Adaptive Dynamics on a Chaotic Lattice

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We introduce models incorporating an adaptive self-regulatory feedback mechanism on an extended chaotic lattice. Numerical experiments show that our system gives rise to a host of novel spatiotemporal phenomena characterized by a rich variety of "phases" in parameter space. Moreover, interestingly, certain phases exhibit distinct scaling properties and $1/f$ noise.

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The dynamics of networks of chaotic elements is important not only as a model for complex nonlinear systems with many degrees of freedom, but also from the viewpoint of possible engineering applications [1]. Here we introduce a model of adaptive dynamics [2] in a lattice of chaotic elements. Our system is spatially extended, with local nonlinear dynamics along with a self-regulatory process incorporated as threshold dynamics. Such systems are relevant in the context of a variety of physical and biological phenomena (and even in social sciences such as economics). Our motivation for this study is to investigate the wealth of spatiotemporal structures this model is likely to yield, and characterize its "phases," pattern dynamics, and scaling relations in time and space (if any).

We first introduce a *one dimensional unidirectional* model. In our model, time is discrete, labeled by n, space is discrete, labeled by i, $i = 1, N$, where N is system size, and the state variable $x_n(i)$ (which in physical systems could be quantities like energy or pressure) is continuous. Each individual site in the lattice evolves chaotically under a suitable nonlinear map $f(x)$. Here we choose $f(x)$ to be the logistic map, which has widespread relevance as a prototype of chaos. So $f(x) = 1 - ax^2$, $x = [-1.0, 1.0]$, with the nonlinearity parameter a chosen in the chaotic regime ($a = 2.0$ in all subsequent numerical experiments). On this chaotic lattice we incorporate a self-regulatory threshold dynamics [3]. The adaptive mechanism is triggered when a site in the lattice exceeds the critical value x_c $(-1.0 < x_c < 1.0)$, i.e., when a certain site $x_n(i) > x_c$. The supercritical site then relaxes by transporting its excess $\delta x = [x_n(i) - x_c]$ to its neighbor as follows:

$$x_n(i) \to x_c,$$
$$x_n(i+1) \to x_n(i+1) + \delta x. \qquad (1)$$

This algorithm thus induces a unidirectional nonlinear transport down the array (by initiating a domino effect). Note also that the model respects *local conservation* laws. The boundaries are open so that the "excess" may be transported out of the system [4].

Note that the dynamics depends on the algorithm for autonomously updating each site and propagating threshold coupling between sites. Here we have studied the case where these two evolutionary steps are carried out separately. The adaptive dynamics begins after each step in the site dynamics and continues until the system has reached a steady state where all sites are less than critical; i.e., all $x(i) \le x_c$, and the system is stationary, after which the next step in the site dynamics takes place. So the time scales of the two dynamics, the intrinsic chaotic dynamics of each lattice site and the adaptive relaxation, are adiabatically separable. The relaxation mechanism is much faster than the chaotic evolution, and this enables the system to relax completely before the next chaotic iteration [5]. We can also introduce a random driving force in our model. Under this, the system is perturbed at some site j in the lattice: $x_n(j) \to x_n(j) + \sigma$ where σ is the strength of the perturbation and j is chosen at random [6]. The random driving force, likewise, operative at time scales comparable to the chaotic dynamics, is much slower than the adaptive dynamics. (It would be interesting of course to study a variation of the above model with the threshold dynamics being incorporated simultaneously with the intrinsic dynamics or with very small Δ).

The relevant parameters in the model are the critical x_c, the strength of perturbation, σ, and the system size N. The simulations are done with random initial conditions for the $x(i)$ and we allow all transience to die. The quantities of interest are the temporal evolution of (a) the individual sites $x_n(i)$ and (b) the "avalanches" (to borrow the language of self-organized criticality), which are defined as the total number of "active" sites, i.e., sites that have "moved" (or dissipated energy) during the adaptive relaxation, denoted by s. Avalanches are thus equivalent to the total dissipation in the system due to self-regulation. The spatial aspects of interest are the distribution of $x(i)$ and the presence of clustering and coherence in space as indicated by the cluster distribution at any point in time [7].

Results of numerical simulations indicate that the most significant parameter in the model is the critical x_c and the dynamics is determined principally by it. Variation of x_c leads to the emergence of the following "phases" in parameter space: The first phase is the fixed point region which occurs when $x_c < 0.5$. Here the system goes to a coherent state where all sites $x_n(i) = x_c$ for all times (after transience). This phenomenon is independent of the perturbation strength and system size. For finite perturbation strengths ($\sigma > 0$), all avalanches are equal to system size (that is, all sites move in order to relax).

