How Crucial is Small World Connectivity for Dynamics?

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Abstract

We study the dynamical behaviour of the collective field of chaotic systems on small world lattices. Coupled neuronal systems as well as coupled logistic maps are investigated. We observe that significant changes in dynamical properties occur only at a reasonably high strength of nonlocal coupling. Further, spectral features, such as signal-to-noise ratio (SNR), change monotonically with respect to the fraction of random rewiring, i.e. there is no optimal value of the rewiring fraction for which spectral properties are most pronounced. We also observe that for small rewiring, results are similar to those obtained by adding small noise.

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1   Introduction

The inadequacy of treating coupled systems as finite dimensional lattices on one hand and fully random networks on the other, has become evident in recent times. Various networks, ranging from collaborations of scientists to metabolic networks, have been studied and shown not to fit in either paradigm. Some alternatives have been suggested, the most popular of which are small-world networks [1] and scale free networks [2]. In the small world model, one starts with a structure on a lattice, for instance regular nearest neighbour connections. Then each link from a site to its nearest neighbor is rewired randomly with probability p, *i.e.* the site is connected to another randomly chosen lattice site. This model is proposed to mimic real life situations in which non-local connections exist along with predominantly local connections. The geometrical properties of these lattices have been extensively studied. Many studies have observed the following [3]: starting from a one dimensional chain at $p = 0$, one obtains long-range order at any finite rewiring probability with same critical exponents as in the mean-field case, namely in the thermodynamic limit the behavior for any $p \neq 0$ is the same as the behavior for $p = 1$ for these models. Newman and Moore recover critical exponents for percolation on small world

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lattices which are the same as for the Bethe lattice, i.e. an infinite dimensional case [4]. For the XY model, Medvedyeva *et al* conjecture that the critical exponents are the same as for the mean field case[5]. They have confirmed it for $p \geq 0.03$ and there is good reason to believe that it is true for any $p > 0$ (The obvious difficulty is that one needs to simulate larger and larger lattices at small p.) Similar conclusions are reached for the Ising model on small world networks as well [6].

So while there is much evidence that random nonlocal connections, even in a small fraction, makes a big difference to geometrical properties like characteristic path length, its implications for dynamics is still unclear and even conflicting. So the first question we will probe here is this: *does one see dynamical changes at very low values of p*, namely does the behavior change as soon as non-local shortcuts are put in place (as observed in equilibrium models).

Now, while the dynamics of coupled oscillators and coupled maps on regular lattices has been extensively investigated, there have been only a few studies on the dynamics on small world lattices. Most of these have focussed on exact synchronization. For instance, in Coupled Map Lattices (CML), with add-on non-local links, it was observed that synchronization occurs in the thermodynamic limit for infinitesimal p [7]. Likewise Barahona *et al* [8] investigated coupled oscillators with small world connections with add-on links, and observed that in cases where synchronization occurs, the fraction p of nonlocal edges required to synchronize the system decreases monotonically with lattice size and is very small for large lattices. However, for CMLs where non-local links were added at the cost of existing regular links, transition to exact synchronization was observed at finite p [9, 10]. This finite p transition was similar to the transition to self-sustained oscillations evident in a model of infection spreading [11]. It was also shown in these CMLs that the magnitude of lyapunov exponents and the coupling parameter range over which exact synchronization occurs, varies monotonically with p [9]. Further, a study on coupled Chate-Manneville minimal maps [12] indicates that the critical exponents for the transition to turbulence change monotonically as a function of rewiring probability p. Here one can find situations in which the critical exponent drops to zero at some value of p, thus changing the nature of the transition from second order to first order, though the change is still monotonic. Another investigation on stochastic resonance on small world networks falls on similar lines [13]. So all these studies indicate that dynamical features vary monotonically with p, and interpolate between the limits of regular and random connections without in any sense being "optimal" or more pronounced at some intermediate value of p [14].

Surprisingly however, a few studies indicate a non-monotonic change in dynamics and special features at small values of p. He *et al* reported that the spirals which were unstable on regular networks get stabilized at very small p [15]. However, it is known that small spatial noise stabilizes spirals and small world connectivity could be playing the same role[16] since the nonlocal connections will join sites well inside one phase with the other phase and vice versa.

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Another interesting study by Lago-Fernandez *et al* demonstrated that the power spectra of the collective field of coupled Hodgin-Huxley elements shows a non-monotonic increase in strength of spectral peaks at low frequency. Thus they argued that small world connectivity gives something special which is amiss in regular or random lattices [17]. Though this trend held true for their particular model system, it is not clear if small world connectivities will have similar consequences in general. Numerical evidence from more varied sources is required, in the absence of analytical results, to settle this question.

In view of the above, the second interesting question we wish to address through our case studies here is as follows: is there evidence for dynamical features at some intermediate value of p, that, in some sense, does not interpolate between the random and regular limits. Thus this paper will attempt to provide some more examples from coupled dynamical systems, including another prototypical neuronal model, in order to *shed more light on the validity of the conjecture that small world connections yield special dynamical features that are absent in both the regular and the random limits*.

