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Local-to-global coupling in chaotic maps

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We study coupled systems whose elements are chaotic maps, with the coupling ranging from "local" (with the interaction spreading over K neighbors) to "global" (mean-field-type coupling). We find that well-defined peaks emerge in the power spectrum of the mean field, indicating a subtle coherence among the elements, as the extent of coupling, i.e., K, is increased. We observe that the significant quantity here is not the ratio of the number of elements coupled K, to lattice size N, but the magnitude of K. After a critical value of K equal to K_c, the coupling takes on a "global" character, and is practically indistinguishable from mean-field interaction. Interestingly, the value of $K_c \approx N_c$, where N_c is the critical lattice size after which the power spectra in globally coupled systems saturate. We also find that the mean-square deviation of the mean field grows linearly with coupling strength, up to K_c.

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Global coupling in dynamical systems yields a host of very novel features. This class of complex systems is of considerable importance in modeling phenomena as diverse as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, and even evolutionary dynamics, biological information processing, and neurodynamics [1]. The ubiquity of globally coupled phenomena has thus made it a focus of much recent research activity [2–6].

A coupled map (CM) is a dynamical system of N elements evolving according to local mappings, and a coupling term, involving neighboring elements. A globally coupled map (GCM) is one where the interaction term is of a "mean-field" type, through which the global information influences the individual elements. It is thus analogous to a mean-field version of coupled-map lattices [2]. The simplest form of a GCM is

$$x_{n+1}(i)=f(x_n(i))+\epsilon H_n \ , \tag{1}$$

where n is a discrete time step, i is the index of the elements ($i=1,2,\ldots,N$; N is the system size), ϵ is the coupling parameter, and the mean field H_n is given as

$$H_n=\frac{1}{N}\sum_{j=1}^{N} f(x_n(j)) \ .$$

It has been noticed that one-dimensional GCM's (for example, globally coupled logistic maps) have two conflicting trends: destruction of coherence due to the chaotic divergences of the individual elements, and a synchronizing force through global averaging [2]. This means that as a function of the coupling ϵ, the dynamics can go from a phase of completely incoherent chaotic motion, through phases of partial synchronization, to a phase of global synchronization, where the synchronized motion can be chaotic or regular. A very surprising result was found by Kaneko [3]: in the fully "turbulent" phase, where coherence is destroyed by chaos in the individual maps and there is no explicit manifestation of correlation among the elements, a subtle collective behavior emerges. This is reflected in the development of significant peaks in the power spectrum of the mean field, which clearly indicates a collective "beating" pattern, and hence a partial order in the dynamics. Further, if all the state variables took quasirandom values almost independently, one would expect the mean field to obey the central-limit theorem and the law of large numbers. If this were true, the mean-square deviation (MSD) ($\equiv\langle H^2\rangle-\langle H\rangle^2$) would decrease as N^{-1}, where N is the number of elements coupled, and the mean field would converge to a fixed value as $N\rightarrow\infty$. Examination of the above expectation in one-dimensional maps showed that the mean field violated the law of large numbers [3]. In fact, the MSD stopped decreasing after a critical value of N, equal to N_c. This result, too, then, indicates the emergence of a certain coherence in the system.

In this paper we study a coupled map with different degrees of coupling. The model we introduce is a set of N logistic maps displaying chaos, where each local map is coupled to K elements. First, we discuss the model and then examine phenomenologically the dynamics of the mean field, with respect to K and N. Our motivation in studying this system is to examine the transition from "local" to "global" interaction.

The CM, in general terms, is given as

$$x_{n+1}(i)=f(x_n(i))+\epsilon h_n^K(i) \ , \tag{2}$$

where the local map is

$$f(x)=1-ax^2 \tag{3}$$

and the coupling interaction is

$$h_n^K(i)=\frac{1}{K+1}\sum_{\Delta=-K/2}^{+K/2} f(x(i+\Delta)) \ , \tag{4}$$

with the i's arranged cyclically. The extent of interaction is given by the number of neighboring elements K that are coupled to each individual element, and the system goes over to a GCM in the limit $K+1$ equal to N.