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This behavior can be understood from the x_n vs x_{n+1} graph of a single logistic map. For $x_n < 0.5$ this graph lies above the 45° line indicating that in this region $x_{n+1} > x_n$. So, the configuration $x(i) = x_c$, $i = 1, N$ is stable, as the subsequent iteration will make all $x(i) > x_c$ and the adaptive feedback will come into effect and the system will be brought back to x_c. It is also easy to see that the size of the avalanche will thus be equal to N. [Finite perturbation will not alter this picture as the effect of the driving force is simply to enhance the x at some random site, say j, which leads to $x_n(j) \rightarrow x_n(j) + \sigma > x_c$.

When $x_c = 0.5$ we still have a coherent state with all $x(i) = x_c$, but the avalanches are now uniformly distributed about the mean ($= N/2$), and the power spectrum of the temporal evolution of the avalanches is that of white noise. The dynamical system at this point in parameter space is marginally stable. Here $x_{n+1}(i) = f(x_n(i)) = x_n(i)$, and the adaptive dynamics is triggered off only by the random driving force. When the perturbation strength is zero there are never any active sites in the lattice (i.e., all avalanches are of size zero).

When $0.5 < x_c < 1.0$, the dynamics of each individual site is attracted to a *cycle* whose periodicity depends on x_c. Here we briefly give some examples of the varied phenomena emerging from tuning x_c: for $0.5 < x_c < 0.8$, we get distinct two-cycles in the temporal evolution of the avalanches and individual sites, for $x_c = 0.84$ we have a four-cycle, for $x_c = 0.86$ a six-cycle, for $x_c = 0.88$ a seven-cycle, for $x_c = 0.9$ a ten-cycle, and for $x_c = 0.98$ a four-cycle again. Typically, small perturbation strengths allow spatial coherence and for large perturbations the spatial profile breaks up into clusters of various sizes (many small clusters and a few large ones). The details of the different spatiotemporal phases will be published in a subsequent long paper.

In the above phase, especially for large x_c ($x_c \rightarrow 1.0$), we have an interesting phenomenon: here *system size N is significant and plays a crucial role in spatiotemporal organization.* For small arrays, the system exhibits *approximate* periodicities, with the avalanches and individual sites displaying long periods of laminar flow (cyclic evolution) with intermittent irregular bursts. (Also note that the sites at the edges are interrupted more by irregular bursts than the sites in the center of the array.) Remarkably now, these periodicities get sharper with increasing N, and for large N (typically $\simeq 100$) the cycles get *exact.* Figure 1 shows the power spectrum of the temporal evolution of avalanches for a small array ($N = 25$) and for a large array ($N = 1000$). It is clearly evident that the large lattice supports exact cycles (sharp clean peak) while only an approximate periodicity exists in the small lattice (broad and noisy peak). For small lattices, the other remarkable fact is that the low frequency end of the power spectrum falls as an approximate power law, $S(f) \sim 1/f^{\phi}$, $0 < \phi < 1$ [8]. This phenomenon is the ubiquitous $1/f$ noise seen in a wide variety of natural processes ranging from resistance fluctuations to sand flow in an hourglass, and even in traffic and stock market movements [9]. We shall subsequently see an extension of this model where this approximate $1/f$ behavior gets more distinct and very clearly defined.

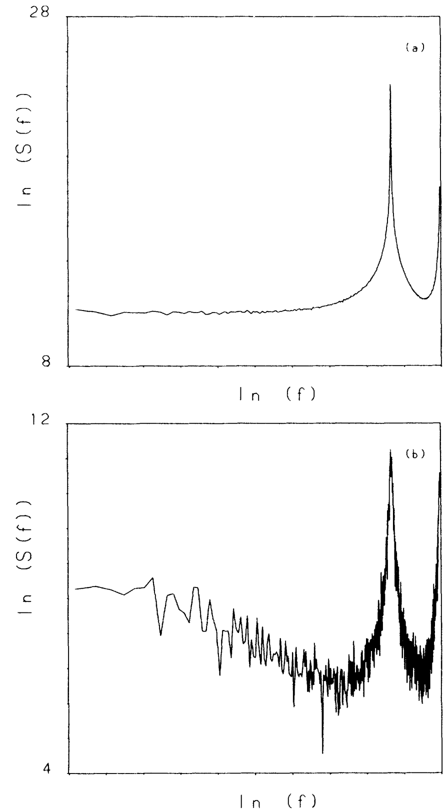

FIG. 1. Power spectra of avalanches in the unidirectional model with parameters $x_c = 0.98$, $\sigma = 0.2$ and lattice size equal to (a) 1000 and (b) 25. Here we average over 8 runs of length 1024 each. The frequency range is $(0, 0.5]$. The best fit slope at the low frequency end of spectrum (b) is 0.77 ± 0.05.

