P(-\infty, t]$ and let H_t denote the closed linear span of the set $\{\psi(u), u\in\mathscr{k}_t\}$. Then each of the sets

$$\{\Omega\} \cup \{A^{\dagger}_{u_1}(t)\dots A^{\dagger}_{u_n}(t)\Omega,\quad u_1, u_2, \dots, u_n \in \mathscr{k},\quad n = 1, 2, \dots\},$$
$$\{\Omega\} \cup \{F^{\dagger}_{u_1}(t)\dots F^{\dagger}_{u_n}(t)\Omega,\quad u_1, u_2, \dots, u_n \in \mathscr{k},\quad n = 1, 2 \dots\}$$

is total in H_t.

Proof The first set contains the vacuum vector and all the n-particle vectors arising from $\mathscr{k}_t$. Hence it spans the Fock space $\Gamma(\mathscr{k}_t) = H_t$. It follows from (14) that the second set is total in H_t.

Theorem 1 The operators $\{Fu(t), F^{\dagger}_u(t), u\in\mathscr{k}_t\}$ restricted to the subspace H_t constitute an irreducible CAR representation of the Hilbert space $\mathscr{k}_t$ for each $-\infty < t \leqslant \infty$.

Proof Let $\Gamma_a(\mathscr{k}) = \mathbf{C}\oplus\mathscr{k}\oplus\dots\oplus\mathscr{k}^{@^n}\oplus\dots$ be the fermion Fock space where $@^n$ denotes n-fold skew symmetric tensor product. The canonical irreducible representation of CAR over $\mathscr{k}$ in $\Gamma_a(\mathscr{k})$ is completely characterized upto unitary equivalence by the existence of a vacuum which is cyclic for the algebra generated by the creation operators. Thus the required result follows from Proposition 3, 5 and 6.

Theorem 2 Let Ω and Ω_- be the vacuum vectors respectively in $\Gamma(\mathscr{k})$ and $\Gamma_a(\mathscr{k})$. Suppose P is a continuous spectral measure on $\mathbf{R}$ whose values are orthogonal projections on $\mathscr{k}$. Then there exists a unitary isomorphism $\Xi_P\colon \Gamma_a(\mathscr{k}) \to \Gamma(\mathscr{k})$ satisfying

$$\Xi_P\Omega_- = \Omega$$

$$\Xi_P \bigwedge_{j=1}^{n} P(-\infty, t]u_j = (n!)^{-1/2}\sum_{\sigma\in S_n}\varepsilon(\sigma)\int_{-\infty<S_1\dots<S_n<t} dA^{\dagger}_{u_{\sigma(1)}}(s_1)\dots dA^{\dagger}_{u_{\sigma(n)}}(s_n)\Omega$$

for $n = 1, 2, \dots, u_1, u_2, \dots, u_n \in \mathscr{k}$ and $-\infty \leqslant t \leqslant \infty$, where $\bigwedge_{j=1}^{n}$ denotes the skew symmetric tensor product in the order $1, 2, \dots, n$.

Proof By theorem 1 we have for every n

$$\langle F^{\dagger}_{u_1}(t)\ldots F^{\dagger}_{u_n}(t)\Omega, F^{\dagger}_{v_1}(t)\ldots F^{\dagger}_{v_n}(t)\Omega\rangle$$
$$= \det((\langle u_i, P(-\infty,t]v_j\rangle)) = n!\left\langle \bigwedge_{j=1}^{n} u_j, \bigwedge_{j=1}^{n} P(-\infty,t]v_j\right\rangle .$$

The required result follows immediately from Proposition 6.

We now specialise to the case when $\mathscr{k} = L_2(\mathbf{R}^\nu)$. Express any point in $\mathbf{R}^\nu$ as $(s, x_1, x_2, \ldots, x_{\nu-1})$ and consider the absolutely continuous spectral measure P on $\mathbf{R}$ in $\mathscr{k}$ defined by

$$[P(E)f](s, x_1, \ldots, x_{\nu-1}) = \chi_E(s) f(s, x_1, \ldots, x_{\nu-1}),$$

where χ_E denotes the indicator of the Borel set $E \subset \mathbf{R}$.

