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Under what kind of parametric fluctuations is spatiotemporal regularity the most robust?

MANISH DEV SHRIMALI[1], SWARUP PORIA[2] and SUDESHNA SINHA[3,*,†]
[1]The LNM Institute of Information Technology, Jaipur 302 031, India
[2]Department of Mathematics, University of Calcutta, 92, A.P.C. Road, Kolkata 700 009, India
[3]Indian Institute of Science Education and Research (IISER) Mohali, Transit Campus: MGSIPAP Complex, Sector 26, Chandigarh 160 019, India
*Corresponding author. E-mail: sudeshna@imsc.res.in

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Abstract. It was observed that the spatiotemporal chaos in lattices of coupled chaotic maps was suppressed to a spatiotemporal fixed point when some fractions of the regular coupling connections were replaced by random links. Here we investigate the effects of different kinds of parametric fluctuations on the robustness of this spatiotemporal fixed point regime. In particular we study the spatiotemporal dynamics of the network with noisy interaction parameters, namely fluctuating fraction of random links and fluctuating coupling strengths. We consider three types of fluctuations: (i) noisy in time, but homogeneous in space; (ii) noisy in space, but fixed in time; (iii) noisy in both space and time. We find that the effect of different kinds of parametric noise on the dynamics is quite distinct: quenched spatial fluctuations are the most detrimental to spatiotemporal regularity; spatiotemporal fluctuations yield phenomena similar to that observed when parameters are held constant at the mean value, and interestingly, spatiotemporal regularity is most robust under spatially uniform temporal fluctuations, which in fact yields a larger fixed point range than that obtained under constant mean-value parameters.

Keywords. Coupled map lattice; parametric fluctuations; networks; spatiotemporal dynamics; synchronization.

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1. Introduction

One of the important prototypes of extended complex systems are nonlinear dynamical systems with spatially distributed degrees of freedom, or alternately, spatial

†On leave from the Institute of Mathematical Sciences, Taramani, Chennai 600 113, India

systems composed of large numbers of low-dimensional nonlinear systems. The basic ingredients of such systems are: (i) creation of local chaos or local instability by a low-dimensional mechanism and (ii) spatial transmission of energy and information by coupling connections of varying strengths and underlying topologies.

The coupled map lattice (CML) is such a model, capturing the essential features of the nonlinear dynamics of extended systems [1]. A very well-studied coupling form in CMLs is the nearest-neighbour coupling. However, some degree of randomness in spatial coupling can be closer to physical reality than strict nearest-neighbour scenarios. In fact, many systems of biological, technological and physical significance are better described by randomizing some fraction of the regular links [2,3]. Here we focus on a ring of coupled chaotic maps whose coupling connections are dynamically rewired to random sites with probability p, namely at any instance of time, with probability p, a regular link is switched to a random one [4–13].

It has recently been found that such random coupling yields a spatiotemporal fixed point in a network of chaotic maps [13]. That is, the strongly unstable fixed point of the local chaotic map is stabilized under increasing randomness in the coupling connections. Thus interestingly, the inherent chaos present in the individual local units is suppressed by dynamically switched random links, giving rise to a global spatiotemporal fixed point attractor.

In this paper we study the effect of parametric fluctuations on the synchronization properties of such networks. We consider different types of noise in the parameters: (i) spatial, (ii) temporal and (iii) both spatial and temporal. Keeping the local dynamics always fully chaotic, we focus on parametric noise in the interaction parameters between the nodes in the network. In particular, we consider fluctuations in the fraction of random links p in the system, and fluctuations in the coupling strength of the different links. That is, we study perturbations in both the geometry of the network connections, as reflected in noisy p, as well as in the strength of the links.

The outline of the paper is as follows: we first describe the model in §2. Then we present numerical results in §3, followed by some analysis in §4. Finally we conclude with a summary of our results in §5.

2. Model

We consider a network of N coupled logistic maps. The sites are denoted by integers $i = 1, \ldots, N$, where N is the size of the lattice. On each site is defined a continuous state variable denoted by $x_n(i)$, which corresponds to the physical variable of interest.

