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Stochastic dilation of minimal quantum dynamical semigroup

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Abstract. A necessary and sufficient condition is formulated for minimal quantum dynamical semigroups to be conservative. The paper also provides a Markovian dilation of the minimal semigroups, as a contractive solution of an associated quantum stochastic differential equation in Boson-Fock space, which is isometric if and only if the minimal semigroup is conservative. Using the reflection principle of Brownian motion a necessary and sufficient condition for the contractive solution to be co-isometric is also obtained.

Keywords. Quantum dynamical semigroup; Markovian cocycle; quantum stochastic differential equation.

1. Introduction

Feller [8] proved the existence of a unique minimal semigroup $P_t, t \geqslant 0$ on l_1 associated with the Fokker-Planck equation:

$$\frac{\mathrm{d}}{\mathrm{d}t} p_{ik}(t) = \sum_j p_{ij}(t)\Omega_{jk}, t \geqslant 0, p_{ik}(0) = \delta_{ik} \tag{1}$$

subject to the Markov condition:

$$\Omega_{jk} \geqslant 0 \text{ for } j \neq k \quad \text{and} \quad \sum_{k \neq j} \Omega_{jk} = -\Omega_{jj} < \infty. \tag{2}$$

Exploiting the special nature of l_1, Kato [14] constructed the minimal semigroup in the framework of semigroup theory. It was also shown in [8, 14] that the minimal semigroup is conservative i.e. $\| P_t y \|_1 = \| y \|_1$, for all $y \in l_1^+$ if and only if

$$B_\lambda \equiv \{x \in l_\infty^+, \sum_k \Omega_{jk} x_k = \lambda x_j\} = \{0\}, \quad \text{for some } \lambda > 0. \tag{3}$$

In this paper we consider the quantum mechanical Fokker-Planck equation in $\mathscr{T}$, the Banach space of trace class operators in $\mathscr{H}_0$:

$$\rho(0) = \rho, \quad \rho(t)' = Y\rho(t) + \rho(t)\, Y^* + \sum_{k \in S} Z_k \rho(t) Z_k^* \tag{4}$$

subject to

$$Y + Y^* + \sum_{k \in S} Z_k^* Z_k = 0, \tag{5}$$

where Y, $Z_k, k\in S\subset \mathbb{Z}_+$ are densely defined operators in $\mathscr{H}_0$ and $\rho\in\mathscr{T}_h$, the real Banach space of self-adjoint elements in $\mathscr{T}$. Davies [4], following essentially Kato's method, constructed the minimal dynamical semigroup $\sigma_t^{\min}(t\geqslant 0)$ in $\mathscr{T}_h$ as a solution to (4)–(5). In this context, we formulate a condition similar to (3) as the necessary and sufficient one for the preservation of trace under the action of $\sigma_t^{\min}$. We also provide a Markovian dilation of $\sigma^{\min}$ in the sense of Accardi [1,2] as a contractive solution of an associated Hudson–Parthasarathy equation [11, 12, 19]. The solution is isometric if and only if $\sigma^{\min}$ is trace preserving. Finally using Journé's reflection principle [13, 17] we also obtain a necessary and sufficient condition for the contractive solution to be co-isometric. Some results on the related dilation problem may be found in Chebotarev [3] and Fagnola [6, 7]. The method employed here is different from that in [3, 7].

The paper is organized as follows: In § 2 we describe the framework of quantum stochastic calculus and a class of contractive cocycles satisfying quantum stochastic differential equation (qsde) with bounded coefficients and also recall [11, 17, 20] the necessary and sufficient condition for the solution to be isometric, co-isometric or unitary. Section 3 is devoted to exactly the same questions as in § 2, this time with unbounded coefficients subject to some conditions. Many of the results in this section are quoted without proof since they are published elsewhere [19]. In § 4 we consider the problem mentioned at the beginning.

2. Contractive bar-cocycles

All the Hilbert spaces that appear here are assumed to be complex and separable with inner product $\langle\cdot,\cdot\rangle$ linear in the second variable. For any Hilbert space H, we denote by $\Gamma(H)$ the symmetric Fock space over H and $B(H)$ the C^* algebra of all bounded linear operators in H. For any $u\in H$, we denote by $e(u)$ the exponential vector in $\Gamma(H)$ associated with u. The family $\{e(u):u\in\mathscr{M}\}$ is total for any dense linear manifold $\mathscr{M}$ in H and linearly independent in $\Gamma(H)$.

We fix two Hilbert spaces $\mathscr{H}_0$ and k and write

$$\mathscr{H}=\mathscr{H}_0\otimes\Gamma(L^2(\mathbb{R}'_+,k)).$$

It is clear that for any pair of linear manifolds $\mathscr{D}$ and $\mathscr{M}$ dense in $\mathscr{H}_0$ and $L^2(\mathbb{R}_+,k)$ respectively, the algebraic tensor product $\mathscr{D}\otimes\varepsilon(\mathscr{M})$ is dense in $\mathscr{H}$, where $\varepsilon(\mathscr{M})$ is the linear manifold generated by the vectors $e(u):u\in\mathscr{M}$. We also denote the vacuum conditional expectation on $\mathscr{H}_0$ by $\mathbb{E}_0$.

For the basic notions in boson stochastic calculus such as *adapted, regular, bounded, contractive, isometric, co-isometric* and *unitary* process, we refer to [11, 21]. The notion of Markovian cocycle was first introduced in [1]. However in this paper we follow the definition introduced in [13] and call it *bar-cocycle* to avoid confusion.

We fix an orthonormal basis $\{e_i:i\in S\}$ in k and set $E^i_j=|e_j\rangle\langle e_i|:i,j\in S$. With respect to this basis we define the basic quantum stochastic processes $\{\Lambda^i_j:i,j\in\bar{S}:=S\cup\{0\}\}$ as in [18, 23]. Then quantum Ito's formula [11] can be expressed as:

$$\mathrm{d}\Lambda^i_j\mathrm{d}\Lambda^k_\ell=\hat{\delta}^i_\ell\mathrm{d}\Lambda^k_j \tag{6}$$

for all $i, j, k, l \in \bar{S}$ where

$$\hat{\delta}^i_\ell = \begin{cases} 0 & \text{if } \ell = 0 \text{ or } i = 0 \\ \delta^i_\ell & \text{otherwise.} \end{cases}$$

We denote by $u^j(s) = \langle e_j, u(s) \rangle$, $u_j(s) = \overline{u^j(s)}$ for $j \in S$ and $u_0(s) = u^0(s) = 1$. Choose $\mathscr{M} \equiv \{u \in H : u^j(\cdot) = 0$ for all but finitely many $j \in S\}$ and set $N(u) = \{j; u^j(\cdot) \not\equiv 0\}$. So $\# N(u) < \infty$ for $u \in \mathscr{M}$.

We also denote by $\mathscr{L}_R$ the class of elements $L \equiv (L^i_j \in \mathscr{B}(\mathscr{H}_0), i, j \in \bar{S})$ such that for each $j \in \bar{S}$ there exists a non-negative constant c_j (depending on L) satisfying

$$\sum_{i \in \bar{S}} \| L^i_j f \|^2 \leqslant c_j^2 \| f \|^2 \tag{7}$$

for all $f \in \mathscr{H}_0$. For any $L \in \mathscr{L}_R$ define the family of bounded linear operators $\{\mathscr{L}^i_j, i, j \in \bar{S}\}$ on $\mathscr{H}_0$ by

$$\mathscr{L}^i_j = L^i_j + (L^j_i)^* + \sum_{k \in S} (L^k_i)^* L^k_j$$

where the necessary convergence follows from (7).