In view of Theorems 1 and 2 we can realise the fermion field operators in ν variables in the boson Fock space over $L_2(\mathbf{R}^\nu)$ through (4) and (5). In such a construction we have taken the first coordinate as a distinguished one but in view of Proposition 4 change of coordinates through permutations or rotations yields only an equivalent fermion field.

3. Positive temperature boson-fermion relations in several dimensions

As at the end of the last section we consider $\mathscr{k} = L^2(\mathbf{R}^\nu)$ and the spectral measure P of multiplication by the indicator in the first variable. In order to construct the positive temperature boson and fermion fields we introduce the Fock spaces

$$\tilde{H} = \Gamma(\mathscr{k})\otimes\Gamma(\mathscr{k}) = \Gamma(\mathscr{k}\oplus\mathscr{k}), \quad \tilde{H}_t = \Gamma(\mathscr{k}_t)\otimes\Gamma(\mathscr{k}_t) = \Gamma(\mathscr{k}_t\oplus\mathscr{k}_t) \tag{15}$$

where $\mathscr{k}_t$ is the range of the projection $P(-\infty, t]$. $\tilde{H}_t$ is to be looked upon as a subspace of $\tilde{H}$. For any $\phi \in \mathscr{k}$ let

$$A^{(1)}_{\phi}(t) = A_{\phi}(t)\otimes 1, \quad A^{(2)}_{\phi}(t) = 1\otimes A_{\phi}(t) \tag{16}$$

where $A_\phi(t) = A_{P,\phi}(t)$ is defined by (1) and 1 denotes the identity operator in $\Gamma(\mathscr{k})$. Let α, β be two bounded complex valued measurable functions on $\mathbf{R}^\nu$ satisfying the conditions

$$|\alpha|^2 - |\beta|^2 = 1, \quad |\alpha\beta| > 0 \text{ everywhere.} \tag{17}$$

Define the operators

$$\tilde{A}_\phi(t) = A^{(1)}_{\bar{\alpha}\phi}(t) + A^{(2)\dagger}_{\beta\bar{\phi}}(t), \tag{18}$$

$$\tilde{A}^{\dagger}_\phi(t) = A^{(1)\dagger}_{\bar{\alpha}\phi}(t) + A^{(2)}_{\beta\bar{\phi}}(t), \tag{19}$$

$$\tilde{A}(\phi) = \tilde{A}_\phi(\infty), \quad \tilde{A}^{\dagger}(\phi) = \tilde{A}^{\dagger}_\phi(\infty). \tag{20}$$

Then the following commutation relations hold:

$$[\tilde{A}_\phi(t), \tilde{A}_\psi(t)] = 0, \quad [\tilde{A}^{\dagger}{}_\phi, \tilde{A}^{\dagger}_\psi] = 0,$$
$$[\tilde{A}_\phi(t), \tilde{A}^{\dagger}_\psi(t)] = \langle\phi, P(-\infty, t]\psi\rangle \tag{21}$$

for all $-\infty \leqslant t \leqslant \infty$ and $\phi, \psi \in \mathscr{h}$. In particular, $\{\tilde{A}(\phi), \tilde{A}^\dagger(\psi), \phi, \psi \in \mathscr{h}\}$ is a representation of CCR in $\tilde{H}$. Let

$$\tilde{W}_\phi = W_{\bar{\alpha}\phi} \otimes W_{-\beta\bar{\phi}}, \quad \phi \in \mathscr{h}, \tag{22}$$

where $\phi \to W_\phi$ is the Weyl representation of $\mathscr{h}$ in $\Gamma(\mathscr{h})$ defined by the relations

$$W_\phi \psi(f) = \exp(-\tfrac{1}{2}\|\phi\|^2 - \langle \phi, f \rangle)\psi(f+\phi) \quad \text{for all } f \in \mathscr{h}. \tag{23}$$