The evolution of this lattice, under interactions with the nearest neighbours, is given by

$$x_{n+1}(i) = (1-\epsilon) f(x_n(i)) + \frac{\epsilon}{2}\{x_n(i+1) + x_n(i-1)\}. \quad (1)$$

The strength of the coupling is given by ϵ. The local on-site map is chosen to be the fully chaotic logistic map: $f(x) = \alpha x(1-x)$ with $\alpha = 4$, as this map has widespread relevance as a prototype of low-dimensional chaos.

We consider the above system with its coupling connections rewired randomly with probability p. Namely, at every update we will connect a site with probability p to randomly chosen sites, and with probability $(1 - p)$ to nearest neighbours, as in eq. (1). That is, at every instant, a fraction p of randomly chosen nearest-neighbour links are replaced by random links. The case of $p = 0$ corresponds to the usual nearest-neighbour interaction, while $p = 1$, corresponds to completely random coupling. This type of connectivity has been observed in a range of natural and human-engineered systems [3].

In this work, we will focus on the effect of fluctuations in the interaction parameters, i.e. noisy coupling strength ϵ, and fraction of random links p [14–17]. Now, one can have four distinct scenarios (denoting the relevant parameter as A):

(i) $A_n(i) \equiv A_0$.

Here the parameter is constant. We will denote this case of zero fluctuations by C.

(ii) $A_n(i) = A_0 \pm \delta A \eta^i \equiv A(i)$.

Here δA is the strength of the fluctuation in the parameter around mean value A_0 and η^i is a zero-mean random number. So here the parameters are random in space but remain frozen in time, i.e. the parameters are spatially fluctuating but temporally invariant. In this case then, we have quenched disorder or a basic inhomogeneity of the network links. We will denote this case of spatial parametric fluctuations by S.

(iii) $A_n(i) = A_0 \pm \delta A \eta_n \equiv A_n$.

Again, δA is the strength of the fluctuation in the parameter around mean value A_0 and η_n is a zero-mean random number. So the fluctuation is a function of time but is site-independent, i.e. the noise in the parameter is synchronous for all the elements, namely parameter A is spatially uniform, though random in time. This kind of a situation may arise when the system is quite uniform intrinsically, but is subject to a common perturbation, for instance from a common environmental influence, like say fluctuations in the ambient temperature. We will denote this case of temporal parametric fluctuations by T.

(iv) $A_n(i) = A_0 \pm \delta A \eta_n^i$.

Here the fluctuations are functions of both time and space. Such a scenario describes a situation where the system is both inhomogeneous in space and noisy in time. We will denote this case of spatiotemporal parametric fluctuations by ST.

Here we consider η to be uniformly distributed in the interval $[0, 1]$. We have simulated the system, for all the above cases, with fluctuations in coupling strength (i.e. $A \equiv \epsilon$) and in the fraction of random links (i.e. $A \equiv p$). The initial conditions of the individual elements were randomly chosen in the interval $[0, 1]$, and sufficient transients were removed before looking at the spatiotemporal profile of the network.

