Fluctuations in the time periods of a model chaotic system

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We study the fluctuations in the density of time periods of the closed orbits of a model chaotic system. The work is motivated by a similar study on integrable systems [D. Biswas (unpublished)], where the existence of universalities in the fluctuation measures is established. We find that the nearest-neighbor spacing distribution exhibits linear repulsion and is well approximated by a one-parameter additive random matrix model. The spectral rigidity is more revealing and clearly shows that the fluctuations lie between those of Poisson-distributed periods and the Gaussian orthogonal ensemble of matrices. The question of universalities, however, remains open. This study should serve as a guide in settling the issue.

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Studies carried out on the fluctuations of the quantum spectra in time-independent Hamiltonian systems reveal the existence of universality classes [1]. It is now known [2,3] that their origin lies in the nature of the underlying classical dynamics. Thus systems as diverse as a hydrogen atom in a strong uniform magnetic field and the abstract hyperbola billiard display similarities in their spectral fluctuations due to the chaotic nature of the respective classical systems. Generic integrable systems on the other hand belong to an altogether different universality class. The levels here are Poisson distributed, while those for the chaotic case exhibit fluctuations that are similar to those of the eigenvalues of random matrices chosen from ensembles that reflect the presence or absence of antiunitary symmetries.

Most of our current understanding is based on the semiclassical periodic orbit theory [4], which allows one to express the density of states as an infinite sum over periodic orbits alone. Thus the spectrum can in principle be obtained from a knowledge of the time periods and the associated stability parameters. The inverse problem has also attracted some attention and it is now established [5–7] that, in homogeneous systems, the energy eigenvalues alone contain information about the lengths and stability properties of individual periodic orbits. Recently, Biswas [7] has extended this duality further for integrable systems and shown that the fluctuations in the density of time periods contain universalities that are identical to those for the energy eigenvalue spectrum. Thus a sequence of "unfolded" time periods (mean density unity) in a generic integrable system has a nearest-neighbor level-spacing distribution (NNSD) that is Poissonian. Higher-order correlations behave in an identical fashion as well. Interestingly, the outer scale in the time period spectrum is shown [7] to depend on the ground-state energy of the quantal system. It is of considerable interest to investigate the extence of universalities (if any) for the chaotic case. With this in mind, we study the fluctuations in the density of time periods for a model chaotic system.

The hypothetical Riemann ζ system [3] is a Hermitian operator with eigenvalues E_n, which are the nontrivial solutions of $\zeta(\frac{1}{2}+iE)=0$, where ζ is the Riemann ζ function, $\zeta(s)=\sum_{n=1}^{\infty} 1/n^s$. The connection with classical mechanics follows from the striking similarity between the density of E_n,

$$d(E)=\langle d(E)\rangle-\frac{1}{\pi}\sum_{p}\sum_{k}\frac{\ln(p)}{p^{k/2}}\cos[Ek\ln(p)] \quad (1)$$

and Gutzwiller's semiclassical approximation to the density of energy eigenstates [4]. The hypothetical time periods are thus $k\ln(p)$ where k is the repetition number. It is trivial to verify that the periodic orbits for the system do proliferate exponentially as in other chaotic systems. There are two other important properties for chaotic systems without any symmetry that are obeyed by these periodic orbits. The first is a classical sum rule, which constitutes a relationship between the periods and stability parameters [8] while the second is a semiclassical sum rule that follows from the relationship between the orbits and the energy eigenvalues E_n in the semiclassical limit [3,9]. In a sense these are consistency conditions that must be satisfied so that the hypothetical time periods belong to a Hamiltonian system.

We shall work with the hypothetical time periods, $k\ln(p)$ since a large number of these can be easily generated. For $k=1$ (primitive orbits alone), the unfolding can be achieved using the mean integrated density of time periods,

$$N_{\rm av}(T)=\frac{e^T}{T}\left[1+\frac{1}{T}+\cdots+(n-1)\frac{1}{T^n}\right]. \quad (2)$$

If multiple repetitions are taken into consideration, a suitable polynomial fit is used to achieve a unit mean density.