Then we have the Weyl commutation relations

$$W_\phi W_\psi = W_{\phi+\psi} \exp - i \operatorname{Im} \langle \phi, \psi \rangle; \tag{24}$$

$$\tilde{W}_\phi \tilde{W}_\psi = \tilde{W}_{\phi+\psi} \exp - i \operatorname{Im} \langle \phi, \psi \rangle, \quad \phi, \psi \in \mathscr{h} \tag{25}$$

in $\Gamma(\mathscr{h})$ and $\Gamma(\mathscr{h}) \otimes \Gamma(\mathscr{h})$ respectively. Whereas

$$\langle \Omega, W_\phi \Omega \rangle = \exp - \tfrac{1}{2}\|\phi\|^2, \tag{26}$$

we have

$$\langle \tilde{\Omega}, \tilde{W}_\phi \tilde{\Omega} \rangle = \exp[-\tfrac{1}{2}(\|\alpha\phi\|^2 + \|\beta\phi\|^2)], \tilde{\Omega} = \Omega \otimes \Omega. \tag{27}$$

Since $|\alpha|^2 + |\beta|^2 = 1 + 2|\beta|^2 > 1$, it is clear that $\tilde{W}$ is a quasi-free non-Fock representation of positive temperature.

Following the notations at the end of the last section we write for any $\phi, \psi \in \mathscr{h}$

$$\langle \phi, \psi \rangle_0(s) = \int_{\mathbf{R}^{\nu-1}} \bar{\phi}(s, \mathbf{x})\psi(s, \mathbf{x})\,d\mathbf{x}, \quad \mathbf{x} = (x_1, \ldots, x_{\nu-1}),$$

$$\|\phi\|_0^2(s) = \langle \phi, \phi \rangle_0(s).$$

Let $\tilde{W}_\phi(t) = \tilde{W}_{P(-\infty, t]\phi}$. Then $\{\tilde{W}_\phi(t), t \in \mathbf{R}\}$ is an adapted unitary process satisfying the quantum-stochastic differential equation

$$d\tilde{W}_\phi(t) = \{d\tilde{A}^\dagger_\phi(t) - d\tilde{A}_\phi(t) - \tfrac{1}{2}[\|\alpha\phi\|_0^2(t) + \|\beta\phi\|_0^2(t)]\,dt\}\,\tilde{W}_\phi(t).$$

Furthermore $\phi \to \tilde{W}_\phi(t)$, $\phi \in \mathscr{h}_t$ satisfies the Weyl commutation relations in $\tilde{H}_t$.

Proposition 7 The set $\{\tilde{W}_\phi(t)\tilde{\Omega}, \phi \in \mathscr{h}_t\}$ is total in the subspace $\tilde{H}_t$. The map $\phi \to \tilde{W}_\phi(t)$, $\phi \in \mathscr{h}_t$ is a reducible projective unitary representation in $\tilde{H}_t$ of the additive group $\mathscr{h}_t$ for each $-\infty < t \leqslant \infty$.

Proof Let

$$\mathscr{k}_t = \{\bar{\alpha}\phi \oplus (-\beta\bar{\phi}), \phi \in \mathscr{h}_t\}.$$

Then $\mathscr{k}_t$ is a real linear manifold in $\mathscr{h}_t \oplus \mathscr{h}_t$ and by (17) $\mathscr{k}_t + i\mathscr{k}_t$ is a dense linear manifold in $\mathscr{h}_t \oplus \mathscr{h}_t$. Hence the set $\{\psi(u), u \in \mathscr{k}_t\}$ is total in $\tilde{H}_t$. Since

$$\begin{aligned}\tilde{W}_\phi(t)\tilde{\Omega} &= c(t)\psi(P(-\infty, t]\bar{\alpha}\phi \oplus P(-\infty, t](-\beta\bar{\phi})) \\ &= c(t)\psi(\bar{\alpha}\phi \oplus (-\beta\bar{\phi})) \quad \text{for } \phi \in \mathscr{h}_t,\end{aligned}$$

where $c(t)$ is a nonvanishing scalar, this proves the first part. To prove the second part we have only to note that the unitary operators $\tilde{W}_\phi$ and $W_{\beta\psi}\otimes W_{-\bar{\alpha}\bar{\psi}}$ commute for all $\phi, \psi \in \mathscr{h}$.