Fix $L \in \mathscr{L}_R$. Then there exists a unique adapted process $X \equiv \{X(t), t \geqslant 0\}$ satisfying the following qsde:

$$\mathrm{d}X = \sum_{i, j \in \bar{S}} L^i_j \mathrm{d}\Lambda^j_i(t) X(t), \; X(0) = I \tag{8}$$

on $\mathscr{H}_0 \otimes \varepsilon(\mathscr{M})$. Moreover X is isometric whenever $L \in \mathscr{I}_R$, where $\mathscr{I}_R \equiv \{L \in \mathscr{L}_R$, $\mathscr{L}^i_j = 0$, for all $i, j \in \bar{S}\}$. For a complete account of these facts the reader is referred to [11, 16, 17, 18, 20, 21].

Observe that for all $i, j \in \bar{S}$, $\mathscr{L}^i_j = (\mathscr{L}^j_i)^*$ and $\mathscr{L}_{S'} \equiv ((\mathscr{L}^i_j))_{i, j \in S'}$ is a self adjoint operator on the Hilbert space $\mathscr{H}_0 \otimes l_2(S')$ for any finite subset S' of S. We set

$$\mathscr{L}^-_R \equiv \{L, \mathscr{L}_{S'} \leqslant 0, \quad \text{for all } S' \subset S, \# S' < \infty\}.$$

Hence $\mathscr{I}_R \subset \mathscr{L}^-_R \subset \mathscr{L}_R$.

The following proposition gives a necessary and sufficient condition for X to be contractive.

PROPOSITION 2.1.

Fix $L \in \mathscr{L}_R$. Consider the family $X \equiv \{X(t), 0 \leqslant t < \infty\}$ of operators satisfying (8). *The following statements are valid:*

(i) *X has a contractive extension if and only if $L \in \mathscr{L}^-_R$;*

(ii) *X has an isometric extension if and only if $L \in \mathscr{I}_R$.*

Proof. By (6) and (8) we have

$$\begin{aligned} &\langle X(t) f e(u), X(t) g e(v) \rangle - \langle f e(u), g e(v) \rangle \\ &= \int_0^t \left\langle X(\tau) f e(u), \sum_{i, j \in \bar{S}} u_i(\tau) v^j(\tau) \mathscr{L}^i_j X(\tau) g e(v) \right\rangle \mathrm{d}\tau, 0 \leqslant t \end{aligned} \tag{9}$$

for all $f, g \in \mathscr{H}_0, u, v \in \mathscr{M}$. For finitely many vectors $f_\alpha \in \mathscr{H}_0$, ${}^\alpha u \in \mathscr{M}$ let $\psi := \Sigma_\alpha f_\alpha e({}^\alpha u) \cdot \| e({}^\alpha u) \|^{-1}$. It is convenient to introduce the Hilbert space $H = \oplus_\alpha H_\alpha$ with $H_\alpha = \oplus_{j \in N({}^\alpha u)} \mathscr{H}$, the vector in $H: \Psi(t) = \oplus_\alpha \psi_{f_\alpha, {}^\alpha u}(t)$ with $\psi_{f,u}(t) = \oplus_{j \in N(u)} u^j(t) X(t) f e(u) \cdot \| e(u) \|^{-1}$, and the bounded operator $\mathbb{L}$ in $H: \mathbb{L}^\alpha_\beta = \mathscr{L}$ for all α, β. Then from (9) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} \| X(t)\psi \|^2 = \langle \Psi(t), \mathbb{L}\Psi(t) \rangle. \tag{10}$$

Also observe that $\mathscr{L}$ is negative semi-definite if and only if $\mathbb{L}$ is negative semi-definite. Hence from (10) it is clear that the map $t \to \| X(t)\psi \|, t \geqslant 0$ is decreasing whenever $L \in \mathscr{Z}^-_R$. This completes the proof of the sufficiency part of (i). Conversely, let X be contractive so that $(\mathrm{d}/\mathrm{d}t) \| X(t)\psi \|^2_{|t=0} \leqslant 0$. Fix any finite set of vectors $g_\alpha \in \mathscr{H}_0$, $\alpha \in S'$, where $S' \subset \bar{S}, \# S' < \infty$. Taking continuous functions ${}^\alpha u \in \mathscr{M}$ so that ${}^\alpha u^j(0) = \delta^\alpha_j$ and

$$f_\alpha = \begin{cases} g_\alpha, & \text{if } \alpha \neq 0, \\ g_0 - \sum_{\beta \neq 0} g_\beta, & \text{if } 0 \in S', \alpha = 0 \\ -\sum_{\beta \neq 0} g_\beta, & \text{if } 0 \notin S', \alpha = 0 \end{cases}$$

in (10) we have

$$\sum_{\alpha, \beta \in S'} \langle g_\alpha, \mathscr{L}^\alpha_\beta g_\beta \rangle \leqslant 0.$$

Hence $L \in \mathscr{Z}^-_R$. This completes the proof of (i). The proof of (ii) is very similar to that of (i). ■

For any $L \equiv (L^i_j : i, j \in \bar{S})$ with L^i_j densely defined closed operators in $\mathscr{H}_0$ we define $\tilde{L} \equiv \{ \tilde{L}^i_j : i, j \in \bar{S} \}$ by

$$\tilde{L}^i_j = (L^j_i)^*, \quad i, j \in \bar{S}.$$

and set

$$\tilde{\mathscr{Z}}_R \equiv \{ L, \tilde{L} \in \mathscr{Z}_R \}, \quad \tilde{\mathscr{Z}}^-_R = \{ L, \tilde{L} \in \mathscr{Z}^-_R \} \text{ and } \tilde{\mathscr{I}}_R = \{ L, \tilde{L} \in \mathscr{I}_R \}.$$

As a consequence of Proposition 2.1 and 'time reversal principle' [17], we have the following theorem, which we state without proof. The proof can be found in [17, 20].

Theorem 2.2. *Suppose $Z \in \mathscr{Z}_R \cap \tilde{\mathscr{Z}}_R$. Then there exists a unique regular $(\mathscr{H}_0, \mathscr{M})$-adapted process $V = \{ V(t), 0 \leqslant t < \infty \}$ satisfying*

$$dV(t) = \sum_{i, j \in \bar{S}} V(t) Z^i_j d\Lambda^j_i(t), \quad V(0) = I \tag{11}$$

on $\mathscr{H}_0 \otimes \varepsilon(\mathscr{M})$. Moreover the following hold:

(i) *The following statements are equivalent: (a) V has a contractive extension; (b) $Z \in \mathscr{Z}^-_R$; (c) $Z \in \tilde{\mathscr{Z}}^-_R$.*

In such a case V is a strongly continuous bar-cocycle.

(ii) *V has an isometric extension if and only if $Z \in \mathscr{I}_R$;*

(iii) *V has a co-isometric extension if and only if* $Z\in\tilde{\mathscr{I}}_R$;
(iv) *V has a unitary extension if and only if* $Z\in\mathscr{I}_R\cap\tilde{\mathscr{I}}_R$.

