

Non-universal dependence of spatiotemporal regularity on randomness in coupling connections

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Abstract

We investigate the spatiotemporal dynamics of a network of coupled nonlinear oscillators, modeled by sine circle maps, with varying degrees of randomness in coupling connections. We show that the change in the basin of attraction of the spatiotemporal fixed point due to varying fraction of random links p , is crucially related to the nature of the local dynamics. Even the *qualitative* dependence of spatiotemporal regularity on p changes drastically as the angular frequency of the oscillators change, ranging from monotonic increase or monotonic decrease, to non-monotonic variation. Thus it is evident here that the influence of random coupling connections on spatiotemporal order is highly non-universal, and depends very strongly on the nodal dynamics.

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Keywords:

I. INTRODUCTION

The dynamics of spatially extended systems has been a focus of intense research activity in the past two decades. In recent years it has become evident that modeling large interactive systems by finite dimensional lattices on one hand and fully random networks on the other, is inadequate, as various networks, ranging from collaborations of scientists to metabolic networks, do not fit in either paradigm [1]. Some alternate scenarios have been suggested, and one of the most popular ones is the small-world network [2]. Here one starts with a structure on a lattice, for instance regular nearest neighbor connections. Then each link from a site to its nearest neighbor is rewired randomly with probability p , *i.e.* the site is connected to another randomly chosen lattice site. This model is proposed to mimic real life situations in which non-local connections exist along with predominantly local connections.

There is much evidence that random non-local connections, even in a small fraction, significantly affects geometrical properties, like characteristic path length [2]. However its *implications for dynamical characteristics is still unclear and even conflicting*. While the dynamics of coupled oscillators and coupled maps on regular lattices (known as ‘coupled map lattices’ or CMLs) has been extensively investigated [3], there have been far fewer studies on the spatiotemporal dynamics of nonlinear elements on networks of different topologies [4]. Most studies so far have indicated that the regularity of systems increase monotonically with p [5].

In this paper we will provide evidence of a system where the dependence of spatiotemporal regularity on the degree of randomness in coupling connections is highly *non-universal*. We will show how this dependence ranges from *monotonically increasing* to *monotonically decreasing*, via *non-monotonic variation*, as the local dynamics changes. Thus we will demonstrate that the interplay between local dynamics and connectivity acts in non-trivial and non-intuitive ways, and so even the qualitative effect of random links on spatiotemporal regularity can be completely reversed by changing the nodal dynamics.

II. THE MODEL

Here we consider nonlinear oscillators coupled to nearest neighbors on a regular ring, with some fraction p of the regular links rewired randomly. The individual sites (nodes) are

modeled by sine circle maps, which have widespread relevance for oscillatory phenomena [6], and are given as:

$$f(x) = x + \Omega - \frac{K}{2\pi} \sin(2\pi x)$$

wherein K measures the strength of the nonlinear term, and Ω represents the natural frequency of the map in the absence of nonlinearity (i.e. when $K = 0$). We restrict our studies to the parameter region: $0 \leq \Omega \leq \frac{1}{2\pi}$ and $K = 1$. In this region, the single sine circle map settles down to the spatiotemporal fixed point, $x^* = \frac{1}{2\pi} \sin^{-1}(\frac{2\pi\Omega}{K})$.

Under diffusive coupling, such a coupled sine-circle map lattice, is given as:

$$x_{n+1}(i) = (1 - \epsilon)f(x_n(i)) + \frac{\epsilon}{2}[f(x_n(i-1)) + f(x_n(i+1))] \pmod{1} \quad (1)$$

where $i = 1, \dots, L$ denotes the site index, n denotes the time index, and ϵ represents the coupling strength between the sites ($0 \leq \epsilon \leq 1$). Periodic boundary conditions have been used, namely one has a ring of oscillators.

Earlier studies of this coupled map lattice have been carried out for various types of initial conditions [7]. In particular, the evolution of this coupled map lattice with random initial conditions shows interesting spatiotemporal dynamics including spatiotemporal fixed points, spatial and spatiotemporal intermittency, and spatiotemporal chaos [8].

Now we introduce randomness in this regular lattice by rewiring the nearest neighbor links with a probability p to randomly chosen sites on the lattice. Here we consider the behavior of the system under *static rewiring*, namely, the randomness in spatial coupling is quenched or “frozen” in time. Ensembles of such randomly rewired systems are studied.

III. RESULTS

A. Basin of attraction for the spatiotemporal fixed point

We study the spatiotemporal dynamics of this system, starting from random initial conditions under varying rewiring probabilities p ($0 \leq p \leq 1$), for different realizations of static rewiring.