In order to construct the positive temperature fermion field operators in terms of the positive temperature boson field operators $\tilde{A}(\phi)$ and $\tilde{A}^\dagger(\phi)$ we introduce the stochastic integrals

$$\tilde{F}_\phi(t) = \int_{-\infty}^t \tilde{J}(s)\,\mathrm{d}\tilde{A}_{\rho\phi}(s), \quad \tilde{F}^\dagger_\phi(t) = \int_{-\infty}^t \tilde{J}(s)\,\mathrm{d}\tilde{A}^\dagger_{\rho\phi}(s), \tag{28}$$

where

$$\rho = (|\alpha|^2 + |\beta|^2)^{-1/2}, \quad \tilde{J}(s) = J(s)\dot{\otimes} J(s), \tag{29}$$

($J(s) = J_P(s)$ being defined by (2) in $\Gamma(\mathscr{h})$). It is to be noted that (28) defines the processes $\tilde{F}_\phi$ and $\tilde{F}^\dagger_\phi$ on the domain $\tilde{\mathscr{E}} = \mathscr{E}\otimes\mathscr{E}$ which is the linear manifold generated by exponential vectors in $\tilde{H}$. Furthermore they are adjoint to each other on $\tilde{\mathscr{E}}$.

Proposition 8 For any $\phi \in \mathscr{h}$, $-\infty \leqslant t \leqslant \infty$

$$\tilde{J}(t)\tilde{F}_\phi(t) + \tilde{F}_\phi(t)\tilde{J}(t) = 0 \text{ on } \tilde{\mathscr{E}}. \tag{30}$$

Proof Since $\tilde{H} = \Gamma(\mathscr{h}\oplus\mathscr{h})$ we have for any $f = f_1 \oplus f_2$, $g = g_1 \oplus g_2$

$$\langle \psi(f), \tilde{J}(t)\tilde{F}_\phi(t)\psi(g)\rangle = \int_{-\infty}^t \langle \psi(R_t f), \psi(R_s g)\rangle \langle \rho\phi, \alpha g_1 - \beta \bar{f_2}\rangle_0(s)\mathrm{d}s$$

and

$$\langle \psi(f), \tilde{F}_\phi(t)\tilde{J}(t)\psi(g)\rangle = \int_{-\infty}^t \langle \psi(f), \psi(R_s R_t g)\rangle \langle \rho\phi, -\alpha g_1 + \beta \bar{f_2}\rangle_0(s)\,\mathrm{d}s.$$

Adding these two relations we obtain (30).

Proposition 9 The operators $\tilde{F}_\phi(t)$ are bounded for every $\phi\in\mathscr{h}$, $-\infty \leqslant t \leqslant \infty$.

Proof We proceed along the same lines as in the proof of Proposition 2 and use the quantum Ito's formula

$$\mathrm{d}A^{(i)}_\phi \mathrm{d}A^{(j)}_\psi = \mathrm{d}A^{(i)\dagger}_\phi \mathrm{d}A^{(j)\dagger}_\psi = \mathrm{d}A^{(i)\dagger}_\phi \mathrm{d}A^{(j)}_\psi = 0,$$

$$\mathrm{d}A^{(i)}_\phi \mathrm{d}A^{(j)\dagger}_\psi = \delta_{ij}\langle \phi, P(\mathrm{d}t)\psi\rangle, \quad i,j = 1,2.$$

We get

$$\langle \tilde{F}^\dagger_\phi(t)\psi(f), \tilde{F}^\dagger_\psi(t)\psi(g)\rangle + \langle \tilde{F}_\psi(t)\psi(f), \tilde{F}_\phi(t)\psi(g)\rangle = \langle \psi(f), \psi(g)\rangle \langle \phi, P(-\infty, t]\psi\rangle$$

for all $f, g \in \mathscr{h}\oplus\mathscr{h}$, $-\infty \leqslant t \leqslant \infty$. Hence

$$\|\tilde{F}^\dagger_\phi(t)\xi\|^2 + \|\tilde{F}_\phi(t)\xi\|^2 = \|\xi\|^2 \langle \phi, P(-\infty, t]\phi\rangle, \xi \in \tilde{\mathscr{E}}.$$

This completes the proof.