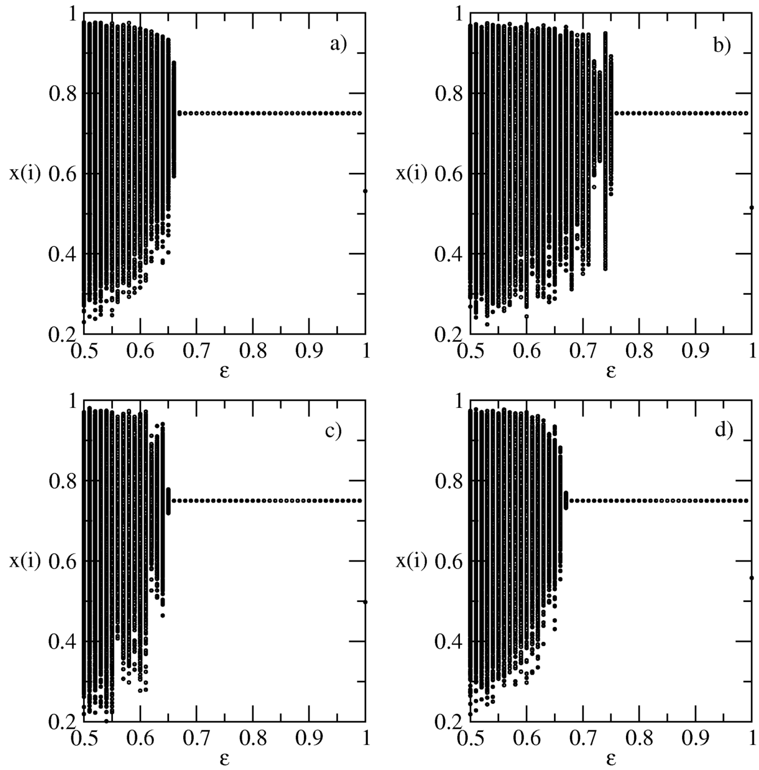

Figure 1. State of the system with respect to coupling strength ϵ, under different types of fluctuations in p: (**a**) constant $p = 0.5$, (**b**) spatial fluctuations, (**c**) temporal fluctuations and (**d**) spatiotemporal fluctuations. Here strength of fluctuations $\delta p = 0.5$ in cases (**b**)–(**d**), around a mean value of $p_0 = 0.5$.

3. Robustness of the spatiotemporal fixed point under parametric fluctuations

Figure 1 displays the bifurcation diagrams of the system under different types of fluctuations in the fraction of random links p, around a mean value of p_0. It is evident that the spatiotemporal fixed point is quite robust under parametric fluctuations in general, as the range of the spatiotemporal fixed point in coupling parameter space does not reduce much under noisy p. It can also be clearly seen that the bifurcation profile of the system under spatiotemporal fluctuations in p (figure 1d) is very similar to the system under constant $p = p_0$ (figure 1a). Further, we observe that quenched spatial fluctuations reduce the range of stability of the spatiotemporal fixed point most significantly (figure 1b). On the other hand, temporal fluctuations in p do not degrade the stability of the fixed point regime, and are most conducive to spatiotemporal regularity. In fact, interestingly, the fixed point range obtained under temporal parametric fluctuations (figure 1c) is larger than that obtained from the constant case (figure 1a). Qualitatively, similar