Specifically we obtain the basin of attraction for the spatiotemporal fixed point, B , by calculating the fraction of rewiring configurations which lead to a spatiotemporally steady state for different (Ω, ϵ) values. Figure 1 shows a gray scale plot of B in a large region of

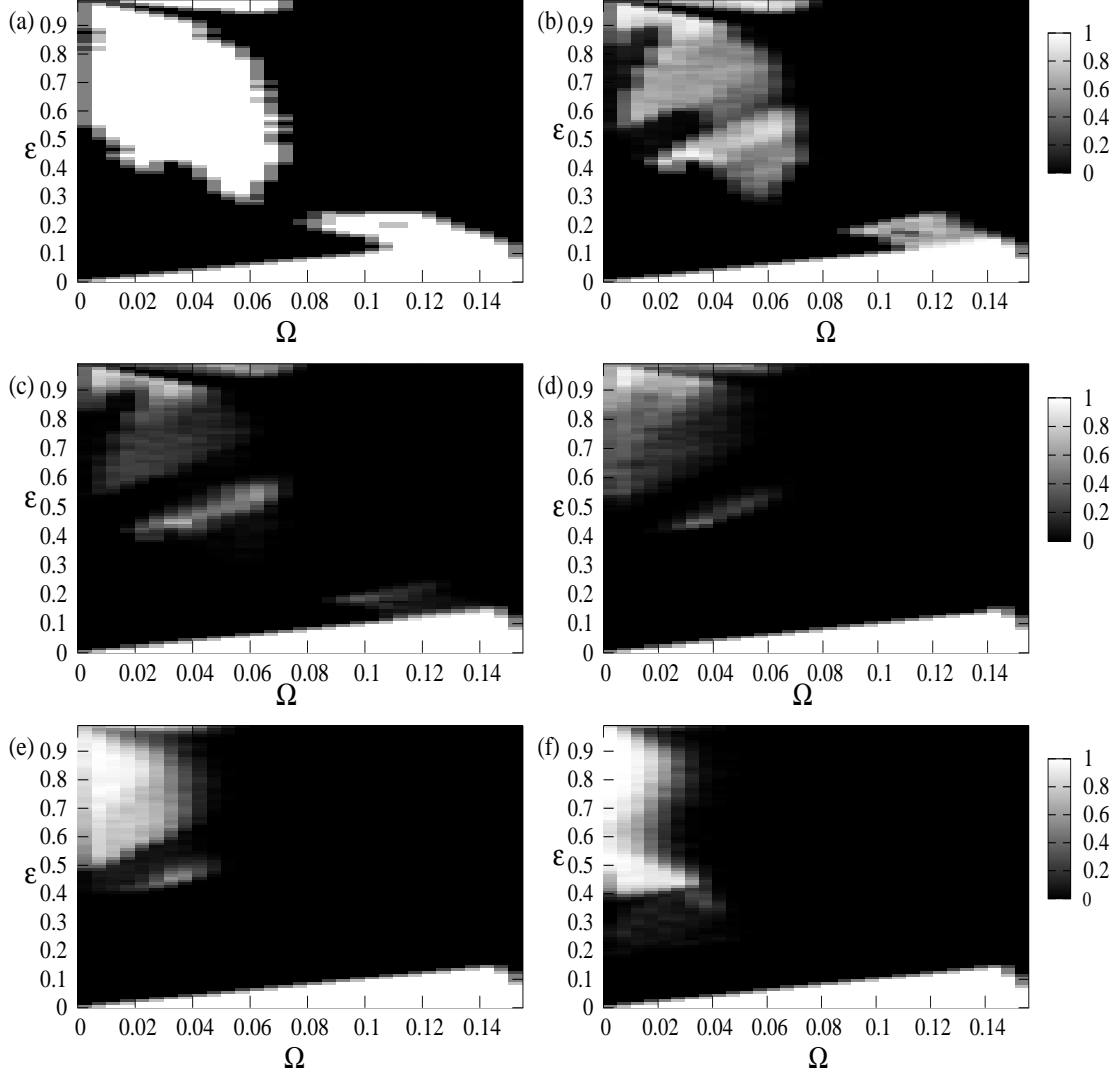


FIG. 1: Basin of attraction of the spatiotemporal fixed point, x^* , obtained for different random rewiring probabilities: (a) $p = 0.0$, (b) $p = 0.04$ (c) $p = 0.1$, (d) $p = 0.3$, (e) $p = 0.5$, and (f) $p = 0.8$. These plots have been obtained for 50 rewiring configurations for a lattice of size $L = 200$ after discarding 5000 transients.