Proposition 9 enables us to extend the operators $\tilde{F}_\phi(t)$, $\tilde{F}^\dagger_\phi(t)$ uniquely to the whole space $\tilde{H}$. Hereafter we denote these extensions by the same symbols.

Theorem 3 The operators $\{\tilde{F}_\phi(t), \tilde{F}_\phi^\dagger(t), \phi \in \mathscr{k}\}$ satisfy the following for every $-\infty \leqslant t \leqslant \infty$:

(i) $\tilde{F}_\phi^\dagger(t)$ is the adjoint of $\tilde{F}_\phi(t)$,

(ii) $[\tilde{F}_\phi(t), \tilde{F}_\psi(t)]_+ = 0$,

(iii) $[\tilde{F}_\phi(t), \tilde{F}_\psi^\dagger(t)]_+ = \langle \phi, P(-\infty, t]\psi \rangle$.

Proof This is proved exactly like Proposition 3.

Proposition 10 For any $\phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n \in \mathscr{k}$, $-\infty \leqslant t \leqslant \infty$

$$\langle \tilde{\Omega}, \tilde{F}_{\phi_1}^\dagger(t) \ldots \tilde{F}_{\phi_m}^\dagger(t)\tilde{F}_{\psi_1}(t) \ldots \tilde{F}_{\psi_n}(t)\tilde{\Omega} \rangle = 0 \quad \text{if } m \neq n$$
$$= \det((\langle \psi_j, P(-\infty, t] \, |\beta|^2\rho^2\phi_i \rangle)) \quad \text{otherwise.} \tag{31}$$

Proof Let $m \neq n$. Suppose H_n denotes the n-particle subspace $\mathscr{k}^{\textcircled{s}n}$ in $\Gamma(\mathscr{k})$. Then $\tilde{F}_{\psi_1}(t) \ldots \tilde{F}_{\psi_n}(t)\tilde{\Omega} \in H_0 \otimes H_n$. Since different particle subspaces are mutually orthogonal the proposition is proved in this case.

Let $m = n$. Since $\tilde{J}(s)\tilde{\Omega} = \tilde{\Omega}$ we have from (28) and (18)

$$\tilde{F}_\phi(t)\tilde{\Omega} = \Omega \otimes A_{\beta\rho\bar{\phi}}^\dagger(t)\Omega = \Omega \otimes F_{\beta\rho\bar{\phi}}^\dagger(t)\Omega,$$

where $F_\phi^\dagger(t)$ is defined by (4). By quantum Ito's formula $d\tilde{A}_\phi(t)\, d\tilde{A}_\psi(t) = 0$ and hence

$$d\tilde{F}_{\phi_1}\tilde{F}_{\phi_2} \ldots \tilde{F}_{\phi_n}\tilde{\Omega} = \sum_j (-1)^{n-j}\tilde{F}_{\phi_1} \ldots \tilde{F}_{\phi_{j-1}}\, \tilde{F}_{\phi_{j+1}} \ldots \tilde{F}_{\phi_n}(dA^{(1)}_{\bar{\alpha}\rho\phi_j} + dA^{(2)\dagger}_{\beta\rho\bar{\phi}_j})\Omega \otimes \Omega$$
$$= \sum_{j=1}^{n} (-1)^{n-j}\tilde{F}_{\phi_1} \ldots \tilde{F}_{\phi_{j-1}}\tilde{F}_{\phi_{j+1}} \ldots \tilde{F}_{\phi_n}\, dA^{(2)\dagger}_{\beta\rho\bar{\phi}_j}\tilde{\Omega}.$$