3. A class of qsde with unbounded coefficients

In this section we recall some results from [19] which will enable us to deal with more general quantum evolutions satisfying (11) on $\mathscr{D}\otimes\varepsilon(\mathscr{M})$, where $\mathscr{D}$ is a common dense domain of the family $Z\equiv\{Z^i_j, i,j\in\bar{S}\}$ of operators in the initial Hilbert space $\mathscr{H}_0$.

We denote by $\mathscr{Z}^-(\mathscr{D})$ the class of elements $Z\equiv\{Z^i_j, i,j\in\bar{S}\}$ such that Z^0_0 is the generator of a strongly continuous contractive semigroup with $\mathscr{D}$ as a core and assume furthermore that

$$\text{(a) } \mathscr{D}\subseteq\mathscr{D}(Z^i_j);\quad (i,j\in\bar{S}); \tag{12}$$

(b) there exists a sequence $Z(n)\in\mathscr{Z}^-_R\cap\tilde{\mathscr{Z}}^-_R, n\geqslant 1$ so that for all $f\in\mathscr{D}$, $i,j\in\bar{S}$

$$s=\lim_{n\to\infty} Z^i_j(n)f=Z^i_jf. \tag{13}$$

Let $Z\in\mathscr{Z}^-(\mathscr{D})$. From Lemma 3.1 in [19] we observe that for each $f\in\mathscr{D}, j\in\bar{S}$ there exists a constant $c_j(f)\geqslant 0$ such that

$$\sup_{n\geqslant 1}\sum_{i\in\bar{S}}\|Z^i_j(n)f\|^2\leqslant c_j(f) \tag{14}$$

and

$$\sum_{i\in\bar{S}}\|Z^i_jf\|^2\leqslant c_j(f). \tag{15}$$

For any $X\in\mathscr{B}(\mathscr{H}_0)$ we define the bilinear forms $\mathscr{L}^i_j(X)(i,j\in\bar{S})$ on $\mathscr{D}$ by

$$\langle f,\mathscr{L}^i_j(X)g\rangle=\langle f,XZ^i_jg\rangle+\langle Z^j_if,Xg\rangle+\sum_{k\in S}\langle Z^k_if,XZ^k_jg\rangle$$

where the necessary convergence follows from (15) and Cauchy–Schwarz inequality. We set for $\lambda>0$

$$\beta_\lambda\equiv\{X\geqslant 0: X\in\mathscr{B}(\mathscr{H}_0);\quad \mathscr{L}^0_0(X)=\lambda X\},$$

and denote by $\mathscr{I}$ the class of elements $Z\in\mathscr{Z}^-(\mathscr{D})$ such that

$$\mathscr{L}^i_j(I)=0\quad\text{for all } i,j\in\bar{S}.$$

Fix a sequence $Z(n)\in\mathscr{Z}^-_R\cap\tilde{\mathscr{Z}}^-_R$ satisfying (12) and (13). We denote by $V^{(n)}\equiv\{V^{(n)}(t):t\geqslant 0\}$ the unique regular $(\mathscr{H}_0,\mathscr{M})$ adapted contractive process satisfying (11) with $Z(n)$ as its coefficients. We state the following propositions without proof, referring to [19, 20] for the proofs.

PROPOSITION 3.1.

Let $Z\in\mathscr{Z}^-(\mathscr{D})$ *and* $V^{(n)}$ *be as above. Then*
(i) $w-\lim_{n\to\infty} V^{(n)}(t)=V(t)$ *exists for all* $t\geqslant 0$,

(ii) *$V=\{V(t):t\geqslant 0\}$ is the unique strongly continuous contractive barcocycle satisfying (11) on $\mathscr{D}\otimes\varepsilon(\mathscr{M})$;*
(iii) *V is isometric only if $Z\in\mathscr{I}$;*
(iv) *if $Z\in\mathscr{I}$ and $\beta_\lambda\equiv\{0\}$ for some $\lambda>0$ then V is isometric.*

Remark 3.2. Suppose for each $n\geqslant 1$, $V^{(n)}$ is a regular contractive $(\mathscr{H}_0,\mathscr{M})$-adapted process satisfying (11) on $\mathscr{D}\underline{\otimes}\varepsilon(\mathscr{M})$ where $Z(n)$ are densely defined operators on $\mathscr{D}$. Then Proposition 3.1 holds as well for the associated sequence $V^{(n)}$ provided (12)–(14) are valid for Z. We omit the proof since it follows by the method employed for the proof of Proposition 3.3 in [19].

Let $\tilde{Z}\in\mathscr{Z}^-(\tilde{\mathscr{D}})$, for some a dense linear manifold $\tilde{\mathscr{D}}$ in $\mathscr{H}_0$. We denote by $\tilde{\mathscr{I}}$ and $\tilde{\beta}_\lambda$ the classes $\mathscr{I}$ and β_λ respectively, with Z replaced by $\tilde{Z}$.

COROLLARY 3.3.

Consider the contractive cocycle V defined as in Proposition 3.1. Let in addition $\tilde{Z}\in\mathscr{Z}^-(\tilde{\mathscr{D}})$. Then the following hold:
(i) *if V is co-isometric then $Z\in\tilde{\mathscr{I}}$;*
(ii) if *$Z\in\tilde{\mathscr{I}}$ and $\tilde{\beta}_\lambda\equiv\{0\}$ for some $\lambda>0$ then V is co-isometric.*

4. Minimal quantum dynamical semigroup and its dilation

We consider the quantum mechanical Fokker-Planck equation written formally as

$$\rho(0)=\rho,\quad \rho(t)'=Y\rho(t)+\rho(t)\,Y^*+\sum_{k\in S}Z_k\rho(t)Z_k^* \tag{16}$$

subject to

$$Y+Y^*+\sum_{k\in S}Z_k^*Z_k\leqslant 0 \tag{17}$$

for $\rho\in\mathscr{T}_h$, where $Y,Z_k,k\in S$ are densely defined operators in $\mathscr{H}_0$ and $\mathscr{T}_h$ is the real Banach space of all self-adjoint trace class operators in $\mathscr{H}_0$. When Y is a bounded operator, (17) implies that $\{Z_k,k\in S\}$ is a family of bounded operators and the series $\Sigma_{k\in S}Z_k^*Z_k$ converges in strong operator topology. In such a case, for each ρ (16) admits a unique $\mathscr{T}_h$-valued solution $\rho(t)$, $t\geqslant 0$ and the map $\rho\to\sigma_t(\rho)=\rho(t)$, $t\geqslant 0$ is a one parameter contraction semigroup in the Banach space $(\mathscr{T}_h,\|\cdot\|_{\mathrm{tr}})$. On the other hand by Theorem 2.2 (i) there exists a unique regular $(\mathscr{H}_0,\mathscr{M})$-adapted contractive operator valued process $V\equiv\{V(t),t\geqslant 0\}$ satisfying

$$\mathrm{d}V(t)=\sum_{k\in S}V(t)Z^i_j\Lambda^j_i(t),\quad V(0)=I \tag{18}$$

on $\mathscr{H}_0\underline{\otimes}\varepsilon(\mathscr{M})$ where

$$Z^i_j=\begin{cases} S^i_j-\delta^i_j, & i,j\in S,\\ Z_i, & i\in S, j=0,\\ -\sum_{k\in S}Z_k^*S^k_j, & i=0, j\in S,\\ Y, & i=0=j\end{cases} \tag{19}$$

and $S=((S^i_j))$ is a contractive operator in $\mathscr{H}_0\otimes l_2(S)$. The contractive one parameter semigroup $\tau_t:=\mathbb{E}_0[V(t)^*(x\otimes I)\,V(t)]$, $t\geqslant 0$ of completely positive maps [1] and σ_t, $t\geqslant 0$ satisfy the relation

$$\operatorname{tr}(x\sigma_t(\rho))=\operatorname{tr}(\rho\tau_t(x))$$

whenever $t\geqslant 0$, $\rho\in\mathscr{T}_h$, $x\in\mathscr{B}(\mathscr{H}_0)$.