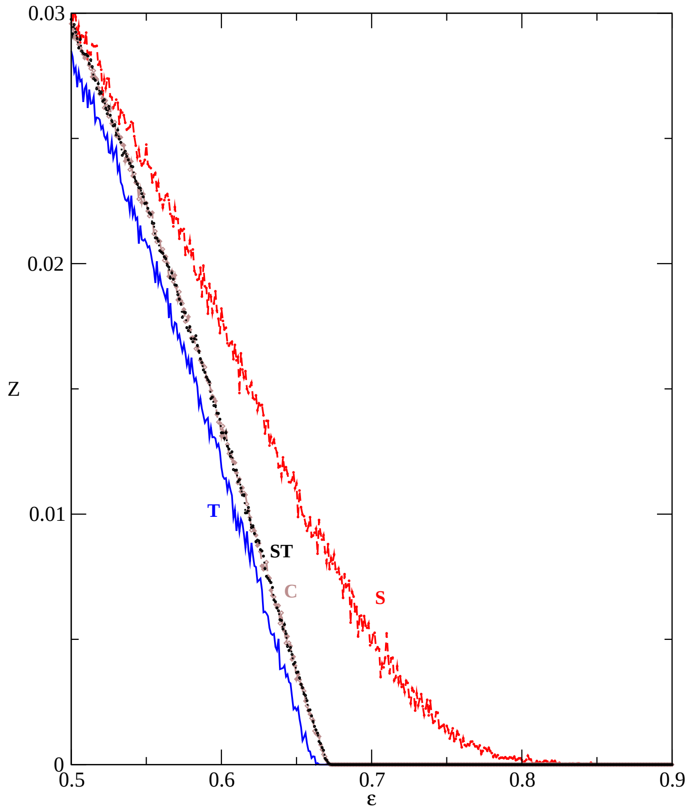

Figure 2. Average synchronization error Z, as a function of coupling strength, for four different cases: (i) p constant at the mean value $p_0 = 0.5$ denoted by C, (ii) spatial fluctuations in p denoted by S, (iii) temporal fluctuations in p denoted by T and (iv) spatiotemporal fluctuations in p denoted by ST. Here the strength of fluctuations $\delta p = 0.5$ in cases (ii)–(iv), that is, p is distributed uniformly in the range [0,1]. Observe that T gives (almost) zero error for the largest range, ST and C give similar trends, while S gives the smallest range, that is, the least robustness for the spatiotemporal fixed point.

bifurcation diagrams with respect to p were obtained under fluctuations in the coupling strength of the different links.

In order to quantify the above observation we calculate the average deviation of the system from a synchronized state, denoted by Z, and defined as

$$Z = \langle\langle (x_n(i) - \bar{x})^2 \rangle\rangle, \tag{2}$$

where $\bar{x}$ is the mean value of x. The averages $\langle\langle \cdots \rangle\rangle$ are over all sites i ($i = 1, \ldots, N$) and over long times n.

Figures 2 and 3 show error Z with respect to coupling strength and the fraction of random links p, under parametric fluctuations in p and ϵ respectively. It is clearly

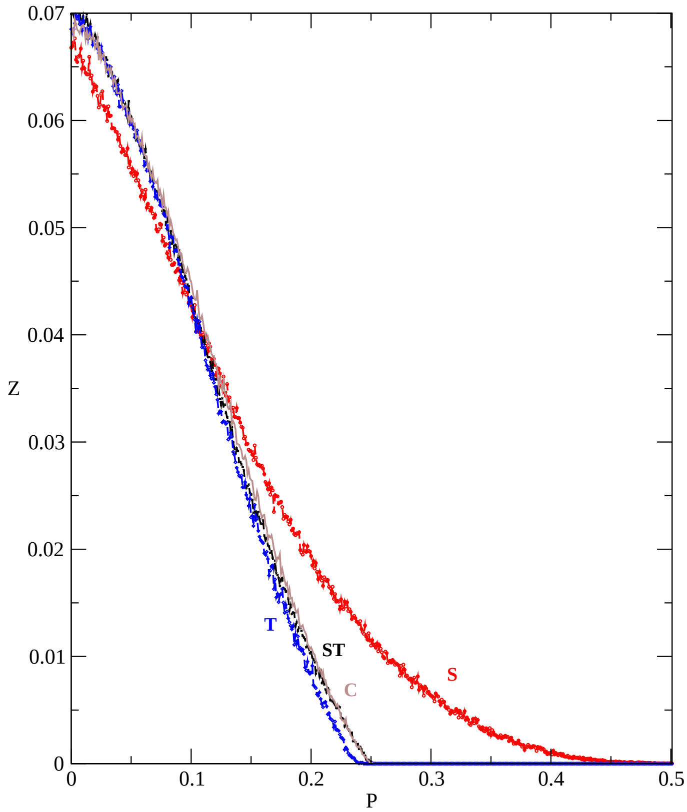

Figure 3. Average deviation of the system from the spatiotemporal fixed point, Z, as a function of the fraction of random links p, for four different cases: (i) ϵ constant at the mean value $\epsilon_0 = 0.8$ denoted by C, (ii) spatial fluctuations in ϵ denoted by S, (iii) temporal fluctuations in ϵ denoted by T and (iv) spatiotemporal fluctuations in ϵ denoted by ST. Here the strength of fluctuations $\delta\epsilon = 0.2$ in cases (ii)–(iv), namely ϵ is distributed uniformly in the range [0.6,1]. Observe that T gives (almost) zero error for the largest range, ST and C give similar trends, while S gives the smallest range, that is, the least robustness for the spatiotemporal fixed point.