We now introduce an extension of the above model to *bidirectional* transport. The modified relaxation algorithm is as follows: if $x_n(i) > x_c$, the supercritical site then relaxes to a value x_0 ($x_0 \leq x_c$) by transporting the excess $[x_n(i) - x_0]$ equally to its two neighbors:

$$\begin{aligned} x_n(i) &\rightarrow x_0, \\ x_n(i+1) &\rightarrow x_n(i+1) + \delta x, \\ x_n(i-1) &\rightarrow x_n(i-1) + \delta x, \end{aligned} \tag{2}$$

where $\delta x = [x_n(i) - x_0]/2$. A new and important parameter here is the quantity x_0. When $x_0 = x_c$ the above model is a direct extension of the unidirectional model. However, when this is the case, for certain configurations the system takes infinitely long to reach steady state. It is not physically realistic though to drive the system chaotically at an infinitely slow rate. So with Δ finite we then have some configurations which have not completely relaxed (i.e., all avalanches have not come to rest) before the subsequent chaotic iteration. Effectively, the dynamics then would be one of "interacting" avalanches, as the time scales of relaxation and chaotic evolution can no longer be separated adiabatically [5].

First we consider the case $x_0 < x_c$ (if x_0 is chosen to be -1.0 then this is always valid [10]). Now, in contrast to the case $x_0 = x_c$, the system reaches a steady state in finite time [$\Delta \sim O(N)$], and so the system has completely relaxed configurations $x_n(i) < x_c$, $i = 1, N$, at all times. For small x_c, each site has an irregular temporal evolution with a white noiselike power spectrum. The avalanches are similar in size, and randomly distributed about their mean value (which is dependent on system size). The average $\langle x \rangle$ of the sites $x_n(i)$, $i = 1, \ldots, N$, is determined by the values of x_c and x_0, and is statistically invariant [11].

When $x_0 = x_c$ the dynamics mimics the unidirectional model closely, and seems to be a noisier or "fuzzier" version of it. Thus we find that for $x_c \leq 0.5$ all the sites are randomly distributed with a small deviation around their mean, which is close to x_c. The avalanches are all of the maximum possible size, $s_{\max}$ [12]. When $0.5 < x_c \sim 0.8$ the sites get attracted to noisy two-cycles, as is evident through the power spectrum of the temporal evolution of the individual sites, which exhibit a broad peak at frequency $\frac{1}{2}$. The avalanches evolve as exact two-cycles, with the size of avalanches alternating between 0 and $s_{\max}$. Further, as we tune x_c we obtain a noisy four-cycle at $x_c \sim 0.84$, and so forth.

As x_c approaches 1.0 the dynamics, in both cases, gets complex and we have the following spatiotemporal scenario: the avalanches, interestingly enough, now come in all sizes. This suggests that energy dissipation (or response to pertubations) takes place at various scales, ranging from small rearrangements, where the disturbance dies out within a few sites, to large events where the transport activity is at a wider scale involving all the elements of the lattice. The distribution of avalanche size s is interesting: for the case of $x_0 = -1.0$, the avalanches are distributed as a power law, $P(s) \sim s^{-\phi_s}$, and the scaling exponent ϕ_s lies between 1.0 to 2.0 [13]. For the case of $x_c = x_0$, the distribution of avalanches is quite novel: now there is an approximate periodicity evident in the evolution of the avalanches (for example, a four-cycle for $x_c = 0.98$), as the system mimics its unidirectional counterpart. A coarse grained distribution $P(s)$ shows that for small sizes the distribution falls as a power law (with scaling exponent $\simeq 1.0$), and then there is an oscillatory hump at large sizes, indicating an enhanced probability of getting very large avalanches. This is related to the fact that there are some avalanches (approximately one in every four for $x_c = 0.98$) that take infinitely long to settle down. So the size of these is very large, bounded only by finite Δ. These then constitute the hump in $P(s)$ at large s.