Note that the change in characteristic length scales under small world connectivities would make a significant change in the characteristic time scales in geometric models, such as those describing epidemics or rumor propagation. In these models the initial disturbance is localized and the time taken for spreading is reduced considerably in presence of nonlocal connections. In coupled chaotic systems, however, the characteristic time scale will be related to largest lyapunov exponent which does not change drastically with nonlocal connections. Also note that in these spatiotemporally chaotic systems, the disturbances are neither localized nor few. This is another important distinction from epidemic models.

Our test systems are the following two networks: (i) coupled Hindmarsh-Rose neurons and (ii) coupled logistic maps. We will study both systems in the parameter regime that shows chaos. Note that the constituents of the networks in the two case studies are very different. One of them is an excitable system, while other becomes chaotic via familiar period-doubling route to chaos.

In particular, we will focus on the spectral properties of the collective field under varying degrees of random rewiring in the two networks. The mean field indicates the degree of independence of different elements in the network. If the elements are uncorrelated and individually chaotic, one would expect the mean field to approach a constant, namely the average value of the components, with the fluctuations decaying with system size. On the other hand if the elements are very coherent, we may see strong departures from this behaviour. Thus we may look at signal-to-noise ratio (SNR) of the peaks in the spectrum of the collective field as some kind of *order parameter* demonstrating the coherent oscillations in the mean field, if any. It is zero when there are no coherent oscillations in the mean field while it has a finite value when it oscillates with some chosen frequency.

As mentioned above, we will explore two questions in this work. First,

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we will examine if dilute rewirings have any significant impact on the spectral properties. Secondly we will try to discern whether or not any non-monotonic changes result in dynamical properties as rewiring fraction p is varied, namely we will address the question: does there exist an optimal p for which certain dynamical features become significantly more pronounced than in either the regular or random limits.

The organization of the paper is as follows. In Section 2 we report on our first test case, namely a network of coupled Hindmarsh-Rose neurons. In section 3, we report our results for a network of coupled logistic maps in the chaotic regime. We summarize our results in Section 4.

2   Network of Coupled Hindmarsh-Rose neurons

In light of the observation that the low frequency spectral peak of the collective field of a network of Hodgkin-Huxley neurons is more pronounced for low p than for either the regular or the random case [17], we have chosen our first case study to be another prototypical neuronal model: the Hindmarsh-Rose model. On lines of Lago-Fernandez *et al* [17], let us cosider a lattice made up of non-identical Hindmarsh-Rose neurons [19]. Let each of them be in chaotic regime and be coupled to its neighbours This system is defined as

$$\frac{dx_{ij}}{dt} = y_{ij} + 3x_{ij}^2 - x_{ij}^3 - z_{ij} + I_{ij} - \epsilon\Delta \tag{1}$$

$$\frac{dy_{ij}}{dt} = 1 - 5x_{ij}^2 - y_{ij} \tag{2}$$

$$\frac{dz_{ij}}{dt} = -rz_{ij} + rS(x_{ij} + 1.6) \tag{3}$$

where Δ is a proportional to laplacian of x variable calculated at site (i, j). For a regular square lattice, $\Delta = 4 * x_{ij} - x_{i1,j} - x_{i2,j} - x_{i,j1} - x_{i,j2}$ where $j1 = j - 1$, $j2 = j + 1$, $i1 = i + 1$, $i2 = i - 1$ (with periodic boundary conditions). However for a small world lattice $i1, i2, j1, j2$ could take random values with probability p.

The parameters in the above equations take values: $r = 0.0021, \epsilon = 0.1$ and $S = 4$. The term $I_{ij} = 3.281 \pm 0.025\eta$ where the η is a random number between 0 and 1.

The collective (mean) field in the above system of coupled neurons can be defined as

$$h = \frac{1}{N}\sum_{i,j} x_{ij}$$

We study this quantity for different system sizes N and different rewiring fractions p.

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Fig. 1 displays the time evolution of the collective field for four values of rewiring fraction p. It seems apparent that there is a monotonic change in the qualitative nature of the dynamics. The trend is as follows: as p increases, i.e. as the network becomes more and more random, the oscillations of the mean field get larger in amplitude and follow the pattern of the individual neurons more closely. This indicates increasing synchronicity as $p \to 1$, though exact synchronisation is never achieved here.

The spectra of the collective field reveal the following trends:

(a) For a regular lattice $(p \sim 0)$ the peak in the power spectrum of the collective field occurs only in the low frequency region.

(b) For fully nonlocal connections $(p \to 1)$ the power spectrum of the mean field shows peaks both in the low as well as the high frequency regimes.

(c) As one changes the rewiring probability p we see more and more peaks in high frequency regime. But the *low frequency peak is neither destroyed nor decreased in strength.* In fact, it increases faster than background noise giving a slightly higher signal to noise ratio.

(d) The background level of the power spectrum grows as p is increased. This is keeping in with expectation, since for higher p the sites will be more correlated.

In Fig. 2 we illustrate the above points by showing the power spectra for the representative cases of $p = 0$, $p = 0.1$ and $p = 0.999$.

In light of the remarks made in introduction, we would like to compare the system with rewired nonlocal couplings with a system with regular couplings influenced by noise. So we simulate the above dynamics on a regular lattice under the influence of noise, i.e. we evolve the dynamics above with an additive noise term $\sigma\eta$, where η is a random number between -1 and 1.