Here our aim is to deal with the dilation problem associated with the Fokker-Planck equations (16)–(17) when the operators Y, Z_k, $k\in S$ are not necessarily bounded operators.

DEFINITION 4.1.

[9, 15] A one parameter family of completely positive maps $\tau\equiv\{\tau_t,t\geqslant 0\}$ on $\mathscr{B}(\mathscr{H}_0)$ is said to be a *quantum dynamical semigroup* if the following hold:

(i) $\tau_0(x)=x$, $\tau_t(\tau_s(x))=\tau_{s+t}(x)$, $s,t\geqslant 0$, $x\in\mathscr{B}(\mathscr{H}_0)$;
(ii) $\|\tau_t\|\leqslant 1, t\geqslant 0$;
(iii) The map $t\to\operatorname{tr}(\rho\tau_t(x))$ is continuous for any fixed $x\in\mathscr{B}(\mathscr{H}_0)$ and $\rho\in\mathscr{T}$, the trace class operators in $\mathscr{H}_0$.
(iv) For each $t\geqslant 0$ the map $x\to\tau_t(x)$ is continuous in the ultra-weak operator topology.

Given a dynamical semigroup τ we define the predual semigroup $\sigma\equiv\{\sigma_t,t\geqslant 0\}$ on $\mathscr{T}$ as

$$\operatorname{tr}(x\sigma_t(\rho))=\operatorname{tr}(\rho\tau_t(x)) \tag{20}$$

wherever $t\geqslant 0, \rho\in\mathscr{T}, x\in\mathscr{B}(\mathscr{H}_0)$. Note that the family σ is uniquely determined if (20) holds for $\rho:=|f><g|$, $f,g\in\mathscr{H}_0$. It is also evident that σ is a strongly continuous one parameter semigroup in the Banach space $(\mathscr{T}, \|\cdot\|_{\mathrm{tr}})$. Conversely, for a strongly continuous one parameter semigroup σ on $\mathscr{T}$, (20) determines a unique dynamical semigroup τ. Moreover for any $t\geqslant 0$, $\operatorname{tr}\sigma_t(\rho)=\operatorname{tr}(\rho), \rho\in\mathscr{T}_h$ if and only if $\tau_t(I)=I$.

The central aim of this section is to exploit the theory developed in § 3 and the construction of the minimal quantum dynamical semigroup, as outlined in Davies [4], in dilating the minimal semigroup in a boson-Fock space.

Before we proceed to the next result we state the following simple but useful lemmas without proof.

Lemma 4.2. *Let* $\text{s.lim}_{n\to\infty}A_n=A$ *and* $\text{s.lim}_{n\to\infty}B_n=B$. *Then* $\lim_{n\to\infty}A_n\rho B_n^*=A\rho B^*$ *in* $\|\cdot\|_{\mathrm{tr}}$ *topology whenever* $\rho\in\mathscr{T}$.

Lemma 4.3. *Let* A_k, $k\geqslant 1$ *and* B_k, $k\geqslant 1$ *be two families of bounded operators such that both the series* $\Sigma_{k\geqslant 1}A_k^*A_k$ *and* $\Sigma_{k\geqslant 1}B_k^*B_k$ *converge in strong operator topology. Then for each* $\rho\in\mathscr{T}_h$ *the series* $\Sigma_{k\geqslant 1}B_k\rho A_k^*$ *converges in* $\|\cdot\|_{\mathrm{tr}}$ *norm topology.*

As in Davies [4], let Y be the generator of a strongly continuous contractive semigroup in $\mathscr{H}_0$ and let Z_k, $k\in S$ be a family of densely defined operators on $\mathscr{H}_0$ such that

$$\mathscr{D}(Y)\subseteq\mathscr{D}(Z_k),\quad k\in S \tag{21}$$

and

$$\langle f, Yf\rangle + \langle Yf, f\rangle + \sum_{k\in S}\langle Z_k f, Z_k f\rangle \leqslant 0 \tag{22}$$

for all $f\in\mathscr{D}(Y)$.

In view of Lemma 4.2 the following relation

$$\kappa_t(\rho) = e^{tY}\rho e^{tY^*}$$

defines a strongly continuous, positive, one parameter, contraction semigroup on $\mathscr{T}_h$, whose generator G is given formally by

$$G(\rho) = Y\rho + \rho Y^*. \tag{23}$$

We introduce the positive one-to-one map π on $\mathscr{T}_h$ defined by

$$\pi(\rho) = (1-Y)^{-1}\rho(1-Y^*)^{-1}.$$

As in [4] we set $\pi\ (\mathscr{T}_h) = \{\pi(\rho),\ \rho\in\mathscr{T}_h\}$ and define the positive linear map $\mathscr{J}:\pi(\mathscr{T}_h)\to\mathscr{T}_h$ by

$$\mathscr{J}(\rho) = \sum_{k\in S} Z_k\rho Z_k^* \tag{24}$$

where the convergence follows from (22) and Lemma 4.3.

PROPOSITION 4.4.

Consider the family $Y, Z_k, k\in S$ of operators satisfying (21) *and* (22). *Then the following hold:*

(i) *$\pi(\mathscr{T}_h)$ is a core for G and* (23) *is valid for all $\rho\in\pi(\mathscr{T}_h)$;*

(ii) *The map $\mathscr{J}$ has a positive extension $\mathscr{J}'$ on $\mathscr{D}(G)$ such that*

$$tr(G(\rho) + \mathscr{J}'(\rho)) \leqslant 0 \tag{25}$$

wherever $\rho\in\mathscr{D}(G)$. Moreover equality holds in (25) *if and only if equality holds in* (22);

(iii) *For each fixed $\lambda>0$, $\mathscr{J}'(\lambda-G)^{-1}$ is a map from $\pi(\mathscr{T}_h)$ into $\mathscr{T}_h$ and has a unique bounded positive extension A_λ in $\mathscr{T}_h$ such that $\|A_\lambda\|\leqslant 1$ and $\mathscr{J}'(\rho) = A_1[1-G](\rho)$ for all $\rho\in\mathscr{D}(G)$;*

(iv) *For any fixed $0\leqslant r<1$, $\pi(\mathscr{T}_h)$ is a core for the operator $W^{(r)} = G + r\mathscr{J}'$ defined on $\mathscr{D}(G)$. Moreover $W^{(r)}$ is the generator of a strongly continuous positive one parameter contraction semigroup $\sigma_t^{(r)}$, whose resolvent at $\lambda>0$ is given by*

$$R_r(\lambda) \equiv (\lambda - W^{(r)})^{-1} = (\lambda - G)^{-1}\sum_{k\geqslant 0} r^k A_\lambda^k, \tag{26}$$

where the series converges in trace norm;

(v) *For each $\rho\geqslant 0$, $t\geqslant 0$ the map $r\to\sigma_t^{(r)}(\rho)$, $r\in[0,1)$ is increasing and continuous;*

(vi) *There exists a positive one parameter strongly continuous contraction semigroup σ_t^{min} on $\mathscr{T}_h$ such that*

$$\lim_{r\uparrow 1}\sigma_t^r(\rho) = \sigma_t^{min}(\rho)$$

for all $\rho\in\mathscr{T}_h$;

(vii) *For each* $\lambda > 0$, $R^{(n)}(\lambda) := (\lambda - G)^{-1} \Sigma_{0\leqslant k\leqslant n} A_\lambda^k \to R(\lambda)$ *strongly as* $n \to \infty$, *where* $R(\lambda) = (\lambda - W)^{-1}$, *W is the generator of* σ_t^{min}.