parameter space, for different values of p . The white areas indicate the parameter regions, where all initial coupling configurations lead to a spatiotemporal fixed point, namely all the sites in the system relax to the fixed point x^* such that $x(i) = x^*$ for all $i = 1 \dots N$ and for all time n . The black regions indicate parameter regions where none of the coupling configurations yield a spatiotemporal fixed point. The gray areas indicate parameter regimes where $0 < B < 1$, i.e. the spatiotemporal fixed point co-exists with other dynamical

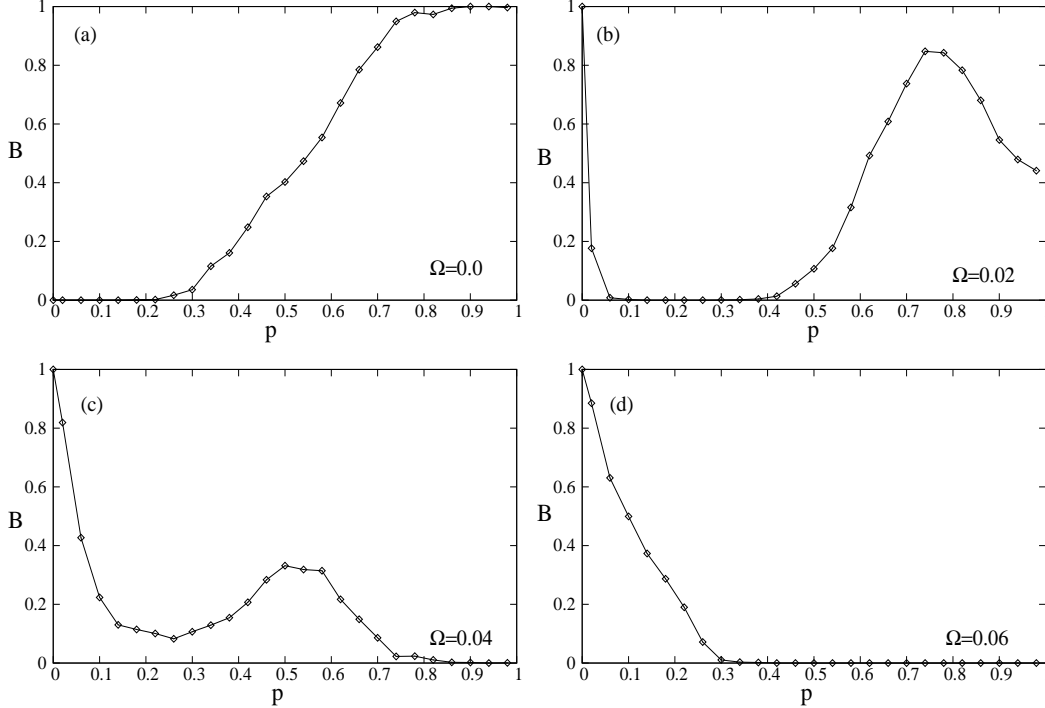


FIG. 2: Variation of basin of attraction B with the rewiring fraction, p plotted for the circle map frequencies $\Omega = 0, 0.02, 0.04, 0.06$ and coupling strength $\epsilon = 0.5$.

behaviour, and the spatiotemporal fixed point is not an attractor of the dynamics for all coupling configurations.

Figure 1(a) shows the basin of attraction when the rewiring fraction is equal to zero, or in other words, the ring has only regular nearest neighbor connections. As the rewiring fraction p is varied, the spatiotemporal fixed point regions also show a change. Figure 1(b)-(f) display the basin of attraction of the spatiotemporal fixed point, for $p = 0.04, 0.1, 0.3, 0.5$ and $p = 0.8$. We see that the dependence of this basin of attraction on the degree of random rewiring is qualitatively very different for different values of Ω and ϵ . Interestingly this variation ranges from monotonic increase to monotonic decrease, as well as non-monotonic behavior, along different “cuts” in (Ω, ϵ) space.