evident from both figures that temporal parametric fluctuations (T) give zero error for the largest range. Spatiotemporal parametric fluctuations (ST) and the case where the parameter is kept constant at the mean value (C) give completely similar trends. As observed earlier, the spatial parametric fluctuations (S) give the smallest range, that is, the spatiotemporal fixed point is least robust under quenched disorder. Note that this enhanced regularity of the system under spatially uniform parametric noise is reminiscent of noise-induced synchronization observed in nonlinear oscillators under common stochastic forcing, in numerical and laboratory experiments [15,18].

We also calculate the critical value of coupling strength after which the spatiotemporal fixed point gains stability, denoted by ϵ_c, and the critical fraction of

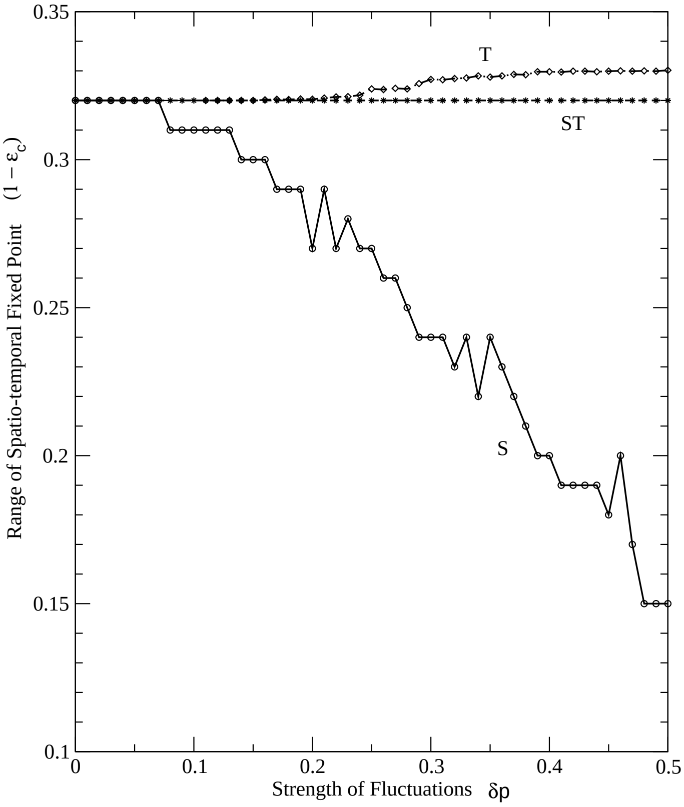

Figure 4. Range of the spatiotemporal fixed point $(1-\epsilon_c)$, where ϵ_c is the critical value of coupling strength after which the spatiotemporal fixed point gains stability, as a function of the parametric fluctuation strength δp. Note that fluctuation strength δp implies that the fraction of random links is distributed uniformly in the range $[p_0 - \delta p,\, p_0 + \delta p]$, with the mean $p_0 = 0.5$. Observe that temporal fluctuations of reasonably large strength δp actually increases the fixed point range, *vis-à-vis* the case of zero fluctuations $\delta p = 0$ (i.e. the constant p_0 case).

random links after which the spatiotemporal fixed point is stable, denoted by p_c. The fixed point range in coupling parameter space is then $(1-\epsilon_c)$, and in the space of p it is $(1-p_c)$.

Figures 4 and 5 show the range of the spatiotemporal fixed point in ϵ and p space, for varying strengths of parametric fluctuations. It is clear that the effect of spatiotemporal fluctuations is quite indistinguishable from the mean field case (namely the case of p_0 and ϵ_0, with δp and $\delta\epsilon$ equal to zero). It is also evident that quenched disorder reduces the stable range the most, while the fixed point range is the largest for spatially uniform temporal fluctuations. In fact remarkably, for temporal fluctuations, the fixed point range for large fluctuation strengths is larger than that obtained for zero fluctuations. These results re-inforce the conclusions drawn from the calculations of the error function Z.

