Nonstatistical behavior of higher-dimensional coupled systems

Sudeshna Sinha, D. Biswas, M. Azam, and S. V. Lawande
Theoretical Physics Division, Central Complex, Bhabha Atomic Research Centre, Bombay 400 085, India
(Received 13 April 1992)

We study a (generalized) globally coupled system whose elements are *two-dimensional* chaotic maps, and find clear evidence of nonstatistical behavior: the mean-square deviation (MSD) of both components of the mean field saturate with respect to an increase in the number of coupled elements, N, after a critical value of N is reached, and their distributions are clearly non-Gaussian. We also find that the power spectrum of both components of the mean field display well-defined peaks, indicating a subtle coherence among different elements, even in the "turbulent" phase. This system is a higher-dimensional example of coupled maps, and its study confirms that the phenomena observed in a wide class of coupled one-dimensional maps (and also in an example of coupled complex maps) are present here as well. This gives more evidence to support that such nonstatistical behavior is probably generic in globally coupled systems. We also investigate the influence of parametric fluctuations on the MSD and power spectra, and find that noise restores the statistical behavior, after a critical value of the number of coupled elements is reached.

PACS number(s): 05.45.+b, 05.90.+m, 87.10.+e

I. INTRODUCTION

Global coupling in dynamical systems yields a host of very novel features. This class of complex systems is of considerable importance in modeling phenomena as diverse as Josephson junction arrays, multimode lasers, vortex dynamics in fluids, and even evolutionary dynamics, biological information processing, and neurodynamics. The ubiquity of globally coupled phenomena has thus made it a focus of much recent research activity [1].

A globally coupled map (GCM) is a dynamical system of N elements evolving according to local mappings and a "mean-field"-type interaction term, through which the global information influences the individual elements. It is thus analogous to a mean-field version of coupled map lattices [2]. The generalized form of a GCM, where each element is of dimension M, is

$$\mathbf{x}_{n+1}(i)=\mathbf{f}_1(\mathbf{x}_n(i))+\epsilon\mathbf{G}\left[\frac{1}{N}\sum_{j=1}^{N}\mathbf{f}_2(\mathbf{x}_n(j))\right] \qquad (1)$$

where $\mathbf{x}$ is an M-dimensional vector of the state variables of each individual element, n is a discrete time step, i is the index of the elements ($i=1,2,\ldots,N$), and $\mathbf{f}_1$, $\mathbf{f}_2$, and $\mathbf{G}$ denote different functions. The mean field $\mathbf{h}$ is the argument of the function $\mathbf{G}$, and ϵ is the coupling parameter.

It has been noticed that one-dimensional GCM's (for example, globally coupled logistic maps) have two conflicting trends: destruction of coherence due to the chaotic divergences of the individual elements, and a synchronizing force through global averaging [2]. This means that as a function of the coupling ϵ the dynamics can go from a phase of completely incoherent chaotic motion, through phases of partial synchronization, to a phase of global synchronization, where the synchronized motion can be chaotic or regular. A very surprising result was found by Kaneko [3]: in the fully "turbulent" phase, where coherence is destroyed by chaos in the individual maps and there is no explicit manifestation of correlation among the elements, a subtle collective behavior emerges. Since all the state variables take quasirandom values almost independently, one may expect that the mean field will obey the central limit theorem and the law of large numbers. If this were true the mean-square deviation (MSD) ($\equiv\langle h^2\rangle-\langle h\rangle^2$) would decrease as N^{-1}, where N is the number of elements coupled, and the mean field would converge to a fixed value as $N\to\infty$; also, for finite N, the distribution of h would be Gaussian. Examination of the above expectation in one-dimensional maps showed that the mean field respected the central limit theorem [3] (at least approximately, see Ref. [4]), but violated the law of large numbers. In fact, the MSD stopped decreasing after a critical value of N. Further, it was observed that the power spectrum of h had broad peaks. This result indicates the emergence of some order, a partial coherence in the dynamics.

In this paper we study an example of a generalized higher-dimensional GCM. This GCM is comprised of individual two-dimensional mappings [5] displaying chaos. First, we discuss the model and give explicitly the form of the (two-dimensional) mean field. Then we examine phenomenologically the dynamics of the components of the mean field, and study the behavior of the MSD with respect to the number of elements coupled. There we find evidence of violation of *both* the central limit theorem and the law of large numbers [6], and broad peaks in the power spectrum of the mean field. Finally, we investigate the influence of static fluctuations on the parameters of the system.