The spectra under different noise strengths σ shows the following trends:

(a) Upto noise strengths $\sigma < 0.001$ the spectral peak remains the same, and arguably even increases a little. So the effect of very small noise is akin to low rewiring fractions $(p \to 0)$.

(b) For larger noise strengths $\sigma > 0.001$, the spectral peak decreases significantly. So here higher noise strengths do not have the same effect as higher rewiring fractions $(p \to 1)$.

(c) As noise increases, the $P(f) \sim 1/f^2$ background gets cleaner and more pronounced.

Fig. 3 illustrates these trends in a lattice of size $N = 151 \times 151$.

We have also studied the system with the connections dynamically rewired, i.e. at very small intervals the connectivity matrix is updated keeping the probability of rewiring fixed at p. Dynamic rewiring yields results qualitatively similar to static rewiring for small p. At larger p however, dynamic rewiring is unlike static rewiring. In fact it's effects are rather similar to that of noise at larger σ. So in dynamically rewired networks, as p increases, the rough oscillatory behaviour of the collective field is rapidly lost and the spectra is dominated by the $1/f^2$ background (see Fig. 4).

Various other questions could be asked about dynamics on networks, like for instance the nature of dynamic phase transitions. Such questions would involve concerns about the relevant order parameter(s), thermodynamic limit in space and asymptotic limit in time and delicate issues in approaching these limits[20]. But here we will not concern ourselves with the asymptotic or thermodynamic limit, and we will draw no conclusions about infinite networks from our studies of finite (albeit quite large) lattices. Indeed from a practical point of view, whether it is a collection of neurons or electronic oscillators, our observations will still be very relevant.

3 Network of coupled logistic maps

We have also studied coupled chaotic logistic maps on a 2-dimensional small world network. This system is given as

$$x_{n+1}(i,j) = (1-\epsilon)f(x_n(i,j)) + \frac{\epsilon}{4}\{f(x_n(i1,j)) + f(x_n(i2,j)) \tag{4}$$
$$+ f(x_n(i,j1)) + f(x_n(i,j2))\}$$

Again $i1, i2, j1, j2$ takes the nearest neighbour values: $i+1$, $i-1$, $j+1$ and $j-1$ with probability $1-p$, and could take random values with probability p. The network has periodic boundary conditions. We carried out simulations for function $f(x) = 1 - rx^2$ for $r = 1.82$, $r = 1.9$ and $r = 2$.

We obtained the mean field, again defined as $\frac{1}{N}\sum_{i,j} x(i,j)$, for the above system, with $N = 201 \times 201$. We have the following observations, for both static and dynamic rewiring:

(a) For $r = 2$, the spectum does not have sharp peaks and spectral features do not change much with rewiring of bonds.

(b) For $r = 1.82$ and $r = 1.9$, there are δ peaks for regular lattices, and as one increases the rewiring probability p the spectral peaks reduce in strength and ultimately vanish.

(c) The background level grows as p is increased, as expected.

(d) The SNR displays monotonic decrease with respect to rewiring probability p.

In the left panel of Fig. 5 we display the power spectrum of the above system with $r = 1.82$ in the local maps, for $p = 0$, $p = 0.1$ and $p = 0.4$. The figure clearly bears out the observations listed above. If the spectral peaks are not very pronounced to begin with at $p = 0$, even very small rewiring probability p will destroy it. However, if the spectral peaks are sharp at $p = 0$, one has to go up to larger values of p to see them vanish.

In Fig. 6 we display the signal-to-noise ratio (SNR) of the spectra at different values of rewiring fraction p, for the case of $r = 1.82$. The figure illustrates that

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the SNR is a monotonic function of p. So evidently no significant dynamical effect is discernable as $p \to 0$. Rather the SNR shows a gradual decrease with increasing p.

We have also simulated the dynamics of a coupled logistic map network on a regular lattice under varying noise strengths, namely:

$$x_{n+1}(i,j) = (1 - \epsilon)f(x_n(i,j)) + \frac{\epsilon}{4}\{f(x_n(i+1,j)) + f(x_n(i-1,j)) \qquad (5)$$
$$+ f(x_n(i,j+1)) + f(x_n(i,j-1)\} + \sigma\xi$$

where ξ is a random number in the range $[-1 : 1]$ and σ is the noise strength, and interestingly we observed a very similar diminishing of spectral peaks with increasing σ.

In the right panel of Fig. 5 we display the power spectrum of the collective field in the above system with $r = 1.82$ in the local maps, for noise strengths $\sigma = 0.001$, $\sigma = 0.01$ and $\sigma = 0.1$. Clearly the noise destroys the spectral peaks much in the same way as increasing p, as evident in the close similarity between the left and right panels of Fig. 5. This indicates that non-local connections act as spatial noise in this dynamical network, with rewiring fraction p playing the role of noise strength σ.