We have simulated Eqs. (2) and (3) with the parameters

$a = 1.99$ and $\epsilon = 0.1$ at different values of K and N. The single map at this value of a is located in the region of completely chaotic behavior. In all cases considered, we have checked to see that the coupled dynamics is not synchronized.

First, we examine the power spectra of the mean field. We find that the mean field reveals the emergence of order as the number of elements coupled, K, is increased. In Figs. 1(a)–1(e), we have plotted the power spectrum for different coupling interactions. It is clear that the

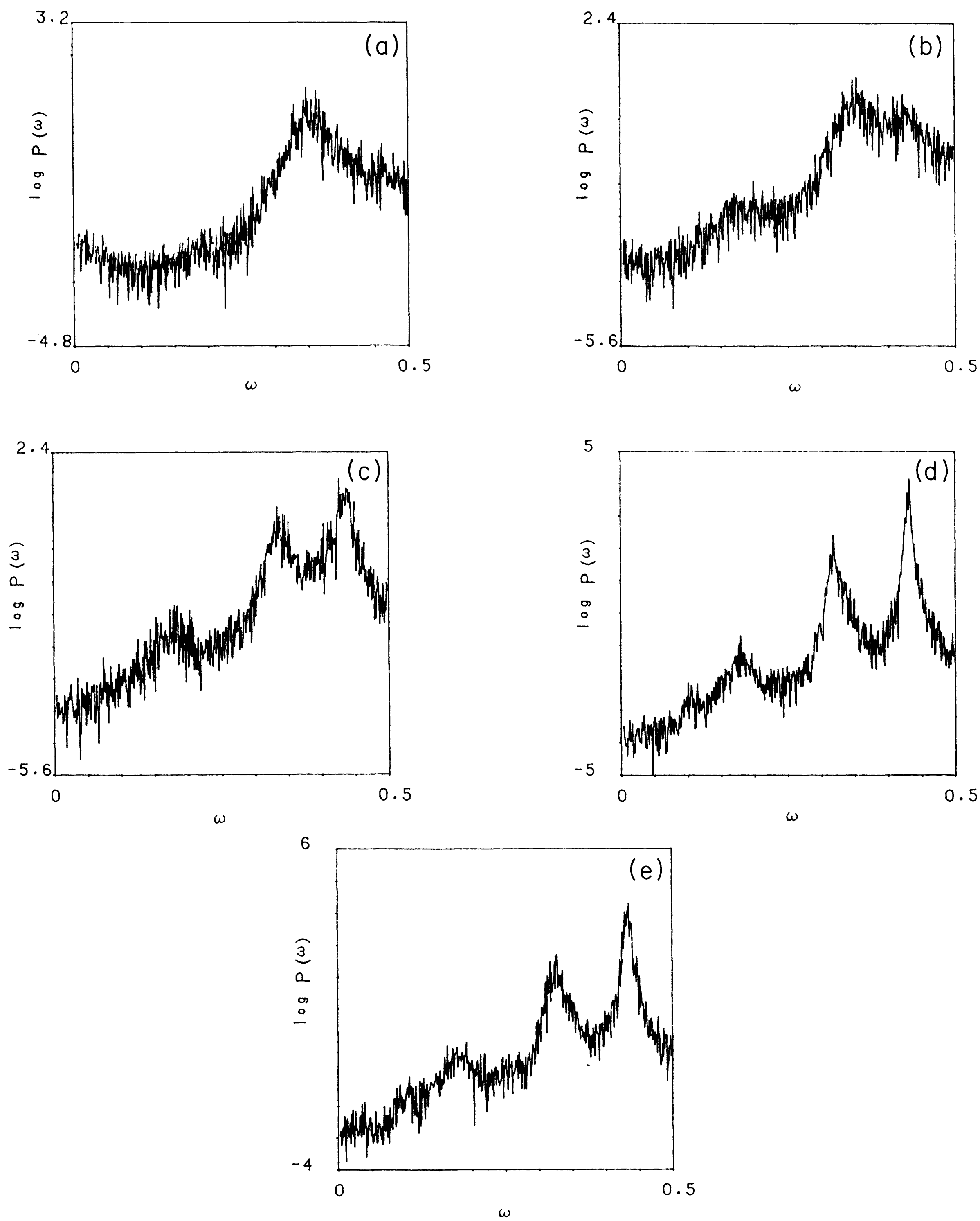

FIG. 1. Power spectra of the mean field for a lattice of size 1000, with the number of elements coupled K equal to (a) 10, (b) 20, (c) 100, (d) 500, and (e) GCM. Here we average over eight runs of length 1024 each. Notice that the peaks have almost saturated in the last two figures.

spectra develop some very prominent peaks as K is increased [7]. We also find that the spectra saturate after a critical value of coupling, $K_c < N$. This suggests that "beats" in the mean field, characteristic of global coupling, emerge as the degree of coupling increases, and interestingly, these features are evident well before K is of the order of N. That is, the coupling is effectively "global" even when the interaction is less than the mean field.

Another interesting observation regards the relevant quantity that governs local to global transitions. What determines the local or global nature of the coupling interaction is not the ratio of the number of elements coupled to lattice size, i.e., K/N, but