II. MODEL

Our system comprises of a set of dissipative mappings given by Graham, Isermahn, and Tel [5], which are similar to the two-dimensional Hénon map. Our motivation in studying this system is that, to the best of our knowledge, it is the first two-dimensional example of globally coupled maps. We would like to ascertain what features of one-dimensional GCM's persist in higher dimensions and what new features emerge. The GCM used for our numerical experiments is the following:

$$x_{n+1}(i)=f_x(x_n(i),y_n(i))+\frac{\epsilon}{N}h_n^x , \quad (2a)$$

$$y_{n+1}(i)=f_y(x_n(i),y_n(i))+\frac{\epsilon}{N}h_n^y , \quad (2b)$$

where

$$f_x(x,y)=1-\alpha x^2/(1+x^4)-\beta y , \quad (3a)$$

$$f_y(x,y)=x , \quad (3b)$$

and the two components of the mean field **h** are

$$h_n^x=\sum_{j=1}^{N} f_x(x_n(j),y_n(j)) , \quad (4a)$$

$$h_n^y=\sum_{j=1}^{N} f_y(x_n(j),y_n(j)) . \quad (4b)$$

The global asymptotic dynamics of the local map given by Eq. (3) is on an attractor for $\beta<1$. When α is varied this map gives rise to the entire repertiore of behavior, ranging from fixed points to chaos, reminiscent of unimodal one-dimensional maps.

III. RESULTS

We have simulated Eq. (2) with the parameters $\alpha=3.4$ and $\beta=0.3$ at different values of N and ϵ. For a single map these parameters are located in the region of completely chaotic behavior [5]. In all case considered, we have checked to see that the coupled dynamics is not synchronized.

First we have checked to see how close to Gaussian the distribution of the components of the mean field are. Figure 1 shows a histogram of the distribution of the x component of the mean field. The y component shows an identical behavior. Clearly the distribution is far from Gaussian, as is evident from the marked asymmetry of the distribution about the mean. This result is similar to the one obtained for a GCM with complex valued local maps (which can be considered a higher-dimensional example of a GCM [6]), and unlike that observed for one-dimensional GCM's, where the distributions are approximately Gaussian (in spite of other nonstatistical features). This lends support to the expectation that for most higher-dimensional GCM's, the distribution of h is probably non-Gaussian, while for one-dimensional GCM's it is closer to Gaussian.

Second, we have calculated the mean-square deviation (MSD) $\mathcal{D}$ of the components of the mean field

$$\mathcal{D}=\frac{1}{T}\sum_{j=1}^{T}(h_j^i-\langle h^i\rangle)^2 \quad (5)$$

as a function of N. Here h_j^i is the ith component of the mean field obtained at iteration j and $\langle h^i\rangle$ is the average

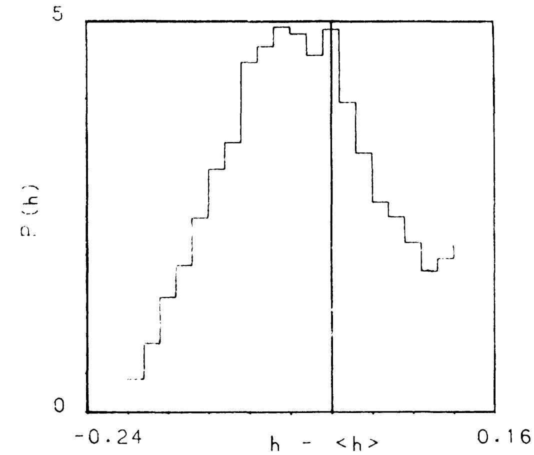

FIG. 1. The normalized probability distribution of the mean field (both components are identically distributed). Note that the distribution is not symmetric about the mean (denoted by a dashed line in the figure), and is clearly not Gaussian. Here and in all other simulations we have used parameters $\alpha=3.4$ and $\beta=0.3$, and there is a transient of 5000 steps. The distribution is built using 5000 iterations.