We evaluate here the nearest-neighbor spacing distribution $P(s)$ and the spectral rigidity Δ_3 for the unfolded time periods. The distribution $P(s)$ is defined such that $P(s)ds$ is the probability of finding pairs of successive unfolded time periods (levels) with spacing between s and $s+ds$. For systems that are classically integrable, $P(s)$ is the Poisson distribution e^{-s}, if the corresponding quantal eigenenergies are nondegenerate [7]. As mentioned earlier, the result is identical to that for the quantal eigenvalue spectrum of generic integrable systems. Of the higher-order correlations, the spectral rigidity Δ_3 is a useful measure and also easy to evaluate. It is defined as the average mean-square deviation of the integrated density of time periods from the best-fitted straight line. Thus

$$\Delta_3(L)=\left\langle \min_{a,b} \int_{-L/2}^{L/2} d\tau[N(x+\tau)-a-b\tau]^2 \right\rangle . \quad (3)$$

The averaging here is performed over an interval large compared to the outer scale $L_{\max}$, which for integrable systems [7] is equal to $\langle d(T)\rangle/\lambda_{\min}$, where $\lambda_{\min}$ is a function of the ground-state energy [the equivalent outer scale for the eigenenergy spectrum is $h\langle d(E)\rangle/T_{\min}$, where $T_{\min}$ is the smallest time period of the corresponding classical system]. For values of $L \ll L_{\max}$, $\Delta_3(L)$ for periodic orbits belonging to generic integrable systems equals $L/15$ [7] as in case of the eigenenergies [3].

Since no theory exists for the density fluctuations of time periods in chaotic systems, we shall assume a form for the outer scale similar to that for integrable systems. The averaging interval then increases exponentially with the time period and hence we shall restrict ourselves to a sequence of 3000 periods after eliminating the first 500. Moreover, since most of them are primitive orbits, we present our results for $k=1$. There are no visible changes if repetitions are allowed as well.

Figure 1 shows a plot of the nearest-neighbor spacing distribution plotted in the interval [0,3]. Since a histogram representation is often misleading, we have expanded $P(s)$ in terms of Laguerre polynomials, $L_n(s)$:

$$P(s)=\frac{1}{N}\sum_{i=1}^{N}\delta(s-s_i) \quad (4a)$$

$$=\sum_n C_n L_n(s)e^{-s} , \quad (4b)$$

where C_n are the coefficients and $N+1$ is the total number of levels. Usually, excellent convergence is obtained by retaining only 15 terms. We have evaluated $P(s)$ using 20 terms and verified its convergence by comparing the cumulative distributions of Eqs. (4a) and (4b).

Clearly the peak in Fig. 1 (curve 1) occurs at about 0.45 and has a value equal to 0.83. A comparison with the predictions of the Guassian orthogonal (curve 3 of Fig. 1) or unitary ensembles is thus meaningless, since the peaks in these cases occur for larger values of s. The best-fit Brody distribution shows considerable deviations as well. We have thus used the distribution,

$$P_\lambda(s)=\left[\frac{su(\lambda)^2}{\lambda}\right]\exp\left[-\frac{u(\lambda)^2s^2}{4\lambda^2}\right] \times\int_0^\infty e^{(-\zeta^2-2\zeta\lambda)}I_0\left[\frac{s\zeta u(\lambda)}{\lambda}\right]d\zeta \quad (5)$$

for an additive random matrix ensemble [10] parameterized by λ. Here I_0 is the modified Bessel function and $u(\lambda)=\sqrt{\pi}U(-1/2,0,\lambda^2)$, where U is the Tricomi function [11]. The fit is good for $\lambda=0.3$, though slight deviations exist near the peak. We have evaluated $P(s)$ for various other ranges of time periods as well, but have noticed no significant changes.