Further, the dependence of spatiotemporal order on rewiring probability, averaged over a large parameter range, varies non-monotonically with p . This is clear from the fact that the extent of the spatiotemporal fixed point basin at intermediate p ($p \sim 0.1 - 0.3$) in Figs.1(c-d) is much smaller than that for low and high p . So the gray-scale basin plots in Figs.1 (c-d) appear far less “white” in general, across large parameter regimes, as compared to Figs.1

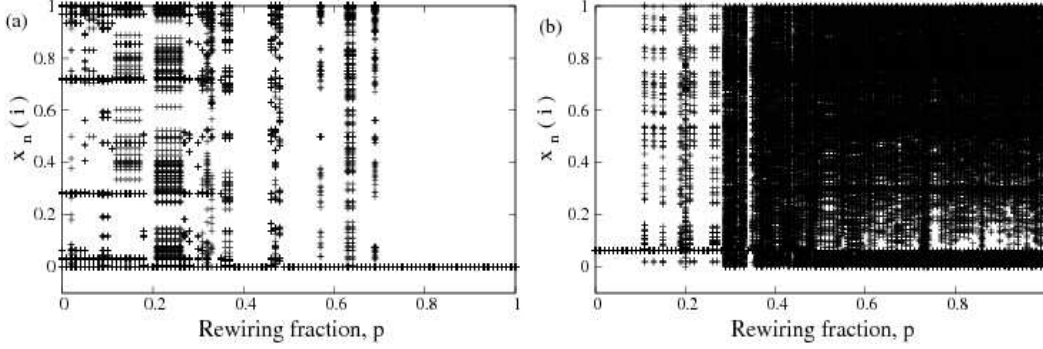


FIG. 3: Bifurcation diagram in which the state variables $x_n(i)$ ($i = 1, \dots, 100$) have been plotted as a function of the fraction of random links, p , for a representative coupling configuration, for the parameter values (a) $\Omega = 0.0$ and (b) $\Omega = 0.06$ at $\epsilon = 0.5$. Here, $n = 1, \dots, 5$ iterations have been plotted after discarding 5000 transients. Note that for regions where $0 < B < 1$, there are rewiring configurations that lead to the spatiotemporal fixed point, co-existing with rewiring configurations that yield spatiotemporal chaos.

(a) and (f).

Figure 2 shows the variation of the basin of attraction B with rewiring fraction p at $\epsilon = 0.5$ and for $\Omega = 0.0, 0.02, 0.04$, and for 0.06 . These plots have been obtained for 50 rewiring configurations for a lattice of size $L = 200$ after discarding 10000 transients. When the natural frequency of the circle map is equal to zero, ($\Omega = 0$), the ring does not yield a spatiotemporal fixed point when coupling connections are completely regular. However, the regularity of the system increases as the rewiring fraction p is increased. This can be seen in Figure 2(a), in which a global spatiotemporal fixed point attractor is obtained for values of the rewiring fraction $p > 0.6$.

The bifurcation diagram in Figure 3(a), showing the spatiotemporal dynamics of the system with respect to the fraction of random links p , further underscores this feature. Here the system has a complex spatial pattern for lower values of random rewiring probability. However, it settles down to the spatiotemporal fixed point ($x^* = 0$, in this case) as the fraction of random links p approaches 1.

In contrast, the variation of the basin of attraction, B , in the case where the frequency Ω is equal to 0.02 is shown in Figure 2(b). Here, we see that the system yields a spatiotemporal fixed point with probability 1 for zero rewiring fraction p , but shows a non-monotonic