obtained over the very large number of iterations T. This MSD does decrease as N grows up to a critical value N_c, and then saturates, as can be observed in Fig. 2, for both components, $i=x,y$, of the mean field. For large values of ϵ the decrease in the MSD differs consistently from the $1/N$ behavior predicted by the law of large numbers. This suggests that the nonstatistical behavior of the process manifests itself well before saturation value N_c. We can then say that the MSD for the mean field **h** decays as a power ϕ, with $\phi\leq 1$, up to a critical value N_c after which it stabilizes. (The qualitative behavior of the two components is similar. There is a slight quantitative

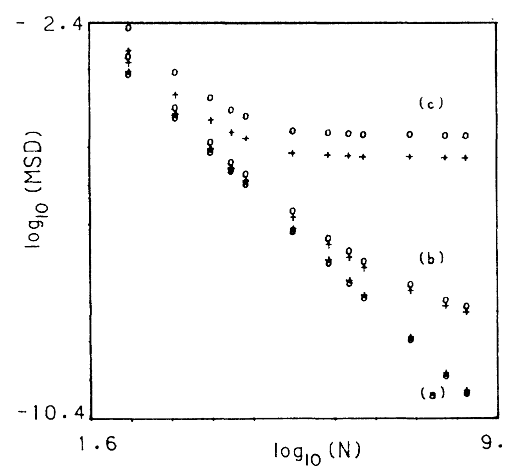

FIG. 2. Mean-square deviation (MSD) of the two components of the mean field vs lattice size N, at three different values of ϵ: (a) $\epsilon=0.0$, (b) $\epsilon=0.1$, and (c) $\epsilon=0.3$. The open circles represent the x component of the mean field and the cross symbol represents the y component. In all cases we have used 5000 iterations.

difference though, and this difference is larger for larger values of the coupling parameter ϵ.)

We have checked the behavior of the MSD with respect to the value of the coupling ϵ. It had been noticed that GCM's where the local maps were modified by the introduction of coupling showed nonmonotonic behavior [7], whereas GCM's where the local maps were not modified by coupling displayed monotonic growth [6]. Here the local maps are not modified by ϵ. Nevertheless, the MSD does not grow monotonically with coupling strength, as is evident from Fig. 3. There is an overall growth in the MSD values, though, and this growth saturates at large values of ϵ. There is a section of the curve near the saturation end where the MSD vs ϵ is monotonic, which displays (gross) scaling, as

$$\mathcal{D}(\epsilon) \approx \mathcal{D}(0)\epsilon^{\nu}, \qquad (6)$$

with $\nu \approx 4$, for both components. This suggests that the exponent probably goes as $2M$, where M, as defined earlier, is the dimensionality of the local maps. This is consistent with the result from one-dimensional GCM's,

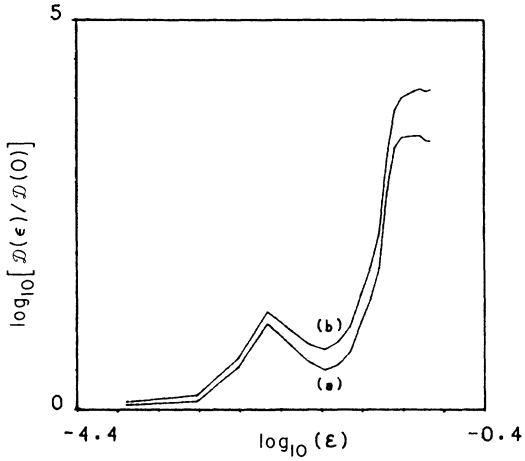

FIG. 3. Mean-square deviation (MSD) vs global coupling parameter ϵ. Here $N = 2000$ and we are using 5000 iterations. Curve (a) is for the x component, and curve (b) for the y component of the mean field.