We have also studied the system with the connections dynamically rewired, i.e. at every iteration n the connectivity matrix is updated keeping the probability of rewiring fixed at p. Dynamic rewiring yields results qualitatively similar to static rewiring for small p. However in a dynamically rewired network the SNR falls much more sharply than for static rewiring. For instance Fig. 7 displays the spectra for dynamic rewiring for the case of $r = 1.82$, $N = 150 \times 150$. Clearly the peaks have vanished for $p = 0.1$ when the connections are dynamically rewired, while they disappear only around $p \sim 0.4$ when the rewired connections are static.

These results arising from a very different system further strengthens our conclusion that $p \to 0$ does not have special implications for dynamical properties. Rather the dynamical characteristics appear to change smoothly and monotonically with respect to rewiring fraction p.

4   Conclusions

It had been observed in a study of the collective field of coupled Hodgin-Huxley neurons that in the small world region the low frequency spectral peak was more pronounced than it was in either the fully regular or fully random case [17]. Our first objective here was to check this feature in another prototypical neuronal model in order to gauge the generality and range of applicability of the above phenomena. So we chose a system of coupled Hindmarsh-Rose neurons as our first case study.

The key results of the study on coupled Hindmarsh-Rose neurons showed that the change in the low frequency spectral peak as a function of random rewiring p is *monotonic*. There was no evidence of any significant increase in spectral strength in the low p regime. So the Hindmarsh-Rose neuron network, unlike the Hodgin-Huxley neuron network of Ref. [17], does not yield special dynamical features in the dilute limit of small world links. In fact the nature of the dynamics of the mean field appears to vary quite smoothly and monotonically throughout the full range of p.

In our second case study we studied a network of coupled logistic maps. Here too the key result was that the spectral features changed monotonically with respect to p and did not show any prominent change at any special value of intermediate p. We also found that the dynamics at small rewiring was very similar to that of regular connections with additive noise.

These observations then provide examples in support of the conjecture that in a large number of coupled dynamical systems the changes in collective behaviour is monotonic with respect to the degree of non-locality in connections, and is not in any way "optimised" at any particular p.

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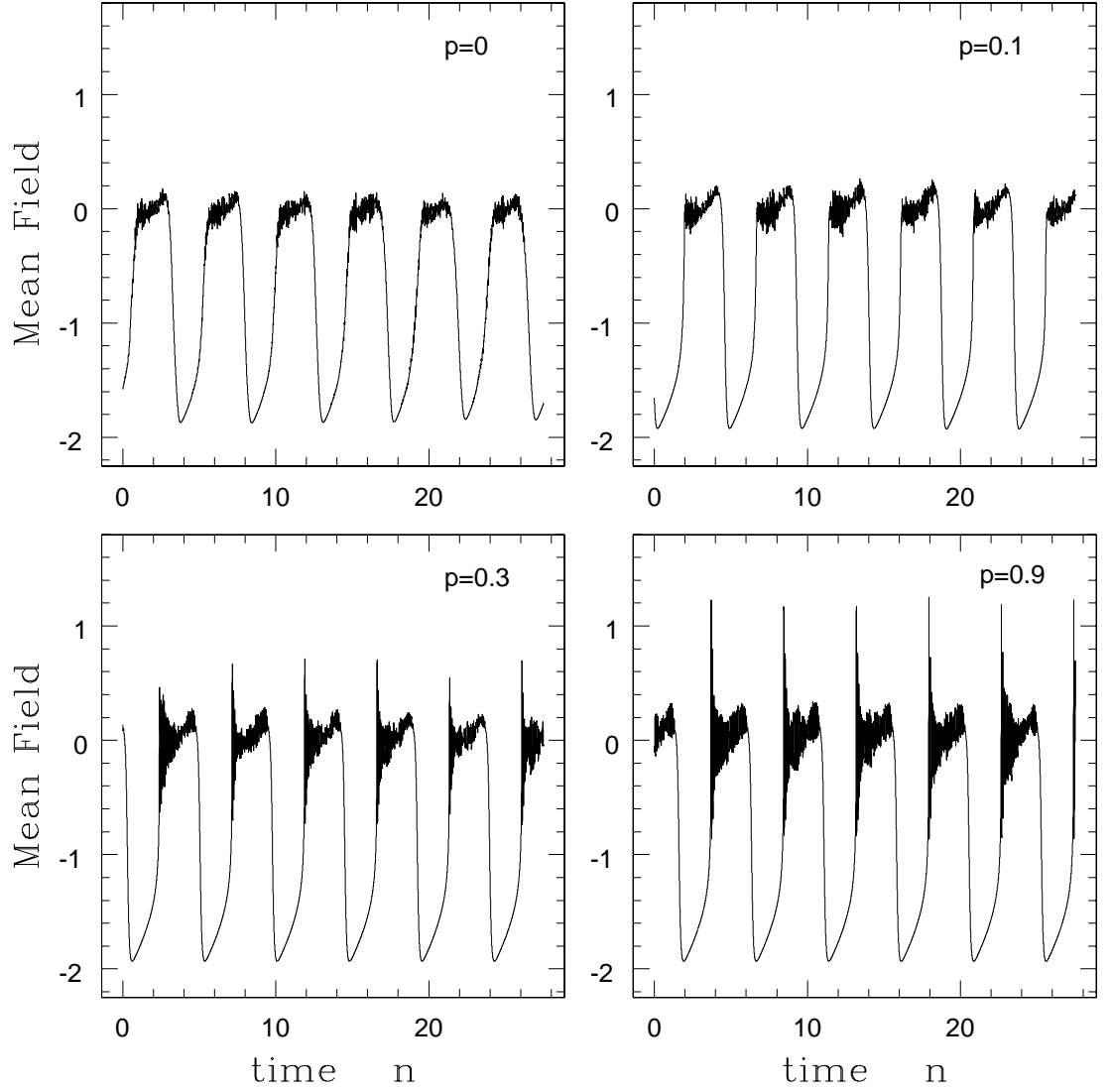[20] See for example, P. Grassberger, Phys. Rev. E **59** R2520 (1999).