Figure 2 shows a plot of the spectral rigidity in comparison with the results of Poisson distributed periods (curve 1) and the Gaussian othogonal ensemble (GOE) of random matrices (curve 3). For values of $L \ll 3$, the agreement with the latter is good (not shown in the figure), though in the expected region of universality ($1 \ll L \ll L_{\max}$) the curve lies between the two extreme cases. In order to have a more quantitative description, we have fitted the numerically obtained Δ_3 for values of L in the range [3,6] to the one-parameter rigidity,

$$\Delta_3(L;\nu)=\Delta_3^{\text{Poisson}}(\nu L)+\Delta_3^{\text{GOE}}(\bar{\nu}L) , \quad (6)$$

where $\nu+\bar{\nu}=1$. The fitted values (curve 2 in the figure) give a good approximation for $\nu=0.2023$.

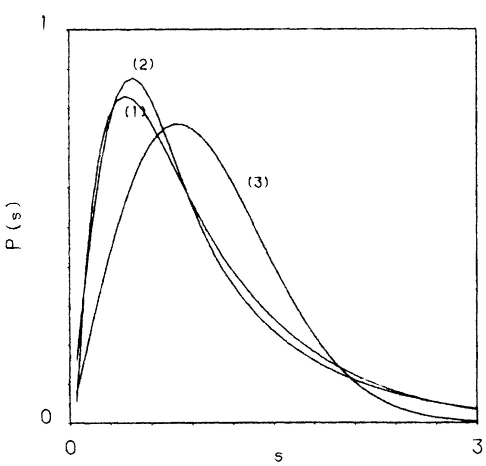

FIG. 1. Nearest-neighbor spacing distribution $P(s)$ of unfolded time periods (curve 1) in comparison with the best-fit additive random matrix distribution [Eq. (5)] for $\lambda=0.3$. Curve 3 is the Wigner surmise.

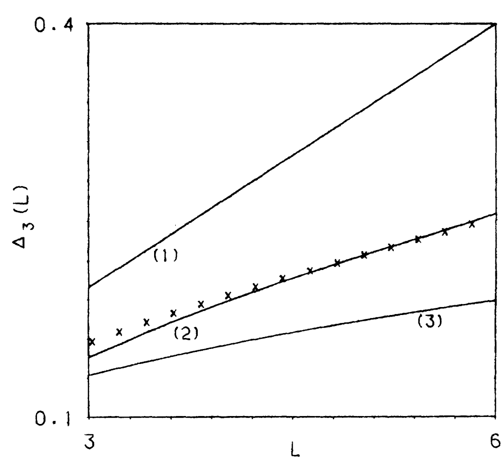

FIG. 2. The spectral rigidity Δ_3 of the unfolded time periods ($\times$) along with the Poisson (curve 1) and Gaussian orthogonal ensemble (curve 3) results. Curve 2 is the best fit of our data to Eq. 6 (see text for details).

In summary, we have carried out a study on the fluctuations in the density of time periods of a model chaotic system. The work is motivated by a similar study on integrable system [5], where the existence of universalities in the fluctuation measures is established. For chaotic systems, the only other study, to the authors' knowledge, has been on the nearest-neighbor spacing distribution of the time periods in a hyperbola billiard [12]. The clustering observed [12] is perhaps due to the symmetries in the system and hence should be considered nongeneric. In the present case, we observe a linear repulsion in the NNSD for small s values followed by a peak at $s \simeq 0.45$ and a rapid decay thereafter. The curve is well approximated by a one-parameter additive random matrix model (in the histogram representation, the fit would be considered excellent), which has incidentally been used recently [13] to study the spacing distribution of a singular billiard system. The spectral rigidity is more revealing, however, for it clearly shows that the fluctuations lie between those of Poisson-distributed periods and the Gaussian orthogonal ensemble of matrices.

We have recently become aware of a similar study by Harayama and Shudo [14] on a class of dispersing billiards (chaotic) addressing the question of universality in the fluctuations of the length spectrum. They obtain a Poisson nearest-neighbor spacings distribution and conjecture that this property is universal, at least in all hyperbolic systems. In the model chaotic system studied in this paper, the essential properties of the length spectrum are identical [15] to those of the Hadamard-Gutzwiller model. In light of this, our results are noteworthy. An analytic study of the fluctuation measures is currently in progress [16].

The question of universalities, if any, in the fluctuations of time periods, however, remains open. This study should serve as a guide in setting the issue.

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