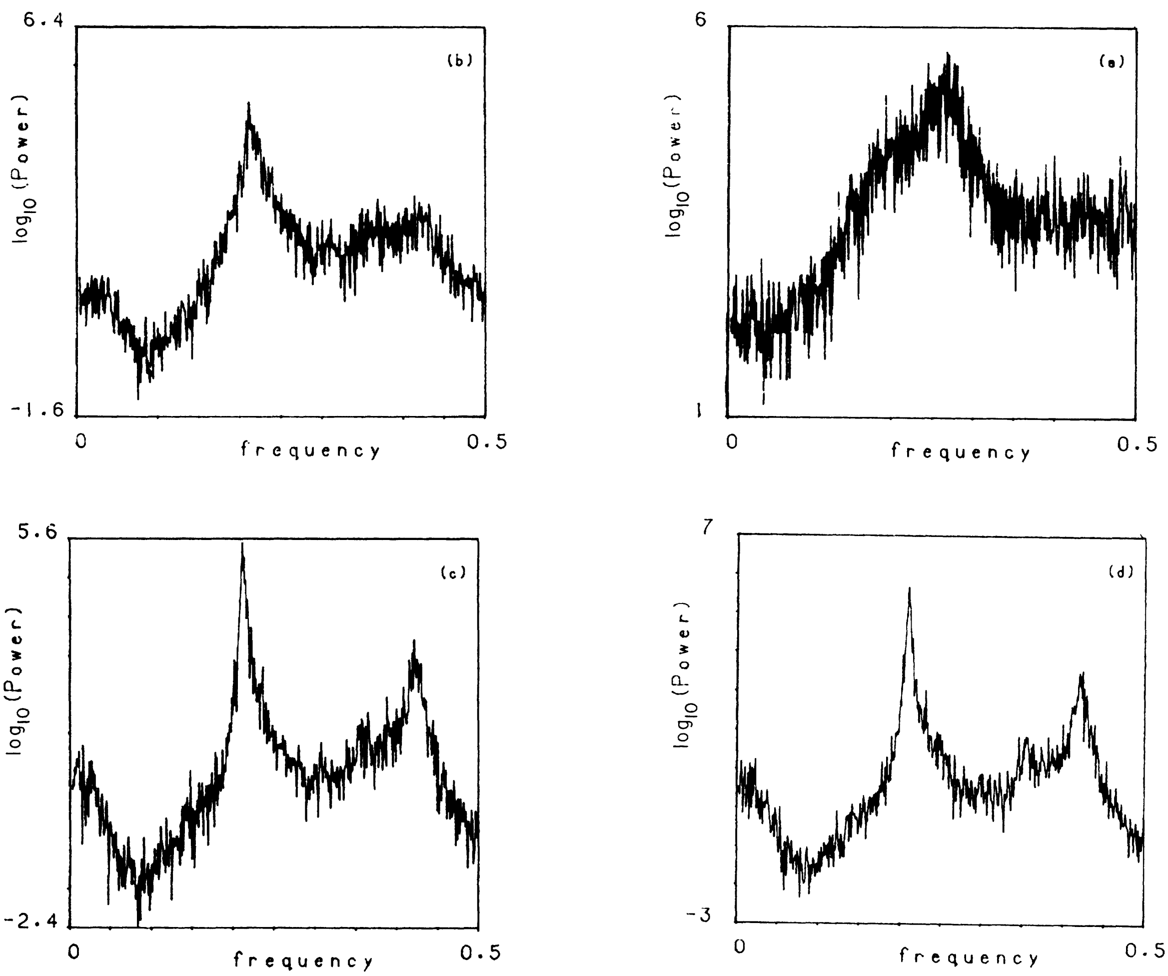

FIG. 4. Power spectra of the mean field for lattice size N equal to (a) 10, (b) 100, (c) 500, (d) 1000. Here we average over eight runs of length 1024 each, and ϵ is fixed at 0.3. Notice that the peaks have almost saturated in the last two figures. Both components of the mean field give identical power spectra.

where the exponent is approximately 2 [4]. However, the scaling may be very gross, and may describe only the overall features. More examples will have to be studied in order to have a conclusive answer to this question.

The Fourier transform of the mean field also reveals the emergence of order as the number of sites in the lattice is increased. In Figs. 4(a)–4(d), we have plotted the power spectrum for four different lattice sizes. It is clear that this power spectrum develops some very prominent peaks as N is increased. The position of the fully developed peaks depend on the value of ϵ. We can quantify the sharpness of the peaks in these power spectra by an autocorrelation function, which is defined by

$$C \equiv \frac{1}{M} \sum_{i=1}^{M} \frac{\sum_{j=1}^{M} P(j+i \bmod M) P(j)}{\sum_{j=1}^{M} P(j) P(j)} , \tag{7}$$

where $P(j)$ is the power at the jth frequency index, and M is the number of discrete points in the spectrum. In practice we use as a measure $S \equiv -\log_{10} C$. This quantity goes to zero for flat spectra and diverges when the spectrum contains only δ-function spikes. This measure is displayed in Fig. 5. Notice that the sharpness of the spectrum saturates after a critical N. As in one-dimensional cases, the saturation of the sharpness of the power spectrum and saturation of the MSD are concurrent. (This is in contrast to the result for the GCM with complex local maps [6] where the MSD had saturated while the sharpness of the power spectra had not.) The two components of the mean field, interestingly, have virtually indistinguishable power spectra. So evidently, all components of the multidimensional mean field develop similar collective beating patterns.