Figure 1: Time evolution of the collective field of networks of coupled Hindmarsh-Rose neurons, after a transience of 10^4, with rewiring fraction $p = $ (a) 0 (b) 0.1 (c) 0.3 and (d) 0.9. The system size is 20×20.
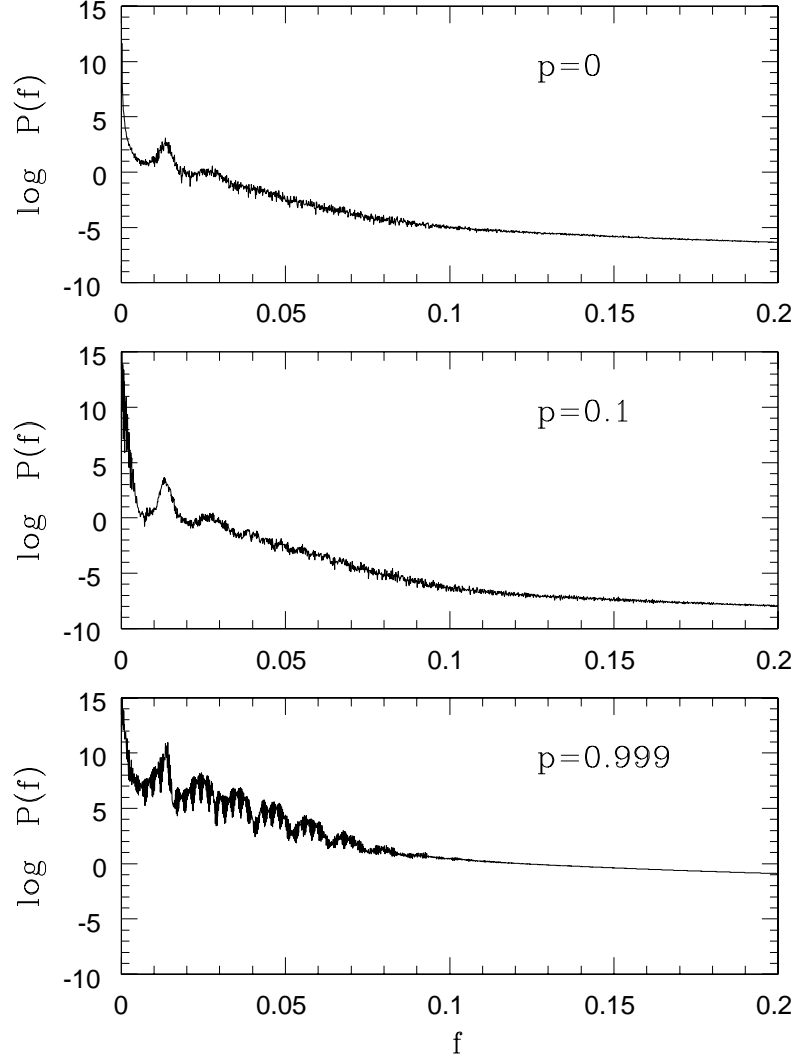
Figure 2: Power spectra of the collective field of networks of coupled Hindmarsh-Rose neurons, with rewiring fraction $p =$ (a) 0 (b) 0.1 and (c) 0.999. Here we average over 25 time series runs of 8192 data points each. The time step is 0.01 in the fourth order Runge-Kutta algorithm employed to evolve the system, and the field at every tenth step is sampled. Thus the longest time scale picked up is $\sim 10^3$. The system size is 151×151. The frequencies are scaled by 0.1.