rather the magnitude of K. After a critical value of K equal to K_c [8], the coupling becomes predominantly global in nature and the collective beating patterns characteristic of globally coupled systems emerge. Figures 2(a) and 2(b) show the power spectra of the mean field of two different CM's with the same fraction K/N, but different values of K (and N). Clearly, the spectra are drastically different, with the spectrum corresponding to $K \approx K_c$ displaying prominent peaks, and the spectrum corresponding to $K \ll K_c$ showing only mild humps. So the significant quantity here, clearly, is the magnitude of K. This observation ties in nicely with the limiting case of $K \cong N$ (i.e., GCM), where we know that the behavior of the system is crucially dependent of the magnitude of N [2–6].

Another feature of the development of peaks in the mean field with increasing coupling strength K is that it bears a striking resemblance to the development of peaks in globally coupled systems with increasing lattice size [5]. In both cases, the spectrum first shows one broad peak, which later resolves into various components (namely, two major peaks). This again is consistent with the observation that the magnitude of coupling (i.e., the absolute value of K, not K/N) governs the nature of the coupled map. That is, the mean field of a GCM of lattice size N shows similar beating pattern as a CM with coupling strength equal to N.

We have also looked at the power spectra of the time series of $h_n^K(i)$ for different elements i. The result for representative cases of $i=250$ and 500, in a lattice of size