Finally, we have considered the effects of static random fluctuations in the values of the parameters of the model. To do this we have simulated a map of the form

$$x_{n+1}(i) = f_x(x_n(i), y_n(i); \alpha(i), \beta(i)) + \frac{\epsilon(i)}{N} h_n^x , \tag{8a}$$

$$y_{n+1}(i) = f_y(x_n(i), y_n(i); \alpha(i), \beta(i)) + \frac{\epsilon(i)}{N} h_n^y , \tag{8b}$$

with f_x and f_y as defined before, and where the now local parameters are defined by

$$\alpha(i) = [1 + \sigma_\alpha \xi(i)] \alpha \tag{9}$$

and similar expressions for $\beta(i)$ and $\epsilon(i)$. Here $\xi(i)$ is a random number between -0.5 and 0.5, and σ is the amplitude of the noise. It has been found [4] that for one-dimensional maps, the introduction of such small static fluctuations does reduce the value of the MSD after the critical N, but that after some larger value of N, the MSD not only saturates but actually grows until it reaches the value where it had saturated in the absence of parametric fluctuations. Surprisingly, what we find here is quite different. In this two-dimensional GCM, noise restores regular statistical behavior of the MSD, after a critical lattice size (see Fig. 7). Until this critical size, the MSD lies between the nonstatistical curve of the system without noise and the $1/N$ line. With increasing noise strength of the MSD lies closer to the statistical prediction. After the critical size, the MSD drops sharply towards the $1/N$ prediction, and statistical behavior is recovered. (This trend is also in contrast to that found in the GCM with complex local maps [6], where noise immediately restores statistical properties.) There was an additional feature of the power spectra of the mean field under noise in one-dimensional systems: it was found that the rough periodicities observed in the mean field actually persisted up to a reasonably large strength of noise [7]. Furthermore, the sharpness of the spectrum actually increased with the addition of noise in the dynamics up to a critical noise strength [7]. Here, too, we observe this surprising feature, as is clearly evident in Fig. 6. The rise in the sharpness of the power spectra with noise amplitude (for small σ) is approximately linear. This strange

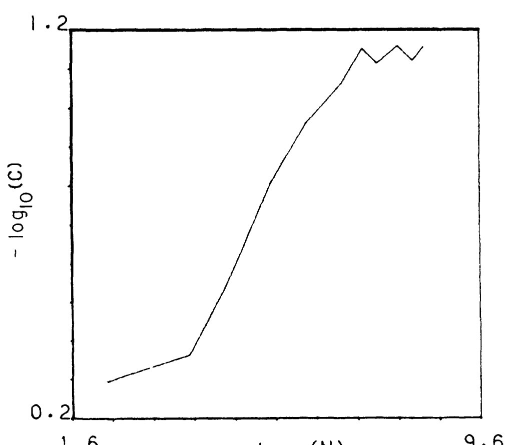

FIG. 5. Measure of the sharpness of peaks in the power spectra, as defined in the text vs the number of elements coupled. Here $\epsilon = 0.3$, and we average over eight runs of length 1024 each.

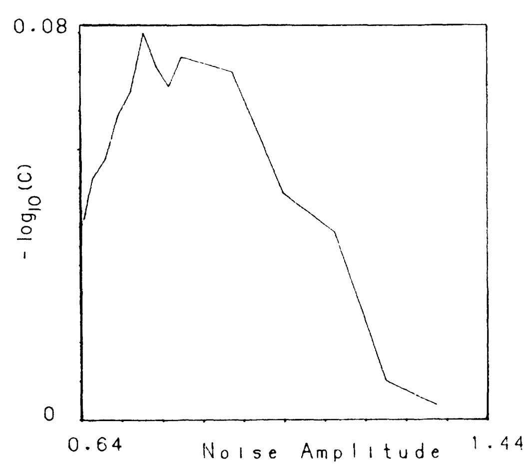

FIG. 6. Measure of the sharpness of peaks in the power spectra, as defined in the text, vs the strength of noise ($\epsilon = 0.3$, $N = 2000$).