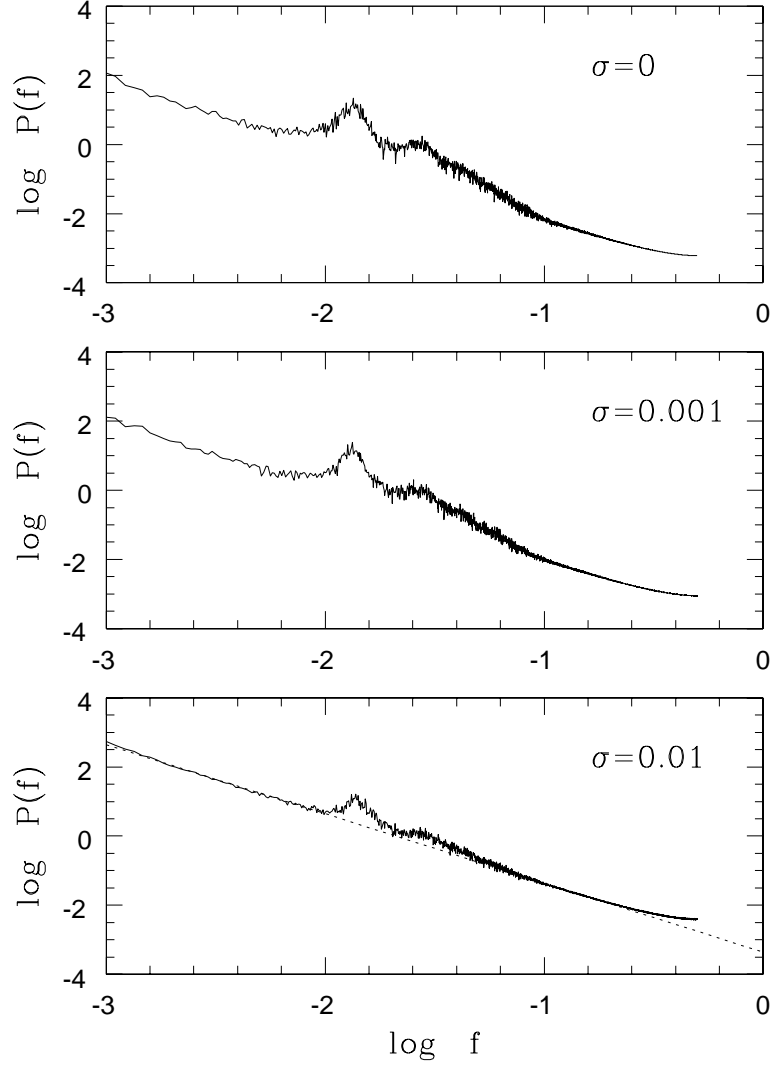
Figure 3: Power spectra of the collective field of regular ($p = 0$) networks of coupled Hindmarsh-Rose neurons, with noise strengths $\sigma =$ (a) 0 (b) 0.001 and (c) 0.01 (with the dotted $P(f) = 1/f^2$ line also displayed). Here we average over 25 time series runs of 8192 data points each. The time step is 0.01 in the fourth order Runge-Kutta algorithm employed to evolve the system, and the field at every tenth step is sampled. The system size is 151×151. The frequencies are scaled by 0.1.
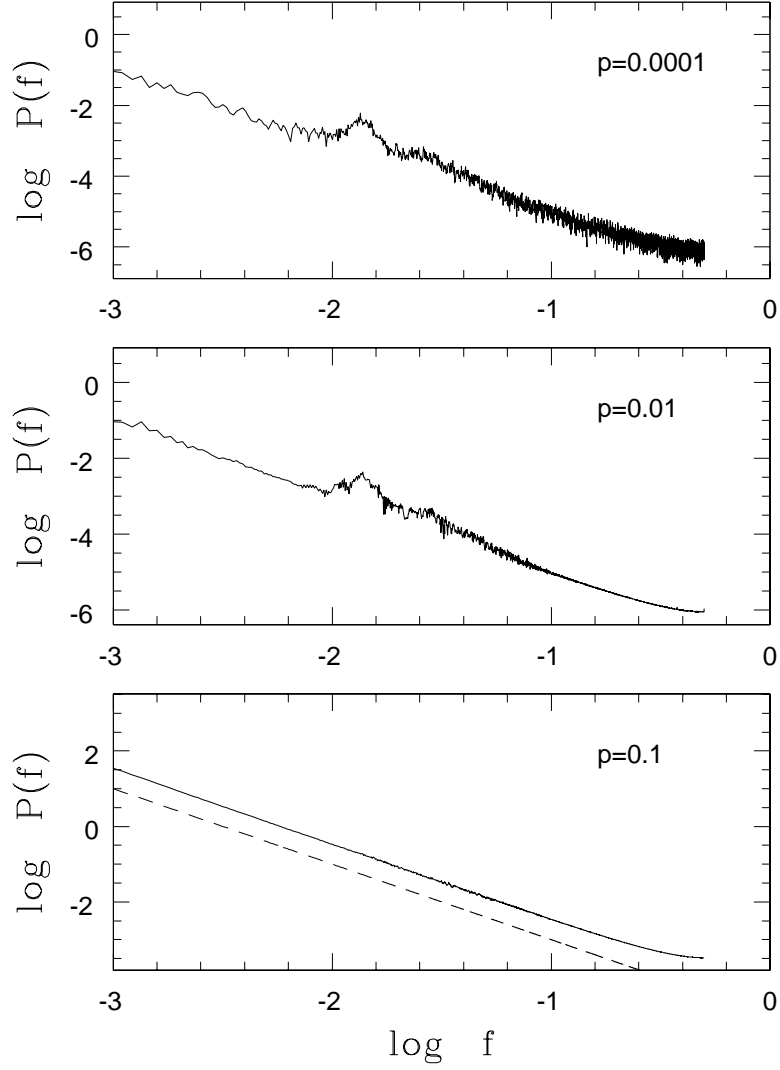
Figure 4: Power spectra of the collective field of coupled Hindmarsh-Rose neurons, with connections dynamically rewired. Three rewiring fractions are