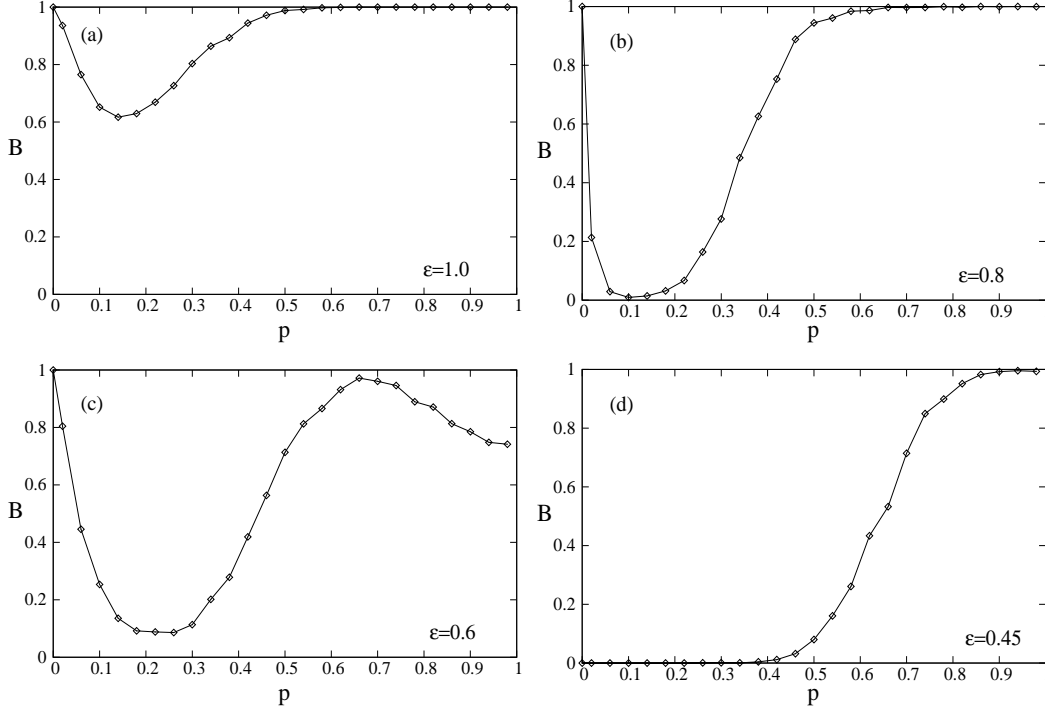


FIG. 4: Variation of basin of attraction, B with the rewiring fraction, p for the circle map frequency, $\Omega = 0.01$ and coupling strengths, $\epsilon = 1.0, 0.8, 0.6$, and 0.45 .

variation as the rewiring fraction p is changed. We see that though the basin of attraction decreases to zero in the interval $p \sim 0.1 - 0.4$, it gradually increases for rewiring fractions, $p > 0.4$, until it again registers a decrease in the large p limit [9]. Hence, the basin of attraction B shows a *non-monotonic* variation with change in p . A similar non-monotonic variation is seen in Figure 2(c), where $\Omega = 0.04$.

In the case of $\Omega = 0.06$, as displayed in Figure 2(d), the basin of attraction *decreases* to zero as the rewiring fraction is increased. In this case, the system settles to the spatiotemporal fixed point x^* for smaller rewiring fractions, but exhibits a complex spatial pattern when the degree of rewiring in the system is increased. This is further illustrated in the bifurcation diagram of the system shown in Figure 3(b). This is exactly the opposite trend to that observed in the case of $\Omega = 0$. So, as the local frequency of the nonlinear oscillator changes, the effect of random rewiring on spatiotemporal properties is completely reversed.

Hence, we see that for the same coupling strength ϵ , and for the same set of rewired configurations, the basin of attraction of the spatiotemporal fixed point shows a very strong dependence on the local dynamics, namely on the frequency Ω of the nonlinear oscillators.

So it is evident that the spatiotemporal regularity depends crucially, not just quantitatively, but also qualitatively, on the nodal dynamics.

Similarly, when the nodal dynamics is fixed and the coupling strength ϵ is varied, we see that the basin of attraction shows a non-monotonic variation with change in rewiring fraction p . This is illustrated in Figure 4 where the basin of attraction has been plotted for $\Omega = 0.01$ and for various representative values of the coupling strength ϵ .

Hence, spatiotemporal regularity of the dynamics on a network depends quite crucially on the interplay between the nodal dynamics and the network topology. That is, coupling configurations with the same degree of randomness may enhance or inhibit spatiotemporal order depending on the properties of the local oscillators.

IV. CONCLUSIONS

In summary, we have investigated the spatiotemporal dynamics of a network of coupled nonlinear oscillators, modeled by sine circle maps, with varying degrees of randomness in coupling connections. We showed that the variation of the basin of attraction of the spatiotemporal fixed point, with increasing fraction of random links p , crucially depends on the nature of the local dynamics. Even the qualitative relationship between spatiotemporal regularity and p changes drastically as the angular frequency of the oscillators change, ranging from monotonic increase or decrease, to non-monotonic variation. Thus it is evident that the influence of random coupling connections on spatiotemporal order is highly non-universal here and depends strongly on the angular frequency of the nodal oscillators. This implies that the delicate interplay between local dynamics and connectivity is crucial in determining the emergence of spatiotemporal order in complex networks of dynamical elements.

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- [9] Note that the average time taken by the system to relax to the spatiotemporal fixed point in the large p regime ($p \sim 0.8 - 1.0$), grows exponentially with p , and the configurations eventually relax to the spatiotemporal fixed point with probability 1, but only at asymptotically long times. This is in contrast to the region of rewiring fractions $p \sim 0.1 - 0.4$, where most initial states do not appear to relax to the spatiotemporal fixed point state, even after very long transience.