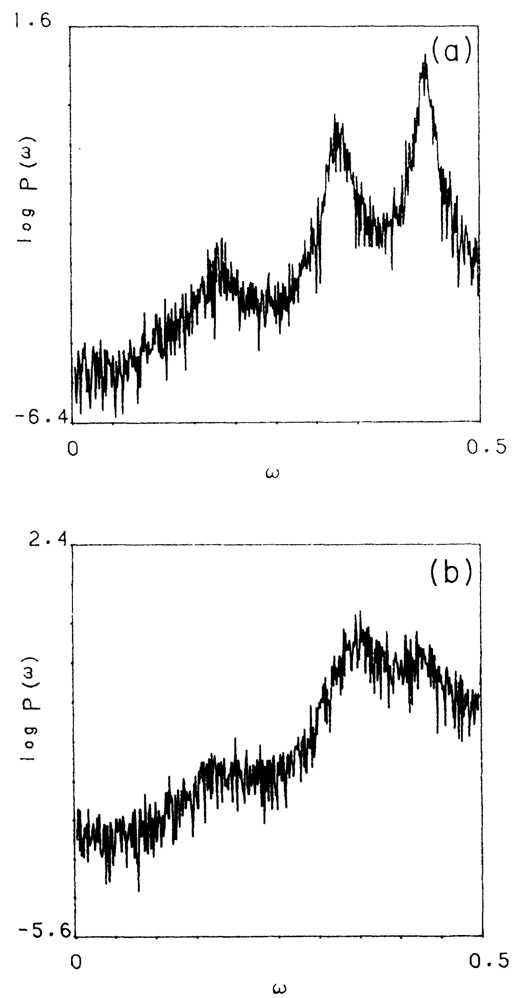

FIG. 2. Power spectra of the mean field for two systems, with identical ratios of number of elements coupled K to lattice size N: (a) $N=5000$, $K=200$; (b) $N=1000$, $K=40$ ($K/N=0.04$). Here we average over eight runs of length 1024 each.

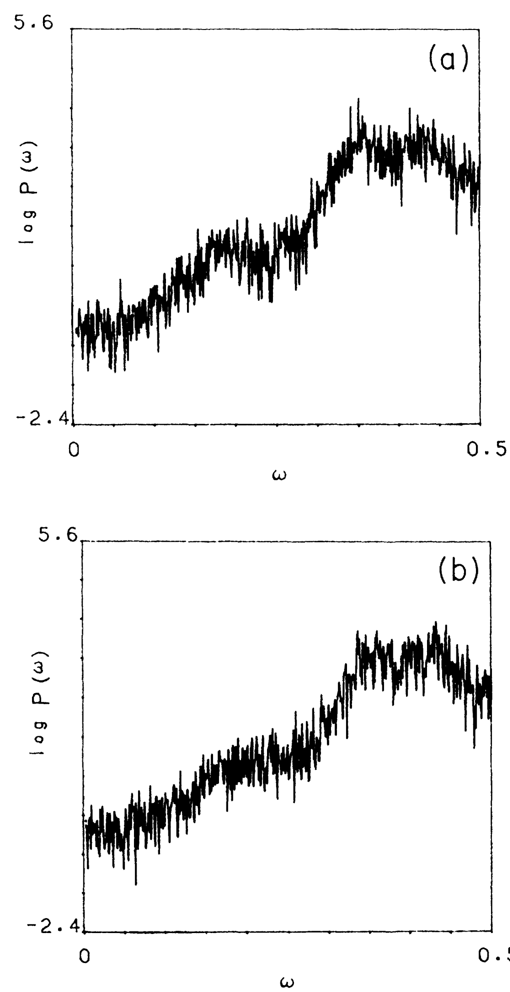

FIG. 3. Power spectrum of the coupling $h_n^K(i)$ for $K=40$, with i equal to (a) 250 and (b) 500. Here we average over eight runs of length 1024 each. (Lattice size, $N=1000$.)

1000, are displayed in Figs. 3 and 4 for two values of K. Clearly, the power spectra are practically indistinguishable for large K. This indicates that for extensive coupling, the dynamics of the coupling interactions of the different lattice elements have identical periodicities. For small K, the spectra of the coupling terms display only very mild humps, but here, too, the positions of these humps are quite similar for the different elements. We may then conclude that the periodicities (rough or pronounced, as the case may be) of $h_n^K(i)$, is similar for all i, and this in turn is similar (as is expected) to that of the mean field. Furthermore, with increasing K, the beating patterns of the individual couplings become more prominent and virtually identical.

When K is small, the spectrum does not change in sharpness as lattice size N is increased (see Fig. 5). This indicates that when the coupling is spread over a small number of elements, the interaction is very local in character, and global information, such as lattice size, does not influence the mean field. On the other hand, when K is closer to K_c, the sharpness of the mean-field power spectrum varies significantly with lattice size. This is evident from Fig. 6, which shows the spectrum at two different values of N, for larger K. (Note that while the sharpness of the peaks changes, the positions occur at identical frequencies.) So the dependence of the sharpness of the spectra of the mean field with respect to lattice size is a good indicator of how "global" the interaction is.