displayed : (a) $p = 0.0001$ (b) $p = 0.01$ and (c)$p = 0.1$ (with the dashed $P(f) = 1/f^2$ line also shown). Here we average over 25 time series runs of 8192 data points each. The time step is 0.01 in the fourth order Runge-Kutta algorithm employed to evolve the system, and the field at every tenth step is sampled. The system size is 151×151. The frequencies are scaled by 0.1.
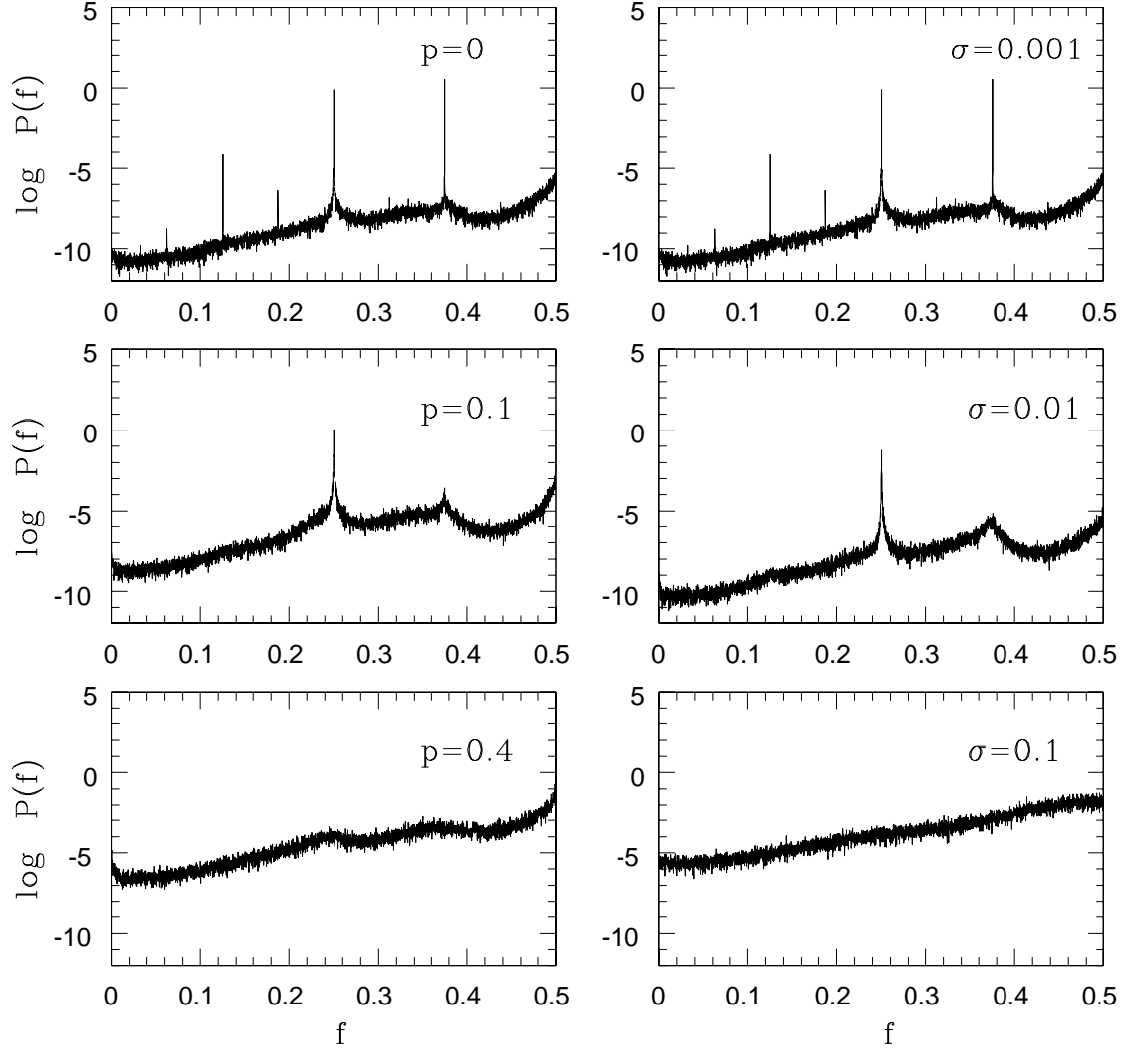
Figure 5: Power spectra of the collective field of coupled logistic map networks: (left) with rewiring fraction $p =$ (a) 0 (b) 0.1 and (c) 0.4; and (right) with $p = 0$ and noise strengths $\sigma =$ (a) 0 (b) 0.01 and (c) 0.1 in Eq. 5. Here system size $N = 350 \times 350$ and we average over 25 time series runs of 8192 data points each.