Proof. For (i)–(vi) see Davies [4]. Now for (vii) we follow Kato [14] (Lemma 7). For each $\lambda > 0$, $0 \leqslant r < 1$ we have

$$R_r^{(n)}(\lambda) := (\lambda - G)^{-1} \sum_{0\leqslant k\leqslant n} r^k A_\lambda^k \leqslant R_r(\lambda) \leqslant R(\lambda).$$

Letting $r \uparrow 1$ we get $R^{(n)}(\lambda) \leqslant R(\lambda)$. But as $R^{(n)}(\lambda)$ is increasing with n, $\text{s.lim}_{n\to\infty} R^{(n)}(\lambda) = R'(\lambda)$ exists and $R'(\lambda) \leqslant R(\lambda)$. We also have $R_r^{(n)}(\lambda) \leqslant R^{(n)}(\lambda) \leqslant R'(\lambda)$. Hence $R_r(\lambda) = \lim_{n\to\infty} R_r^{(n)}(\lambda) \leqslant R'(\lambda)$, $R(\lambda) = \lim_{r\uparrow 1} R_r(\lambda) \leqslant R'(\lambda)$ by (vi). This completes the proof.

Now our aim is to obtain a necessary and sufficient condition for σ to be trace preserving. It is evident that equality in (22) is necessary. We have the following theorem giving sufficient conditions.

Theorem 4.5. *Consider the semigroup* σ_t^{min}, $t \geqslant 0$ *defined as in Proposition* 4.4. *Let* $W_0 = G + \mathscr{J}'$ *with domain* $\pi(\mathscr{T}_h)$ *and let* W_0^* *be the adjoint of* W_0. *Assume furthermore the equality in* (22). *Then the following statements are equivalent:*

(i) $tr(\sigma_t^{min}(\rho)) = tr(\rho)$ *for all* $t \geqslant 0$, $\rho \in \mathscr{T}_h$;
(ii) *for each fixed* $\lambda > 0$, $A_\lambda^n \to 0$ *strongly as* $n \to \infty$;
(iii) *for each fixed* $\lambda > 0$, $(\lambda - W_0)(\pi(\mathscr{T}_h))$ *is dense in* $\mathscr{T}_h$;
(iv) *for each fixed* $\lambda > 0$, *the characteristic equation* $W_0^*(x) = \lambda x$ *has no non-zero solution in* $\mathscr{B}(\mathscr{H}_0)$;
(v) *for any fixed* $\lambda > 0$,

$$\beta_\lambda \equiv \{x \geqslant 0, x \in \mathscr{B}(\mathscr{H}_0): \langle f, xYg\rangle + \langle Yf, xg\rangle + \sum_{k\in S} \langle Z_k f, xZ_k g\rangle = \lambda\langle f, g\rangle \tag{27}$$

hold for all $f, g \in \mathscr{D}(Y)\} = \{0\}$.

Proof. The proof is exactly along the lines of Theorem 3 in [14]. We write $\sigma = \sigma^{\min}$. As in [14] in this context we note that

$$\| R(\lambda)(\rho) \|_{\text{tr}} = \int_0^\infty \exp(-\lambda t) \| \sigma_t(\rho) \|_{\text{tr}} \text{d}t \tag{28}$$

for all $\rho \geqslant 0$, which follows from the resolvent formula $R(\lambda) = \int_0^\infty \exp(-\lambda t)\sigma_t \text{d}t$, $\lambda > 0$. As a simple consequence of the following identity

$$I + \mathscr{J}' R^{(n)}(\lambda) = (\lambda I - G) R^{(n)}(\lambda) + A_\lambda^{n+1} \tag{29}$$

and (22) and (25), we get $\text{tr}(\rho) = \lambda \,\text{tr}(R^{(n)}(\lambda)(\rho)) + \text{tr}(A_\lambda^{n+1}(\rho))$ for $\rho \in \mathscr{T}$. Since $R^{(n)}(\lambda)(\mathscr{T}_+) \subset \mathscr{T}_+$ we have

$$\| \rho \| = \lambda \| R^{(n)}(\lambda)(\rho) \|_{\text{tr}} + \| A_\lambda^{n+1}(\rho) \|_{\text{tr}} \tag{30}$$

for all $\rho \geqslant 0$. Now taking limit as $n \to \infty$ in (30) we get by Proposition 4.4(vii)

$$\lim_{n\to\infty} \| A_\lambda^{n+1}(\rho)\|_{\mathrm{tr}} = \|\rho\| - \lambda \| R(\lambda)(\rho)\|_{\mathrm{tr}}$$

$$= \lambda \int_0^\infty \exp(-\lambda t)(\|\rho\|_{\mathrm{tr}} - \|\sigma_t(\rho)\|_{\mathrm{tr}}) \tag{31}$$

for all $\rho \geqslant 0$, where we have used (28) in the second equality. Since for each fixed $\rho \in \mathscr{T}$ the map $t \to \|\sigma_t(\rho)\|_{\mathrm{tr}}$ is continuous and $\|\sigma_t(\rho)\|_{\mathrm{tr}} \leqslant \|\rho\|_{\mathrm{tr}}$, $t \geqslant 0$ from (31) we conclude that (i) and (ii) are equivalent.

Our next aim is to show that (ii) and (iii) are equivalent for any fixed $\lambda > 0$. From (29) we note that (ii) is equivalent to

$$\lim_{n\to\infty} [\lambda - G - \mathscr{J}'] R^{(n)}(\lambda)(\rho) = \rho$$

for all $\rho \in \mathscr{T}$. Since $R^{(n)}(\lambda)(\rho) \in \mathscr{D}(G)$ we conclude that $[\lambda - G - \mathscr{J}'](\mathscr{D}(G))$ is dense in $\mathscr{T}$. Since $\pi(\mathscr{T}_h)$ is a core for G, for any fixed $\rho \in \mathscr{D}(G)$ we choose a sequence $\rho_n \in \pi(\mathscr{T}_h)$ such that $\rho_n \to \rho$ and $G(\rho_n) \to G(\rho)$ as $n \to \infty$. By Proposition 4.4(iii) we have

$$\|\mathscr{J}'(\rho)\|_{\mathrm{tr}} = \|A_1[1 - G)](\rho)\|_{\mathrm{tr}} \leqslant \|[1 - G](\rho)\|_{\mathrm{tr}} \leqslant \|\rho\|_{\mathrm{tr}} + \|G(\rho)\|_{\mathrm{tr}}$$

for all ρ in $\mathscr{D}(G)$, hence $\mathscr{J}'(\rho) = \lim_{n\to\infty} \mathscr{J}'(\rho_n)$. Thus it is evident that

$$(\lambda - G - \mathscr{J}')(\rho) = \lim_{n\to\infty} (\lambda - G - \mathscr{J}')(\rho_n)$$

Hence $[\lambda - G - \mathscr{J}'](\pi(\mathscr{T}_h))$ is dense in $\mathscr{T}_h$.