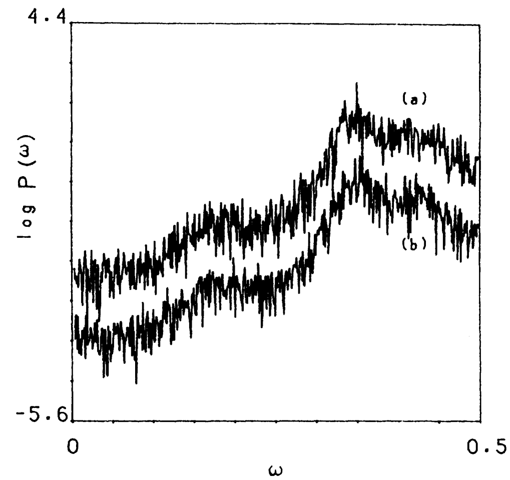

FIG. 5. Power spectra of the mean field for a lattice of size equal to (a) 200 and (b) 1000. Here the number of elements coupled K is 40. Note that the peaks are qualitatively the same for the two widely different lattice sizes. (We average over eight runs of length 1024 each.)

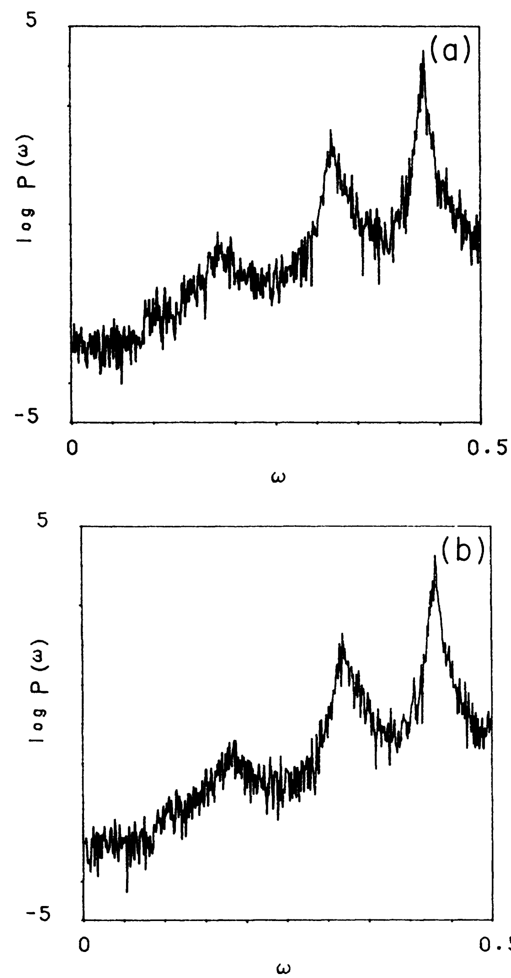

FIG. 4. Power spectrum of the coupling $h_n^K(i)$ for $K=500$ with i equal to (a) 250 and (b) 500. Here we average over eight runs of length 1024 each. (Lattice size, $N=1000$.)

Finally, we have calculated the mean-square deviation σ of the mean field

$$\sigma=\frac{1}{T}\sum_{n=1}^{T}(H_n-\langle H\rangle)^2 \tag{5}$$

as a function of coupling strength K and lattice size N. Here H_n is the mean field obtained at iteration n, and

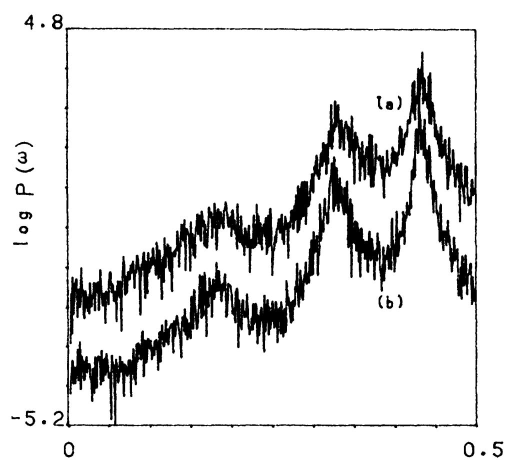

FIG. 6. Power spectra of the mean field for a lattice of size equal to (a) 200 and (b) 1000. Here the number of elements coupled K is 200. Note that the spectrum is significantly sharper for the larger lattice. (We average over eight runs of length 1024 each.)