Spatially the $x(i)$ vs i profile is now no longer convex (as was the case for small x_c). In fact $x_n(i), i = 1, \ldots, N$, breaks up into several clusters [14] of size c, occurring now on all scales. For the case of $x_0 = x_c$ we have a cluster distribution $P(c)$ quite similar to that for avalanches. For small sizes $P(c) \sim c^{-\phi_c}$, $1.0 < \phi_c < 2.0$, and then there is a hump at large cluster size [15]. For the case of $x_0 = -1.0$ we have a well defined power law behavior for the cluster distribution over the entire range of c, $P(c) \sim c^{-\phi_c}$, with the exponent $\phi_c \sim 2.0$. For example, in the case of $x_c = 0.98$, $N = 2500$, and $\sigma = 0.1$ we have $\phi_c = 2.06 \pm 0.04$. In summary, then, the system displays well defined scaling behavior (over a smaller range for the case of $x_0 = x_c$ though) and can provide a model for dynamical states characterized by scale invariance, a phenomenon ubiquitous in natural systems.

The temporal evolution of the avalanches too yields novel phenomena for certain ranges of x_c (x_c is large). For the case of $x_0 = -1.0$, we have a crossover behavior. The lowest frequencies scale as $1/f^0$, and then there is a distinct crossover to $1/f^{\phi}$, where $\phi \sim 1.0$ [see Fig. 2(a)]. For the case of $x_0 = x_c$, something remarkable happens: first, we have a pronounced (though broad and noisy)

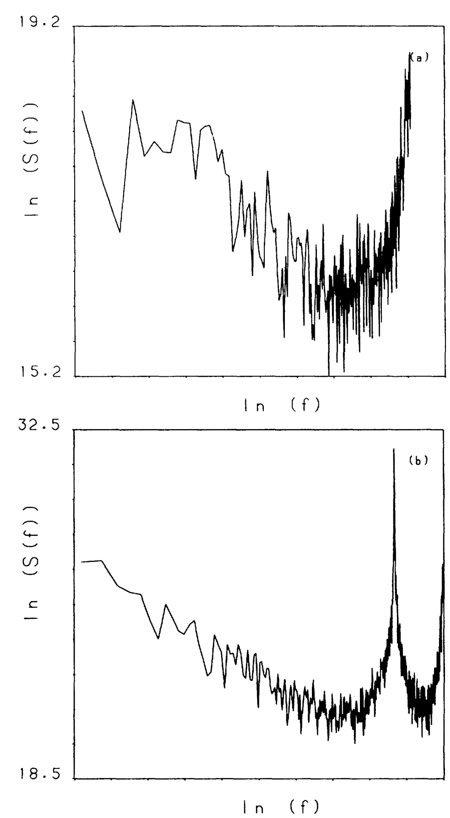

FIG. 2. Power spectra of avalanches in the bidirectional model with parameters $x_c = 0.98$, $\sigma = 0.1$, $N = 100$, and $\Delta = 4N$, for the cases of (a) $x_0 = -1.0$ and (b) $x_0 = x_c$. Here we average over 8 runs of length 1024 each. The frequency range is $(0, 0.5]$. For the lowest frequencies in spectrum (a) the slope is ~ 0.0, and after that there is a crossover and the slope in the next region is $\sim 0.8 \pm 0.2$. The best fit slope at the low frequency end of spectrum (b) is 1.13 ± 0.07.

peak at the frequency corresponding to the cycle found in the unidirectional model at that parameter value, indicating an approximate periodicity in the system [16]. For low frequencies though, interestingly, we now have clear evidence of $1/f$ noise. For instance, for the specific example of $x_c = 0.98$, we have a peak at $f = \frac{1}{4}$, and for low f, $S(f) \sim 1/f^{\phi}$, where the best fit value of the exponent is 1.13 ± 0.07 [see Fig. 2(b)]. This kind of "flicker noise" is widely observed in nature [9] and it is thus of great interest that our model yields this phenomenon. (Note that the spectrum is similar to the one observed in the unidirectional case for small lattices [Fig. 1(b)]. However, the $1/f$ behavior is more pronounced and clearly evident here, even at the lowest frequency end, i.e., when $f \to 0$.)

In conclusion, the model we have proposed provides a rich repertoire of behavioral patterns. Further, we can consider several natural extensions of it, such as different nonlinear evolution functions $f(x(i))$ in the local dynamics (e.g., logistic maps with different nonlinearity parameters, or the circle map and the tent map) or suitable higher dimensional versions [17]. Most importantly, we can proceed to extend this general model to more realistic and explicit examples relevant to physically realizable situations.

[1] J. Crutchfield and K. Kaneko, in *Directions in Chaos*, edited by Hao Bai-Lin (World Scientific, Singapore, 1987), and references therein.