Conversely let (iii) be valid. Since $[I - A_\lambda](\mathscr{T}_h) = [I - A_\lambda][\lambda - G](\mathscr{D}(G)) = [\lambda - G - \mathscr{J}'][\pi(\mathscr{D}(G)] \supset [\lambda - G - \mathscr{J}'][\pi(\mathscr{T}_h)]$, $[I - A_\lambda](\mathscr{T}_h)$ is dense in $\mathscr{T}_h$. Set $C_\lambda^{(n)} = (1/n+1)\Sigma_{0\leqslant k \leqslant n} A_\lambda^k$, which is a uniformly bounded by 1. That $\operatorname{slim}_{n\to\infty} C_\lambda^{(n)} = 0$ is now an easy consequence of $C_\lambda^{(n)}[I - A_\lambda] = (1/n+1)[I - A_\lambda^{n+1}]$. On the other hand, A_λ being a contractive positive map, $\|A_\lambda^m\| \leqslant \|A_\lambda^n\|$ whenever $m \geqslant n$, hence

$$\|C_\lambda^{(n)}(\rho)\|_{\mathrm{tr}} = \frac{1}{n+1} \sum_{0\leqslant k\leqslant n} \|A_\lambda^k(\rho)\|_{\mathrm{tr}} \geqslant \|A_\lambda^n(\rho)\|_{\mathrm{tr}}$$

whenever $\rho \geqslant 0$. Thus we have $A^n(\rho) \to 0$ as $n \to \infty$. This shows that (ii) and (iii) are equivalent.

That (iii) and (iv) are equivalent follows by the definition of adjoint of a densely defined operator and Hahn–Banach theorem. Finally we need to show that an element $x \in \mathscr{D}(W_0^*)$ satisfies $W_0^*(x) = \lambda x$ if and only if x satisfies (27). For any fixed $f, g \in \mathscr{H}_0$ and $x \in \mathscr{D}(W_0^*)$ we have

$$\begin{aligned} \operatorname{tr}(\pi(|f\rangle\langle g|)\, W_0^*(x)) = {} & \sum_{k\in S} \langle Z_k(1-Y)^{-1} f, x Z_k (1-Y)^{-1} g\rangle \\ & + \langle Y(1-Y)^{-1} f, x(1-Y)^{-1} g\rangle \\ & + \langle (1-Y)^{-1} f, xY(1-Y)^{-1} g\rangle. \end{aligned} \tag{32}$$

Since $\mathscr{R}(1-Y)^{-1})=\mathscr{D}(Y)$, that (iv) and (v) are equivalent is a simple consequence of (32). ∎

We use the same symbol for the linear canonical extension of a bounded map that appeared in Proposition 4.4 to the Banach space of all trace class operators. In the case of an unbounded operator, say G we extend it to $\mathscr{D}(G)+i\mathscr{D}(G)$ by linearity. The family of maps $\tau_t^{\min}:=(\sigma_t^{\min})^*$ on the dual space $\mathscr{B}(\mathscr{H}_0)$ is called the *minimal dynamical semigroup*. For further details we refer to [4].

Our next aim is to deal with the dilation problem associated with the Fokker-Planck equation (16) whenever the operators Y, Z_k, $k\in S$ satisfy the following assumption.

Assumption A. Y is the generator of a strongly continuous semigroup on $\mathscr{H}_0$ and Z_k, $k\in S$ is a family of densely defined operators satisfying (21) and (22). There exists a dense linear manifold $\mathscr{D}$ in $\mathscr{H}_0$ so that it is a core for Y and

$$S_j^k(\mathscr{D})\subseteq\mathscr{D}(Z_k^*),\quad k,j\in S$$

where $S\equiv((S_j^k,\ k,j\in S))$ is a contractive operator on $\mathscr{H}_0\otimes l_2(S)$. Furthermore for any fixed $j\in S$, $S_j^i\neq 0$ for finitely many $i\in S$. The last hypothesis ensures that the third expression (19) is meaningful and is indeed verified for most applications [19].

For any $\lambda>0$ we define bounded operators Y_λ, Z_k^λ, $k\in S$ by

$$Y_\lambda=\lambda^2(\lambda-Y^*)^{-1}Y(\lambda-Y)^{-1},\quad Z_k^\lambda=\lambda Z_k(\lambda-Y)^{-1}$$

where boundedness of Z_k^λ, $k\in S$ follows from (22). Moreover for each $\lambda>0$, Y_λ, Z_k^λ, $k\in S$ satisfies (17), hence the series $\Sigma_{k\in S}(Z_k^\lambda)^*Z_k^\lambda$ converges in strong operator topology. On the other hand for each $g\in\mathscr{D}(Y)$ we have $Y_\lambda g\to Yg$ as $\lambda\to\infty$. Taking $f=(I-\lambda(\lambda-Y)^{-1})g$ in (22) we get

$$\|Z_k(I-\lambda(\lambda-Y)^{-1})g\|\leqslant 2\|(I-\lambda(\lambda-Y)^{-1}g\|\,\|Y(I-\lambda(\lambda-Y)^{-1})g\|$$

Hence $Z_k^\lambda g\to Z_kg$ as $\lambda\to\infty$ for all $g\in\mathscr{D}(Y)$ [5].

For any $\lambda>0, 0\leqslant r\leqslant 1$ we define bounded operators $Z(\lambda,r)\equiv\{Z_j^i(\lambda,r), i,j\in\bar{S}\}$ as in (19) with $Y, Z_k, k\in S$ replaced by $Y_\lambda, r^{1/2}Z_k^\lambda, k\in S$ respectively. So for each $0\leqslant r\leqslant 1$ and $\lambda>0$, $Z(\lambda,r)\in\mathscr{Z}_R^-\cap\tilde{\mathscr{Z}}_R^-$. We denote by $V^{(\lambda,r)}=\{V^{(\lambda,r)}(t), t\geqslant 0\}$ the unique regular $(\mathscr{H}_0,\mathscr{M})$ - adapted contractive process satisfying (18) with $Z(\lambda,r)$ as its coefficients.

We also define operators $Z(r)$ on $\mathscr{D}$ as in (19) with Z_k replaced by $r^{1/2}Z_k$, $k\in S$ and write $Z(\lambda,1)=Z(\lambda)$, $Z(1)=Z$. For each $0\leqslant r\leqslant 1$ it is evident that

$$\lim_{\lambda\to\infty} Z_j^i(r,\lambda)f=Z_j^i(r)f,$$

for all $i,j\in\bar{S}$, $f\in\mathscr{D}$.

PROPOSITION 4.6.