$\langle H \rangle$ is the average obtained over the very large number of iterations T. We find (for K not too small) that the σ of the mean field grows *linearly* with the coupling strength K up to $K \cong K_c$ [9]. That is,

$$\sigma(K) \sim K \ .$$

This is clearly evident from Fig. 7, where σ is plotted against K for a lattice of size 1000. Further, we examine the behavior of the σ with respect to lattice size for fixed K. We do not observe any deviation of this σ from $1/N$ up to the K and N values scanned ($K \le K_c$, $N \sim 2000$). So the manifestation of nonstatistical behavior in the σ is not coincident with the development of broad periodicities in the mean field, as the σ does not indicate any obvious nonstatistical trends even when the peaks have developed in the mean-field power spectrum. One expects, though, from the approximately linear growth of σ with K, that the σ will vary as $N^{-\alpha}$, $\alpha \le 1$, with α decreasing as K goes from small coupling to mean-field interaction. However, the deviation from $1/N$ may be apparent only for large lattice sizes and extensive coupling. Due to computational limitations we are unable to check this out.

In summary, here we have investigated various aspects of the dynamics of a coupled system of chaotic logistic maps, with varying interactions, ranging from local to global. An interesting result that emerges from this study is that the coupling takes on a global character (characterized by collective "beats" in the mean field) well before the interactive term in the CM is mean field. Surprisingly, the relevant quantity in the transition from local to global is not the ratio of number of neighbors coupled to the lattice size, i.e., K/N, but rather the magnitude of K. After a critical number of neighbors coupled, K_c, the effects of the coupling and mean-field interaction are practically indistinguishable. The value of K_c is approximately the same as the critical lattice size N_c, at which saturation of the peaks in the spectrum of the mean field occur in globally coupled systems [3].

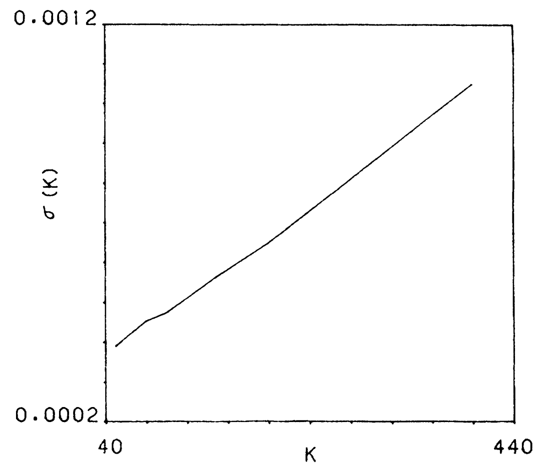

FIG. 7. Mean-square deviation σ of the mean field vs coupling strength K for a lattice of size 1000. In all cases we have used 8000 iterations. Here $50 \le K \le 400$.

Another earlier result [5] that seems to lend support to the above observations is that the partials sums defined as

$$S_m(n) = \frac{1}{m} \sum_{i=1}^{m} x_n(i)$$

also show rough periodicities, with the periodicities getting sharper with increasing m, and then saturating after a critical m [5]. Here our coupling interaction involving K neighbors is somewhat like a partial sum S_K. So it stands to reason that the interaction approximates global mean-field-type interaction after a critical value of K.

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[7] Surprisingly, the power spectrum of the mean field for $K = 500$ is a little sharper than that for the globally coupled case. The dominant frequencies occur at identical positions though [see Figs. 1(d) and 1(e)].

[8] The value of K_c for this system of coupled logistic maps is approximately 500.

[9] The growth of $\sigma(K)$ with respect to K for $K > K_c$ seems to be a little faster than linear.