Hence by induction and Proposition 5

$$\tilde{F}_{\phi_1}(t) \ldots \tilde{F}_{\phi_n}(t)\tilde{\Omega} = \Omega \otimes \sum_{\sigma \in S_n} \varepsilon(\sigma) \int_{-\infty < s_1 < \ldots < s_n < t} dA^\dagger_{\beta\rho\bar{\phi}_{\sigma(1)}}(s_1) \ldots dA^\dagger_{\beta\rho\bar{\phi}_{\sigma(n)}}(s_n)\Omega$$
$$= \Omega \otimes F_{\beta\rho\bar{\phi}_1}(t) \ldots F_{\beta\rho\bar{\phi}_n}(t)\Omega.$$

Thus the left hand side of (31) is equal to

$$\langle F_{\beta\rho\bar{\phi}_1}(t) \ldots F_{\beta\rho\bar{\phi}_n}(t)\Omega, F_{\beta\rho\bar{\psi}_1}(t) \ldots F_{\beta\rho\bar{\psi}_n}(t)\Omega \rangle$$
$$= \det((\langle P(-\infty, t]\beta\rho\bar{\phi}_i, P(-\infty, t]\beta\rho\bar{\psi}_j \rangle))$$
$$= \det((\langle \psi_j, P(-\infty, t] \, |\beta|^2\rho^2\phi_i \rangle)).$$

Proposition 11 The set $\mathscr{P}_t = \{\tilde{\Omega}\} \cup \{\tilde{F}_{\phi_1}^\dagger(t) \ldots \tilde{F}_{\phi_m}^\dagger(t)\tilde{F}_{\psi_1}(t) \ldots \tilde{F}_{\psi_n}(t)\tilde{\Omega}, \phi_1, \ldots, \phi_m, \psi_1, \ldots, \psi_n \in \mathscr{k}, m, n = 0, 1, 2, \ldots\}$ is total in $\tilde{H}_t$ for every t.

Proof Let $H_n(t)$ be the n-particle subspace of $\Gamma(\mathscr{k}_t)$. We write $H_{m,n}(t) = H_m(t) \otimes H_n(t)$ and denote by S_t the closed linear span of $\mathscr{P}_t$. By Proposition 6 and (32) it follows that $H_{0,n}(t) \subset S_t$ for all n. We now proceed by induction. Suppose $H_j(t) \otimes H_n(t) \subset S_t$ for $j \leqslant m$ and $n = 0, 1, 2, \ldots$ It is clear that $\tilde{F}_\phi^\dagger(t)$ maps $H_{m,n}(t)$ into $H_{m+1,n}(t) \oplus H_{m,n-1}(t)$.

Let u be a vector in $H_{m+1,n}(t)$ which is orthogonal to $\tilde{F}_\phi^\dagger(t)\{H_{m,n}(t)\}$ for all $\phi \in \mathscr{k}$. Then by induction hypothesis

$$\left\langle u, \int_{-\infty}^{t} J(s)\,\mathrm{d}A_{\alpha\rho\phi}^{(1)\dagger}(s)v \right\rangle = 0 \text{ for all } v \in H_{m,n}(t), \phi \in \mathscr{k}. \tag{33}$$

The element u can be expressed as a function $u(\xi_1, \ldots, \xi_{m+1}, \eta_1, \ldots, \eta_n)$ in $m+n+1$ variables from $\mathbf{R}^\nu$, which is symmetric in the first $m+1$ variables and the last n variables separately. Similarly v can be expressed as $v(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$ which is symmetric in the first m and the last n variables separately. We denote the first coordinate of the points $\xi_j, \eta_k \in \mathbf{R}^\nu$ by s_j, t_k respectively and put

$$\omega(\xi_1, \ldots, \xi_{m+1}, \eta_1, \ldots, \eta_n) = [\sum_{j=1}^{m} \chi_{(-\infty, s_{m+1}]}(s_j) + \sum_{k=1}^{n} \chi_{(-\infty, s_{m+1}]}(t_k),$$