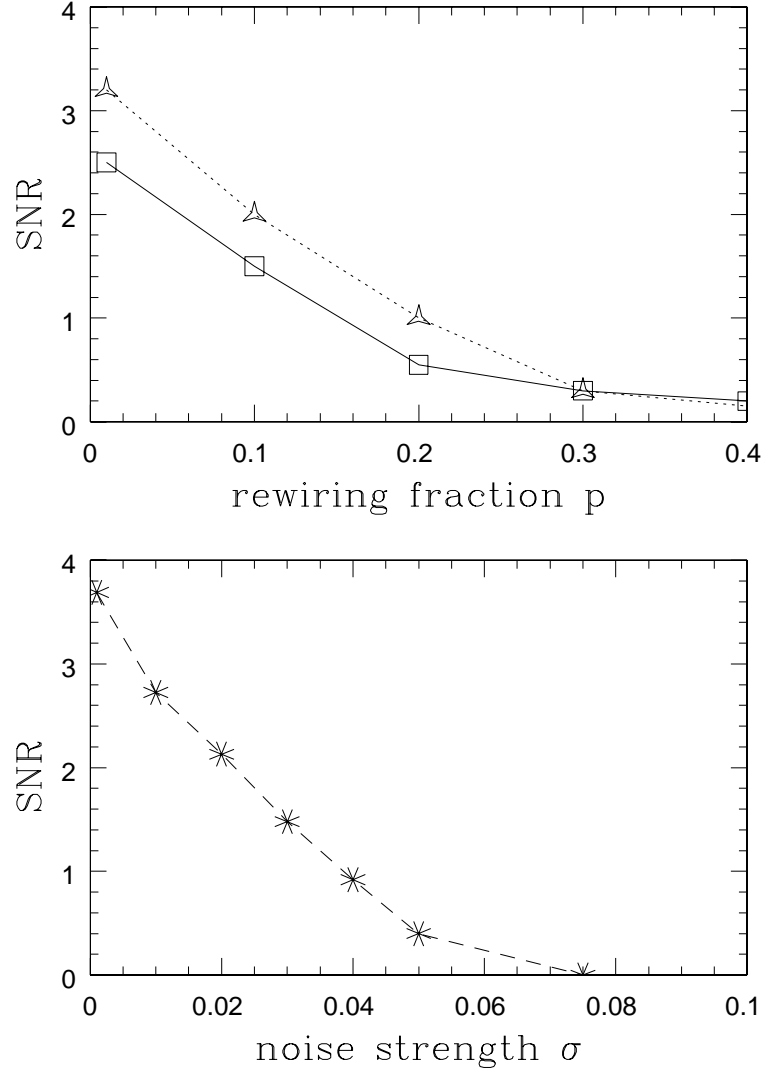
Figure 6: The upper panel displays the signal-to-noise ratio (SNR) of the power spectra of the collective field of coupled logistic maps for different values of rewiring fraction p. The lower panel displays the SNR of the power spectra for a regular network under different noise strengths σ. The system size is 20×20.
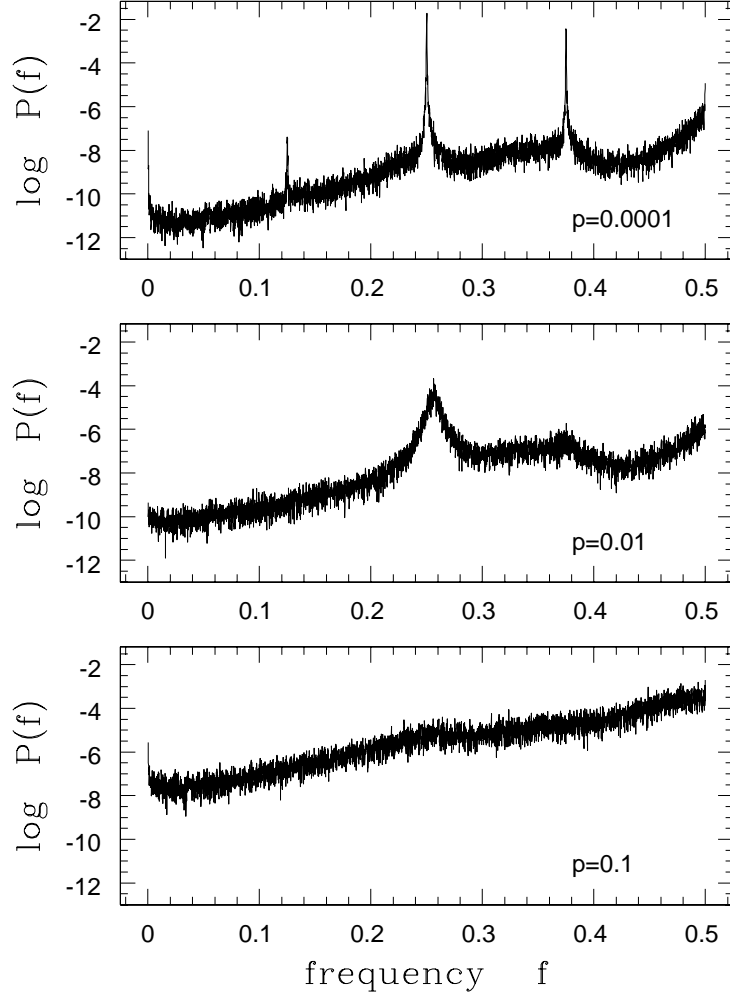
Figure 7: Power spectra of the collective field of coupled logistic map networks with connections dynamically rewired. Three rewiring fractions $p = $ (a) 0.0001 (b) 0.01 and (c) 0.1 are displayed. Here system size $N = 350 \times 350$ and we average over 25 time series runs of 8192 data points each.