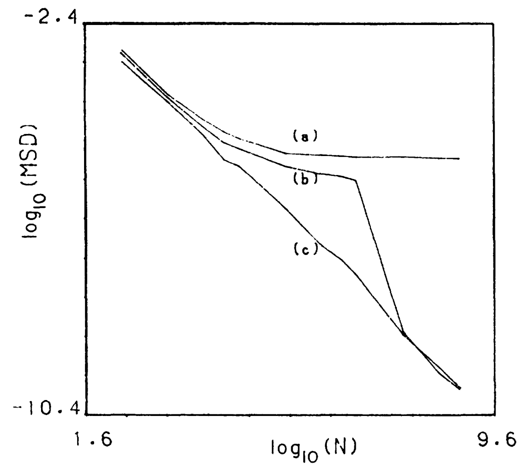

FIG. 7. MSD (for the x component of the mean field) in the presence of static fluctuations in the parameters. We have performed trials with $\epsilon=0.3$, and with $\sigma_\alpha=\sigma_\beta=\sigma_\epsilon$ equal to (a) 0.0, (b), 0.02, and (c) 0.07. The MSD of the two components of the mean field get closer quantitatively under noise.

phenomenon is another instance of stabilization of periodic motion through small noise, resembling in a way the phenomenon of stochastic resonance [8].

IV. CONCLUSIONS

Here we have investigated various aspects of the dynamics of the mean field in a globally coupled system of two-dimensional chaotic maps. This system presents a simple model for the study of global coupling in systems with higher-dimensional local dynamics.

As in previously studied one-dimensional cases and a case with complex local maps, we have found that the mean field shows evidence of the violation of the law of large numbers. These violations are clear after the lattice reaches a critical value N_c. In this regime the mean-square deviation (MSD) stops decreasing with N, and instead saturates to a fixed value. At the same time, the distribution of the mean field is clearly non-Gaussian. There is some evidence of violation of statistical laws even before it reaches N_c, since for large values of the coupling parameter the MSD decays as $N^{-\phi}$, with $\phi < 1$. Another evidence of this anomalous behavior is the emergence of several peaks in the power spectrum for the time sequence of the mean field. This indicates the emergence of a subtle coherence in the system, even though the individual mappings look completely unsynchronized. Finally we found that small random fluctuations in the parameters of the local maps restores statistical behavior after a critical lattice size N_σ. Before N_σ, the MSD of the mean field lies between the nonstatistical prediction (without noise) and the statistical prediction of $1/N$.

ACKNOWLEDGMENTS

One of us (S.S.) would like to thank G. Perez for stimulating discussions on GCM's, and the Theoretical Physics Division, BARC, for supporting this work.

[1] K. Kaneko, Prog. Theor. Phys. **72**, 480 (1984); **74**, 1033 (1985); Prog. Theor. Phys. (Suppl.) **99**, 263 (1989); J. Crutchfield and K. Kaneko, in *Direction in Chaos,* edited by B.-L. Hao (World Scientific, Singapore, 1987); P. Alstrom and R. K. Ritala, Phys. Rev. A **35**, 300 (1987); P. Hadley and K. Wiesenfeld, Phys. Rev. Lett. **62**, 1335 (1989); R. V. Sole and J. Valls, Phys. Lett. A **153**, 330 (1991); C. Bracikowski and R. Roy, Phys. Rev. A **43**, 6455 (1991).

[2] K. Kaneko, Phys. Rev. Lett. **63**, 219 (1989); Physica D **41**, 137 (1990).

[3] K. Kaneko, Phys. Rev. Lett. **65**, 1391 (1990).

[4] K. Kaneko (unpublished).

[5] R. Graham, S. Isermann, and T. Tel, Z. Phys. B **71**, 237 (1989).

[6] G. Perez, S. Sinha, and H. A. Cerdeira, Physica D (to be published).

[7] G. Perez, C. Pando-L., S. Sinha, and H. A. Cerdeira, Phys. Rev. A **45**, 5469 (1992).

[8] L. Gammaitoni *et al.*, Phys. Rev. Lett. **62**, 349 (1989); Phys. Lett. A **142**, 59 (1989). For a comprehensive review, see F. Moss, in *Some Problems in Statistical Physics,* edited by G. H. Weiss (SIAM, Philadelphia, 1992).