χ denoting indicator. Then by the definitions of $\tilde{J}(s)$ and $A_\psi^{(1)\dagger}$ the left hand side of (33) is equal to

$$(m+1)^{1/2} \int\limits_{\substack{s_1 \leqslant t, \ldots, s_{m+1} \leqslant t \\ t_1 \leqslant t, \ldots, t_n \leqslant t}} \bar{u}(\xi_1, \ldots, \xi_{m+1}, \eta_1 \ldots \eta_n) v(\xi_1, \ldots, \xi_m, \eta_1, \ldots, \eta_n)$$

$$\times \alpha\rho\phi(\xi_{m+1})(-1)^{\omega(\xi_1, \ldots, \xi_{m+1}, \eta_1 \ldots \eta_n)} \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_{m+1}\, \mathrm{d}\eta_1 \ldots \mathrm{d}\eta_n = 0.$$

Since ω is symmetric in $(\xi_1, \ldots, \xi_m)$ and $(\eta_1, \ldots, \eta_n)$, v is arbitrary in $H_{m,n}(t)$, ϕ is arbitrary in $\mathscr{k}_t$ and $|\alpha\rho| > 0$ everywhere it follows that $u = 0$. In other words $H_{m+1,n}(t) \subset S_t$.

Proposition 12 The CAR representation $\{\tilde{F}_\phi(t), \tilde{F}_\phi^\dagger(t), \phi \in \mathscr{k}_t\}$ restricted to $\tilde{H}_t$ is reducible for every t.

Proof Consider the unitary operator S defined by

$$S\psi(f \oplus g) = \psi(-g \oplus -f), \quad f, g \in \mathscr{k},$$

and put

$$\tilde{G}_\phi(t) = S\tilde{F}_\phi(t)S^{-1}.$$

Then $\{\tilde{G}_\phi(t), \tilde{G}_\phi^\dagger(t), \phi \in \mathscr{k}_t\}$ is another CAR representation for $\mathscr{k}_t$. From the two relations

$$\mathrm{d}\tilde{F}_\phi = \tilde{J}(\mathrm{d}A_{\rho\bar{\alpha}\phi}^{(1)} + \mathrm{d}A_{\rho\beta\bar{\phi}}^{(2)\dagger}),$$
$$\mathrm{d}\tilde{G}_\phi = -\tilde{J}(\mathrm{d}A_{\rho\bar{\alpha}\phi}^{(2)} + \mathrm{d}A_{\rho\beta\bar{\phi}}^{(1)\dagger})$$

and quantum Ito's formula we have

$$\mathrm{d}[\tilde{F}_\phi(t), \tilde{G}_\psi(t)]_+ = -2\langle \rho\bar{\alpha}\phi, \rho\beta\bar{\psi}\rangle_0\, \mathrm{d}t \text{ for all } \phi, \psi \in \mathscr{k}.$$

Hence

$$[\tilde{F}_\phi(t), \tilde{G}_\psi(t)]_+ = -2\int_{-\infty}^{t} \langle \rho\bar{\alpha}\phi, \rho\beta\bar{\psi}\rangle_0\,(s)\,\mathrm{d}s$$

$$= -2\int_{-\infty}^{t} \langle \phi, \rho^2\alpha\beta\bar{\psi}\rangle_0\,(s)\,\mathrm{d}s.$$

Since

$$[\tilde{F}_\phi(t), \tilde{F}^\dagger_\xi(t)] = \int_{-\infty}^{t} \langle \phi, \xi \rangle_0 (s)\, ds,$$

it follows that

$$[\tilde{F}_\phi(t), \tilde{G}_\psi(t) + 2\tilde{F}^\dagger_{\rho^2\alpha\beta\bar{\psi}}(t)]_+ = 0 \text{ for all } \phi, \psi \in \mathscr{k}.$$

Once again by quantum Ito's formula

$$[\tilde{F}_\phi(t), \tilde{G}^\dagger_\psi(t)]_+ = 0.$$

Combining the last two relations we conclude that

$$[\tilde{F}_\phi(t), \{\tilde{G}^\dagger_\psi(t) + 2\tilde{F}_{\rho^2\alpha\beta\bar{\psi}}(t)\}\{\tilde{G}_\psi(t) + 2\tilde{F}^\dagger_{\rho^2\alpha\beta\bar{\psi}}(t)\}] = 0$$

for all ϕ, $\psi \in \mathscr{k}$. Since the second operator in the above commutator is self-adjoint the required result follows.