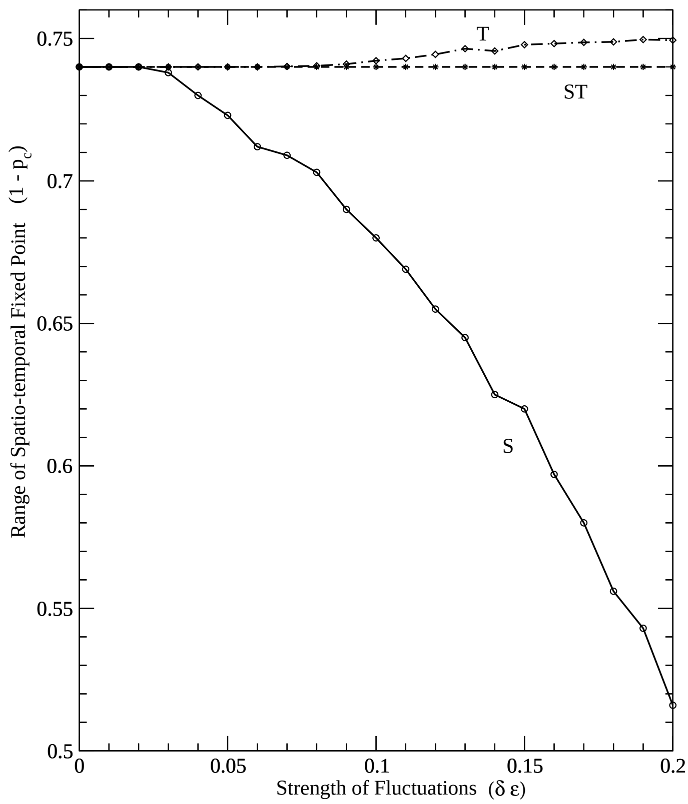

Figure 5. Range of the spatiotemporal fixed point $(1 - p_c)$, where p_c is the critical value of p after which the spatiotemporal fixed point gains stability, as a function of the parametric fluctuation strength $\delta\epsilon$. Note that fluctuation strength of $\delta\epsilon$ implies that coupling ϵ is distributed uniformly in the range $[\epsilon_0 - \delta\epsilon, \epsilon_0 + \delta\epsilon]$, with the mean $\epsilon_0 = 0.8$. Observe that temporal fluctuations of reasonably large strength $\delta\epsilon$, actually increases the fixed point range, *vis-à-vis* the case of zero fluctuations $\delta\epsilon = 0$ (i.e. the constant ϵ_0 case).

Lastly, note that similar trends are also observed when the parametric fluctuations are periodic, not random. For instance, consider the case where the fraction of random links p is distributed in a period-2 cycle: p_1, p_2. For the case of quenched spatial periodic fluctuations (denoting the value of p at site i as $p(i)$), we have $p_1(1), p_2(2), p_1(3), p_2(4), \ldots$ for all time. For temporal periodic fluctuations we have $p(i) = p_1$ for all sites i at time n, followed by $p(i) = p_2$ for all sites at time $n+1$, back to $p(i) = p_1$ for all sites at time $n+2$ etc. For spatiotemporal periodic fluctuations we have the parameters varying as a 2-cycle in space and time, namely $p_1(1), p_2(2), p_1(3), p_2(4), \ldots$ at time n followed by $p_2(1), p_1(2), p_2(3), p_1(4), \ldots$ at time $n+1$ etc. For all these cases one obtains the same qualitative behaviour as the random fluctuation cases, namely quenched spatial periodic fluctuations are the most detrimental to spatiotemporal regularity; space-time periodic fluctuations yield phenomena similar to that with constant mean-values $p_0 = (p_1 + p_2)/2$;

and spatiotemporal regularity is the most robust under spatially uniform periodic fluctuations.