Consider the operators Y, Z_k, $k\in S$ satisfying Assumption A. Then the following hold:

(i) *For each $0\leqslant r\leqslant 1$, $w.\lim_{\lambda\to\infty}V^{(\lambda,r)}(t)=V^{(r)}(t)$ exists for all $t\geqslant 0$ and $V^{(r)}=\{V^{(r)}(t), t\geqslant 0\}$ is the unique regular $(\mathscr{H}_0,\mathscr{M})$-adapted contractive operator valued process*

satisfying (18) on $\mathscr{D}\otimes\varepsilon(\mathscr{M})$ *with* $Z(r)\equiv\{Z^i_j(r),\ i,j\in\bar{S}\}$ *as its coefficients. Moreover* $V^{(r)}$ *is a strongly continuous contractive bar-cocycle;*
(ii) *For each* $t\geqslant 0$ *the map* $r\rightarrow V^{(r)}(t), 0\leqslant r\leqslant 1$ *is continuous is weak operator topology.*

Proof. For each $0\leqslant r\leqslant 1, \lambda>0; Z(\lambda,r)\in\mathscr{Z}^-_R\cap\mathscr{Z}^-_R$ and the triad $Z(r), Z(\lambda,r), \mathscr{D}$ satisfy (12) and (13) with n replaced by λ. By our hypothesis $\mathscr{D}$ is also a core for Y. Hence we conclude (i) by Proposition 3.1 (i)–(ii).

Choose $0\leqslant r,r_n\leqslant 1 (n\geqslant 1)$ such that $r_n\rightarrow r$ as $n\rightarrow\infty$. Since the triad $(Z(r), Z(r_n), n\geqslant 1, \mathscr{D})$ satisfies (12) and (13) on $\mathscr{D}$ and $Z^i_j(r_n)f\rightarrow Z^i_j(r)f$, as $r_n\rightarrow\infty$ for any $f\in\mathscr{D}$, remark 3.2 implies that $w\lim_{n\rightarrow\infty} V^{(r_n)}(t) = V^{(r)}(t), 0\leqslant t<\infty$. This completes the proof. ■

For each $\lambda,\mu>0$, $0\leqslant r,s\leqslant 1$, we define the semigroup $\tau^{(\lambda,\mu,r,s)}$ on $\mathscr{B}(\mathscr{H}_0)$ by

$$\tau^{(\lambda,\mu,r,s)}_t(x) = \mathbb{E}_0[V^{(\lambda,r)}(t)^* x V^{(\mu,s)}(t)],\quad t\geqslant 0,$$

where the semigroup property follows from the cocycle property of the contractive processes $V^{(\lambda,r)}$. The associated pre-dual semigroup $\sigma^{(\lambda,\mu,r,s)}$ on $\mathscr{T}$ is defined as in (20) whose bounded generator $\mathscr{L}^{(\lambda,\mu,r,s)}_*$ is given by

$$\mathscr{L}^{(\lambda,\mu,r,s)}_*(\rho) = Y_\mu\rho+\rho Y^*_\lambda+\sqrt{rs}\sum_{k\in S} Z^\mu_k\rho(Z^\lambda_k)^*,\quad \rho\in\mathscr{T}.$$

For each $0\leqslant r<1$ we also have

$$W^{(r)}(\rho) = Y\rho+\rho Y^*+r\sum_{k\in S} Z_k\rho Z^*_k,\quad \rho\in\pi(\mathscr{T}),$$

where $W^{(r)}$ is described in Proposition 4.4.

We write $r^{(\lambda,r)}$, $\tau^{(\lambda,\mu,r)}$, $\sigma^{(\lambda,r)}$ and $\sigma^{(\lambda,\mu,r)}$ for $\tau^{(\lambda,\lambda,r,r)}$, $\tau^{(\lambda,\mu,r,r)}$, $\sigma^{(\lambda,\lambda,r,r)}$ and $\sigma^{(\lambda,\mu,r,r)}$ respectively. When $r=1$ we omit the symbol r. For each $0\leqslant r,s\leqslant 1$ we also define the one parameter semigroup

$$\tau^{(r,s)}_t := \mathbb{E}_0[V^{(r)}(t)^* x V^{(s)}(t)],\quad t\geqslant 0$$

on $\mathscr{B}(\mathscr{H}_0)$. Again when $r=s=1$ we omit the symbol r.

Our aim is to show that $\sigma^{(\min)}_t$ is the pre-dual map of τ_t for all $t\geqslant 0$, where $\sigma^{\min}$ is defined as in Proposition 4.4. For this we need the following lemma.

Lemma 4.7. *Let* $A_k, k\geqslant 1$ *and* $B_k, k\geqslant 1$ *be two families of bounded operators such that the series* $\Sigma_{n\geqslant 1} A^*_k A_k$ *converges in strong operator topology and* $s\lim_{n\rightarrow\infty} B_n = B$. *Then for each* $\rho\in\mathscr{T}_h$, $\lim_{m,n\rightarrow\infty} C(m,n)\equiv \lim_{m,n\rightarrow\infty}\Sigma_{k\in S}A_kB_m\rho B^*_n A^*_k = \Sigma_{k\in S}A_kB\rho B^*A^*_k\equiv C$ *in* $\|\cdot\|_{tr}$ *norm.*

Proof. Lemma 4.3 implies that C, $C(m,n)$, $m,n\geqslant 1$ are elements in $\mathscr{T}$. For any fixed $m,n\geqslant 1$ and $\rho\in\mathscr{T}$ we have

$$\|C(m,n)-C\|_{\mathrm{tr}}\leqslant\sum_{k\geqslant 1}\{\|A_k(B_m-B)\rho(A_kB_n)^*\|_{\mathrm{tr}}$$
$$+\|A_kB\rho(A_k(B_n-B))^*\|_{\mathrm{tr}}.\}.$$

Hence for $\rho = |f\rangle\langle g|$ we have

$$\begin{aligned}\|C(m,n)-C\|_{tr} &\leqslant \sum_{k\geqslant 1}\{\|A_k(B_m-B)f\|\,\|A_kB_ng\| + \|A_kBf\|\,\|A_k(B_n-B)g\|\}\\ &\leqslant \Big(\sum_{k\geqslant 1}\|A_k(B_m-B)f\|^2\Big)^{1/2}\Big(\sum_{k\geqslant 1}\|A_kB_ng\|^2\Big)^{1/2}\\ &\quad+\Big(\sum_{k\geqslant 1}\|A_kf\|^2\Big)^{1/2}\Big(\sum_{k\geqslant 1}\|A_k(B_n-B)g\|^2\Big)^{1/2}\\ &\leqslant \alpha(\|(B_m-B)f\|\,\|B_ng\| + \|f\|\,\|(B_n-B)g\|) \leqslant \beta\|f\|\,\|g\|\end{aligned}$$

where α, β are some positive constants independent of f, g. Hence the result follows for $\rho = |f\rangle\langle g|$, $f, g \in \mathscr{H}_0$. For a general $\rho = \Sigma_i c_i |f_i\rangle\langle g_i|$, $\|f_i\| = \|g_i\| = 1$, $\Sigma_i |c_i| < \infty$, we use dominated convergence theorem to conclude the required result. ∎

PROPOSITION 4.8.