We now summarize our conclusions in the following theorem.

Theorem 4 Let the operators $\{\tilde{F}_\phi(t), \tilde{F}^\dagger_\phi(t), \phi \in \mathscr{k}, -\infty \leqslant t \leqslant \infty\}$ be defined in terms of the positive temperature boson field operators $\{\tilde{A}_\phi(t), \tilde{A}^\dagger_\phi(t), \phi \in \mathscr{k}, -\infty \leqslant t \leqslant \infty\}$ and the reflection operators $\{\tilde{J}(t), -\infty \leqslant t \leqslant \infty\}$ by the stochastic integrals (28), where $\mathscr{k} = L^2(\mathbf{R}^\nu)$. Let P denote the spectral measure of multiplication by the indicator in the first coordinate in $\mathbf{R}^\nu$. Then for each fixed $-\infty < t \leqslant \infty$, $\{\tilde{F}_\phi(t), \tilde{F}^\dagger_\phi(t), \phi \in P(-\infty, t]\mathscr{k}\}$ restricted to the boson Fock space $\tilde{H}_t = \Gamma(P(-\infty, t]\mathscr{k} \oplus P(-\infty, t]\mathscr{k})$ is a reducible CAR representation for $P(-\infty, t]\mathscr{k}$ which has the vacuum $\tilde{\Omega}$ in $\tilde{H}_t$ as a cyclic vector. Furthermore

$$\begin{aligned} \langle \tilde{\Omega}, \tilde{F}^\dagger_{\phi_1}(t) \ldots \tilde{F}^\dagger_{\phi_m}(t)\tilde{F}_{\psi_1}(t) \ldots \tilde{F}_{\psi_n}(t)\tilde{\Omega}\rangle &= 0 \quad \text{if } m \neq n, \\ &= \det((\langle \psi_j, P(-\infty, t]\,|\beta|^2\rho^2\phi_i\rangle)) \quad \text{if } m = n. \end{aligned} \tag{33}$$

Proof This is just a restatement of theorem 3, Proposition 10–12 put together.

Now we shall compare the CAR representation $\{\tilde{F}(\phi), \tilde{F}^\dagger(\phi), \phi \in \mathscr{k}\}$ obtained from Theorem 4 when $t = \infty$ and the positive temperature CAR representation of Araki and Wyss (1964). Using the isomorphism Ξ_ρ of Theorem 2 the Araki-Wyss representation may be defined by

$$\hat{F}(\phi) = F(\rho\bar{\alpha}\phi) \otimes 1 + J(\infty) \otimes F^\dagger(\rho\beta\bar{\phi}), \tag{34}$$

where α, β satisfy (17), ρ is given by (29), $F(\phi)$, $F^\dagger(\phi)$ denote the zero temperature irreducible CAR representation in $\Gamma(\mathscr{k})$ and $J(\infty)$ is defined by (2) by putting $t = \infty$.

It has been shown by Araki and Wyss that the operators defined by (34) and their adjoints constitute a reducible representation of CAR with $\tilde{\Omega}$ as cyclic vector and expectation values $\langle \tilde{\Omega}, \hat{F}^\dagger(\phi_1) \ldots \hat{F}^\dagger(\phi_m)\hat{F}(\psi_1) \ldots \hat{F}(\psi_n)\tilde{\Omega}\rangle$ are given by the right hand side of (33) when $t = \infty$. In other words the CAR representation $\{\tilde{F}(\phi), \tilde{F}^\dagger(\phi), \phi \in \mathscr{k}\}$ described in Theorem 4 when $t = \infty$ is unitarily equivalent to the Araki-Wyss representation. Since formula (34) involves the reflection $J(\infty)$ it is not possible to replace ϕ by $P(-\infty, t]\phi$ in $\hat{F}(\phi)$ and localize it to an adapted process. On the contrary formula (28) localizes $\tilde{F}(\phi)$ and at the same time realizes it in terms of the positive temperature boson field operators.

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