[2] B. Huberman and E. Lumer, IEEE Trans. Circuits Syst. **37**, 547 (1990); S. Sinha, R. Ramaswamy, and J. Subba Rao, Physica (Amsterdam) **43D**, 118 (1990), and references therein.

[3] This is like certain biological systems (e.g., the synapses of nerve tissue) or certain mechanical systems (e.g., chains of nonlinear springs) rather than chemical systems (e.g., reaction diffusion system) where coupling is diffusive [see R. Kapral, J. Math. Chem. **6**, 113 (1991)].

[4] The dynamics here is reminiscent of the "sandpile" model [P. Bak, C. Tang, and K. Weisenfeld, Phys. Rev. Lett. **59**, 391 (1987)]. Our model is, however, significantly different in that the self regulatory mechanism now occurs on a "chaotic substrate," i.e., we allow for an "intrinsic" or "internal" dynamics at each site.

[5] One self-regulatory update of all the supercritical sites in the lattice, according to Eq. (1) (which may create some other supercritical sites), constitutes one relaxation time step. After Δ such relaxation steps, the system undergoes a chaotic update. In some sense then, time n associated with the chaotic dynamics is measured in units of Δ. When Δ is large enough to allow all sites to relax before the next chaotic update (which can be thought of as "perturbation", see [6]) then we are operating in the "dilute" perturbation limit. [For the unidirectional model this is achieved when$\Delta \sim O(N)$.] As Δ gets smaller we approach a situation where relaxations interfere, a scenario analogous to that of "interacting avalanches" in sandpiles (see T. Hwa, Ph.D. thesis, MIT, 1989).

[6] The perturbation can hit a random site or a specific site. This variation leaves the dynamics qualitatively unchanged. Perturbation strength is not a very important parameter, as few sites become active due to the external perturbation. Most sites become supercritical, hence active, due to the chaotic evolution of the lattice. So the chaotic evolution of the substrate can be regarded as a random "internal" perturbing force.

[7] In certain contexts, some other quantities may be of interest; for instance, the dynamics and distribution of the "drop rate"; i.e., the "excess" transported out of the system during relaxation may be relevant.

[8] Note, however, that the lowest frequencies scale as $1/f^0$, and subsequently there is a crossover to $1/f^{\phi}$, $0 < \phi < 1$ behavior.

[9] For a short review on $1/f$ spectra, or "flicker noise," see W.H. Press, Comments Mod. Phys. C **7**, 103 (1978).

[10] The unidirectional version of this case is quite trivial. It is easy to see that when there is no external perturbation, for all values of x_c, the system will remain at $x_i(n) = -1.0$, for all i at all times n (as $x = -1.0$ is a fixed point of the map). For finite perturbation strengths the dynamics largely remains as above, except for a small set of phase points where the state variables have value $-1.0 < x < x_c$.

[11] The $x(i)$ vs i profile is always convex (except for a few sites), with the sites at the edges having smaller x values than the central sites.

[12] When $x_c < 0.5$ and $\Delta \sim O(N)$, the entire system is very far from its relaxed steady state at most times. In such a situation, periodicities tend to develop in the evolution of the avalanches and sites, and spatially the $x(i)$ vs i profile is convex with supercritical sites in the center. As Δ increases the $x(i)$ profile gets flatter, though still supercritical in the center, and temporally the periodicities give way to a white-noise-like spectrum for $x_n(i)$; the avalanches are now always of the maximum possible size, $s_{\max}$, where $s_{\max}$ depends on N and Δ.

[13] The exact value of the exponents depends on the system parameters, but they all lie in the range given in the text. For example, when $x_c = 0.98$, $N = 100$, $\Delta = 10N$, and $\sigma = 0.1$, we have $\phi_s = 1.6 \pm 0.1$.

[14] The extent of the clusters is defined by the number of consecutive sites for which this inequality holds: $x_n(i) - x_n(i+1) < \epsilon$. Here $\epsilon = 0.1$.

[15] The unidirectional model has a similar distribution of clusters for large x_c. There, too, the clusters have a power law distribution, $P(c) \sim c^{-\phi_c}$, for small c, and then there are some very large clusters which give rise to a hump in $P(c)$ at large sizes.

[16] Here, too, the system size N aids spatiotemporal organization. As N becomes larger, the periodic peak gets more enhanced, i.e., sharper and larger in magnitude. Because of computational limitations we have not checked whether these periodicities get exact, as in the unidirectional case, for very large N.

[17] For details see S. Sinha, in Proceedings of Fractal 93, England, 1993 (to be published).