Consider the family of operators $\{Y, Z_k, k \in S\}$ *satisfying* (21) *and* (22). *Then for each fixed* $0 \leqslant r < 1$ *the following hold*:

(i) *For each* $\lambda, \mu > 0$, $0 \leqslant r, s \leqslant 1$,

$$\sigma_t^{(\lambda,\mu,r,s)} = \sigma^{(\lambda,\mu,(rs)^{1/2})}, \quad t \geqslant 0;$$

(ii) *For* $\rho \in \pi(\mathscr{T})$, $\lim_{(\lambda,\mu)\to\infty} \|\mathscr{L}_*^{(\lambda,\mu,r)}(\rho) - W^{(r)}(\rho)\|_{tr} = 0$ *where the limit is independent of the order of* λ, μ;

(iii) $\lim_{(\lambda,\mu)\to\infty} \|\sigma_t^{(\lambda,\mu,r)}(\rho) - \sigma_t^{(r)}(\rho)\|_{tr} = 0$ *for all* $\rho \in \mathscr{T}$, *where* $\sigma^{(r)}$ *is the map defined as in Proposition* 4.4;

(iv) *The pre-dual map of* $\tau_t^{(r)}$ *is* $\sigma_t^{(r)}$, $t \geqslant 0$;

(v) *For each* $0 \leqslant s < 1$, $\sigma_t^{(r,s)} = \sigma_t^{((rs)^{1/2})}$ *for all* $t \geqslant 0$.

Proof. Since for each fixed $\lambda, \mu > 0$, $\mathscr{L}_*^{(\lambda,\mu,r,s)} = \mathscr{L}^{(\lambda,\mu,(rs)^{1/2}(rs)^{1/2})}$ we conclude (i) by the fact that a bounded generator uniquely determines the semigroup [5].

Now for (ii) first observe that

$$\mathscr{L}_*^{(\lambda,\mu,r)}(\pi(\rho)) = Y_\mu\pi(\rho) + \pi(\rho)\,Y_\lambda^* + r^2\sum_{k\in S} Z_k^1(\mu(\mu-Y)^{-1})\rho(\lambda(\lambda-Y)^{-1})^*(Z_k^1)^*$$

and

$$W^{(r)}(\pi(\rho)) = Y\pi(\rho) + \pi(\rho)\,Y^* + r^2\sum_{k\in S} Z_k^1\rho(Z_k^1)^*$$

for all $\rho \in \mathscr{T}$ and $Y_\mu\pi(\rho) = \mu^2(\mu - Y^*)^{-1}(\mu - Y)^{-1}(Y(1-Y)^{-1}\rho(1-Y^*)^{-1})$. Now (ii) is immediate from Lemma 4.7.

Since $\pi(\mathscr{T})$ is a core for $W^{(r)}$ which is the generator of a strongly continuous contraction semigroup, (iii) is evident from (ii) and a standard result (Corollary 3.18 [5]) in the theory of semigroups.

For any fixed $f, g \in \mathscr{H}_0$, $\lambda, \mu > 0$ we have

$$\mathrm{tr}(x\sigma_t^{\lambda,\mu,r}(|f\rangle\langle g|)) = \langle fe(0), V^{(\lambda,r)}(t)^* x V^{(\mu,r)}(t) ge(0)\rangle$$

Hence (iv) follows from Proposition 4.6(i) and (iii). Finally, we arrive at (v) from (i) and (iii).

The following theorem establishes the main result.

Theorem 4.9. *Let $Y, Z_k, k \in S$ be a family of operators satisfying Assumption A. Consider the family $Z \equiv \{Z_j^i, i, j \in S\}$ defined as in* (19) *on $\mathscr{D}$. Then there exists a unique regular $(\mathscr{D}, \mathscr{M})$-adapted contractive process $V \equiv \{V(t), t \geqslant 0\}$ satisfying* (18) *on $\mathscr{D} \otimes \varepsilon(\mathscr{M})$. Moreover the following hold:*
(i) $\tau_t^{\min}(x) = \mathbb{E}_0[V(t)^* x V(t)]$, where $\tau^{\min}$ is the minimal dynamical semigroup on $\mathscr{B}(\mathscr{H}_0)$ associated with (16) *and* (17)
(ii) Assume furthermore that S is an isometry and the equality in (22) *holds. Then $Z \in \mathscr{I}$. In such a case V is isometric if and only if $\beta_\lambda = 0$ for some $\lambda > 0$, where β_λ is defined as in Theorem* 4.5 (v).

Proof. The first part is a restatement of Proposition 4.6 (i) for $r = 1$.

In view of Proposition 4.8 it is evident that for all $0 \leqslant r < 1$,

$$\begin{aligned}\mathrm{tr}(\sigma_t^{((r)^{1/2})}(|f\rangle\langle g|)x) &= \lim_{s\uparrow 1} \mathrm{tr}(\sigma_t^{((rs)^{1/2})}(|f\rangle\langle g|)x) \\ &= \lim_{s\uparrow 1} \langle fe(0), V^{(r)}(t)^* x V^{(s)}(t) ge(0)\rangle \\ &= \langle fe(0), V^{(r)}(t)^* x V(t) ge(0)\rangle\end{aligned}$$

for any $f, g \in \mathscr{H}_0$. Now taking limit as $r \uparrow 1$ in the above identity we get the required identity for (i) by Proposition 4.4(vi).

That $Z \in \mathscr{I}$ is simple to verify. The 'only if' part of (ii) follows from (i) and Theorem 4.5. For the converse we appeal to Proposition 3.1 (iv). This completes the proof. ■

Now combining Corollary 3.3 and Theorem 4.9 we arrive at necessary and sufficient conditions for V to be co-isometric.

Theorem 4.10. *Consider the family $V \equiv \{V(t), t \geqslant 0\}$ of operators defined as in Theorem 4.9. Suppose the family $\{Y^*, Z_k, k \in S\}$ of operators also satisfy* (21), (22) *and $\tilde{D}$ is a core for Y^* so that $\tilde{\mathscr{D}} \subset \mathscr{D}(Z_k^*)$, $k \in S$. Assume further the equality in* (22) *and S is a co-isometry then $Z \in \tilde{\mathscr{I}}$. In such a case V is co-isometric if and only if $\tilde{\beta}_\lambda = 0$ for some $\lambda > 0$, where $\tilde{\beta}_\lambda$ defined in Corolary* 3.3 *is modified as β_λ was in the statement of Theorem* 4.5(*v*).

Proof. S being a contractive operator we observe that

$$\sum_{k\in S} \| L_k f\|^2 \leqslant \sum_{k\in S} \|Z_k f\|^2$$

for each $f \in \mathscr{D}(Z_k)$, $k \in S$, where $L_k = \Sigma_{l\in S}(S_k^l)^* Z_l$. Hence the family $\{Y^*, L_k, k \in S\}$ also satisfy (21) and (22). Thus $\tilde{Z}\in\mathscr{Z}^-(\tilde{\mathscr{D}})$. The proof is complete once we appeal to Corollary 3.3.

Example 4.11. Let L_k, $k\in S$ be a family of closed operators in $\mathscr{H}_0$ and Y be the generator of a contractive C_0 semigroup satisfying (21) and (22). For each $k\in S$ consider the polar decomposition $L_k = S_k|L_k|$, where S_k is the partial isometry with initial subspace as $\overline{\mathscr{R}(|L_k|)}$, hence $S_k^* L_k = |L_k|$. Now with $Z_k = L_k, S_j^k = \delta_j^k S_k$, define the family of operators $Z\equiv\{Z_j^i, i,j\in\bar{S}\}$ as in (19) on $\mathscr{D}(Y)$. It is evident that Assumption A is valid. In general it is difficult to verify if β_λ or $\tilde{\beta}_\lambda$ or both are trivial. However, when $|L_k|$, $k\in S$ is a family of commuting self-adjoint operators then $\tilde{\beta}_\lambda = 0$ for some (hence for all) $\lambda > 0$. For more explicit examples refer [19].

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