4. Analysis

We shall now attempt to account for the enhanced stability of the homogeneous phase under temporal parametric fluctuations *vis-à-vis* spatial fluctuations. First we recall the stability analysis for constant parameters p and ϵ, following the treatment in [13]. Then we discuss the effects of parametric noise on the stability, by gauging the effects of fluctuating p (or ϵ) on the linear stability conditions.

Now, there exists only one solution for a spatiotemporally synchronized state, namely one with all $x_n(i) = x^*$, where $x^* = f(x^*)$ is the fixed point solution of the local map $f(x)$. For the case of the logistic map here, $x^* = 4x^*(1-x^*) = 3/4$.

To calculate the stability of the lattice with all sites at x^* we will construct an average probabilistic evolution rule for the sites, which becomes a sort of mean field version of the dynamics. So we work from the basic premise that all sites have probability p of being coupled to random sites, and probability $(1-p)$ of being wired to nearest neighbours. Then the averaged evolution equation of a site j is

$$\begin{aligned} x_{n+1}(j) &= (1-\epsilon)f(x_n(j)) + (1-p)\frac{\epsilon}{2}\left\{x_n(j+1) + x_n(j-1)\right\} \\ &\quad + p\frac{\epsilon}{2}\left\{x_n(\xi) + x_n(\eta)\right\}, \end{aligned} \tag{3}$$

where ξ and η are random integers between 1 and N.

To calculate the stability of the coherent state, we perform the usual linearization. Replacing $x_n(j) = x^* + h_n(j)$, and expanding to first order gives

$$\begin{aligned} h_{n+1}(j) &= (1-\epsilon)f'(x^*)h_n(j) + (1-p)\frac{\epsilon}{2}\left\{h_n(j+1) + h_n(j-1)\right\} \\ &\quad + p\frac{\epsilon}{2}\left\{h_n(\xi) + h_n(\eta)\right\} \\ &\approx (1-\epsilon)f'(x^*)h_n(j) + (1-p)\frac{\epsilon}{2}\left\{h_n(j+1) + h_n(j-1)\right\}. \end{aligned} \tag{4}$$

Above, to a first approximation, one considers the sum over the fluctuations of the random neighbours to be completely uncorrelated, thus summing up to zero in the time-averaged picture in eq. (4).

Effectively, this approach is equivalent to considering the interactive term to have a local part weighted by $(1-p)$ and a 'global' (or averaged) part weighted by p arising from the random connections [19], namely,

$$x_{n+1}(j) = (1-\epsilon)f(x_n(j)) + \frac{\epsilon}{2}(1-p)\left\{x_n(j+1) + x_n(j-1)\right\} + \frac{\epsilon}{2}p\langle x\rangle. \tag{5}$$

Linear stability considerations of the above formulation, in the limit of $N \to \infty$, also give rise to eq. (4).

For stability considerations one can diagonalize the expression in eq. (4) using a Fourier transform ($h_n(j) = \sum_q \phi_n(q) \exp(ijq)$, where q is the wave number and j is the site index), which finally leads us to the following growth equation:

$$\frac{\phi_{n+1}}{\phi_n} = f'(x^*)(1-\epsilon) + \epsilon(1-p)\cos q \tag{6}$$

with q going from 0 to π. Considering the fully chaotic logistic map with $f'(x^*) = -2$, one finds that the growth coefficient that appears in this formula is smaller than one in magnitude if and only if [13]

$$\frac{1}{1+p} < \epsilon < 1, \tag{7}$$

i.e. $\epsilon_{\mathrm{c}} = 1/(1+p)$. So the range of stability $\mathcal{R}$ is

$$\mathcal{R} = 1 - \frac{1}{1+p} = \frac{p}{1+p}. \tag{8}$$

The above expression is in very good agreement with numerical results.

Now, when the quantity p is spatially uniform, but fluctuates in time, the above picture remains quite valid. Note that at any instant of time n, the value of p_n is the same for all sites. So the symmetry of the Jacobian is preserved, and it is clear that the time-averaged evolution equation is

$$h_{n+1}(j) = (1-\epsilon)f'(x^*)h_n(j) + (1-\langle p\rangle)\frac{\epsilon}{2}\left\{h_n(j+1) + h_n(j-1)\right\} \tag{9}$$

which is the same as eq. (4), as $\langle p\rangle = p$. So the basic analysis in the case of spatially uniform fluctuating p remains the same as that for eq. (4).

This analysis also helps us to understand why the behaviour of a system periodically cycling over two values p_1, p_2, is similar to that of a system with p equal to the average $\langle p\rangle = (p_1 + p_2)/2$.

The same reasoning holds for fluctuations in coupling strength, in which case one has

$$h_{n+1}(j) = (1-\langle\epsilon\rangle)f'(x^*)h_n(j) + (1-p)\frac{\langle\epsilon\rangle}{2}\left\{h_n(j+1) + h_n(j-1)\right\}, \tag{10}$$

where $\langle\epsilon\rangle = \epsilon$.

The effect of spatiotemporal fluctuations is likely to be described reasonably by the above analysis as well, since temporal averaging will again yield $\langle p\rangle \sim p$ in the effective time-averaged dynamical equations above. However, while the space-uniform temporal fluctuations strictly maintain the symmetry of the dynamical equations of the sites, for spatiotemporal fluctuations the symmetry of the instantaneous dynamical equations of the sites, is broken, though on an average the equations are identical to that of temporal fluctuations. So one expects the stability of the system under spatiotemporal fluctuations to be somewhat diminished compared to the spatially uniform case, for strong fluctuations.

Now we consider the case of fluctuations that are frozen in time but distributed in space. Here the symmetry of the sites is broken, even on an average, namely the individual sites no longer have the same time-averaged probabilistic equation. In general terms, it may be argued, that spatial disorder does not average out in time, and so the time-averaged description of the system in eq. (4) does not capture the stability accurately.

We attempt to understand this case as follows: one can consider, in the limit of large N, the effective dynamics to be governed by the analysis with $\langle p \rangle = p$ given above, plus a random matrix term in the Jacobian, weighted by $\epsilon/2$. The largest eigenvalue of the random matrix part scales as $NC\sigma^2$, where C is the connectivity of the random matrix. Since the random parametric fluctuations give rise to two non-zero entries per row, C of the random matrix arising from spatially fluctuating p is $\sim 2N/N^2 \sim 2/N$. So the largest eigenvalue of the system with quenched disorder will increase by an amount that scales approximately as $\epsilon\sigma^2$ where σ^2 is the variance of the spatial fluctuations about the mean $\langle p \rangle = p$. So, unlike the case of temporal fluctuations, the stable range will decrease under spatial inhomogeneity.

5. Conclusions

In summary, it was observed that lattices of coupled chaotic maps, with coupling connections dynamically rewired to random sites with probability $p > 0$, gave rise to a window of spatiotemporal fixed points in coupling parameter space. Here we investigate the effects of different kinds of parametric fluctuations on the robustness of this spatiotemporal fixed point regime. In particular we study the spatiotemporal dynamics of the network with fluctuating rewiring probabilities and coupling strengths, with the fluctuations being (i) noisy in time, homogeneous in space, as applicable for intrinsically homogeneous systems under common environmental noise; (ii) noisy in space, and fixed in time, namely quenched disorder and (iii) noisy in both space and time.

We find that static spatial inhomogeneity, namely quenched disorder, degrades spatiotemporal regularity most significantly. Spatiotemporal fluctuations yield dynamical properties almost identical to networks with the parameters held constant at the mean values. Interestingly, spatiotemporal regularity is most robust under spatially uniform temporal fluctuations. Such space-invariant temporal parametric noise actually yields a regular range that is larger than that obtained for systems with the parameters held constant at mean value.

So the effect of different kinds of parametric noise on spatiotemporal regularity is quite distinct: quenched spatial fluctuations are the most detrimental to spatiotemporal regularity; spatiotemporal fluctuations yield phenomena similar to that with constant mean values; and spatiotemporal regularity is most robust under spatially uniform temporal fluctuations.

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