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Nonstatistical behavior of coupled optical systems

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We study globally coupled chaotic maps modeling an optical system, and find clear evidence of nonstatistical behavior: The mean-square deviation (MSD) of the mean field saturates with respect to an increase in the number of elements coupled, after a critical value, and its distribution is clearly non-Gaussian. We also find that the power spectrum of the mean field displays well-defined peaks, indicating a subtle coherence among different elements, even in the "turbulent" phase. This system is a physically realistic model that may be experimentally realizable. It is also a higher-dimensional example (as each individual element is given by a complex map). Its study confirms that the phenomena observed in a wide class of coupled one-dimensional maps are present here as well. This gives more evidence to believe that such nonstatistical behavior is probably generic in globally coupled systems. We also investigate the influence of parametric fluctuations on the MSD.

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I. INTRODUCTION

Global coupling in dynamical systems yields a host of very novel features. This class of complex systems is of considerable importance in modeling phenomena as diverse as Josephson junction arrays, multimode lasers, vortex dynamics in fluids and even evolutionary dynamics, biological information processing, and neurodynamics. The ubiquity of globally coupled phenomena has thus made it a focus of much recent research activity [1].

A globally coupled map (GCM) is a dynamical system of N elements evolving according to local mappings and a "mean-field"-type interaction term through which the global information influences the individual elements. It is thus analogous to a mean-field version of coupled map lattices [2]. The general form of a GCM is

$$x_{n+1}(i)=f_1(x_n(i))+\epsilon G\left[\frac{1}{N}\sum_{j=1}^{N} f_2(x_n(j))\right], \qquad (1)$$

where n is a discrete time step, i is the index of elements ($i=1,2,\ldots,N$), and f_1, f_2, and G denote different functions. The mean field h is the argument of the function G, and ϵ is the coupling parameter.

It has been noticed that one-dimensional GCM's (for example, globally coupled logistic maps) have two conflicting trends: destruction of coherence due to the chaotic divergences of the individual elements and a synchronizing force through the global averaging [2]. This means that, as a function of the coupling ϵ, the dynamics of the GCM can go from a phase of completely incoherent chaotic motion, through phases of partial synchronization, to a phase of global synchronization, where this synchronized motion can be chaotic or regular. A very surprising result was found by Kaneko [3]: in the fully turbulent phase, where coherence is completely destroyed by chaos in the individual maps and there is no explicit manifestation of correlation among the elements, a subtle collective behavior emerges. Since all the state variables take quasirandom values almost independently, one may expect that the mean field will obey the central-limit theorem and the law of large numbers. If this were true, the mean-square deviation (MSD) ($\equiv\langle h^2\rangle-\langle h\rangle^2$) would decrease as N^{-1}, where N is the number of elements coupled, and the mean field would converge to a fixed value as $N\rightarrow\infty$; also, for finite N, the distribution of h would be Gaussian. Examination of the above expectation in one-dimensional maps showed that the mean field respected the central-limit theorem [3] (at least approximately, see Ref. [4]) but violated the law of large numbers. In fact, the MSD stopped decreasing after a critical value of N. Further, it was observed that the power spectrum of h had broad peaks. This result indicates the emergence of some order, a partial coherence in the dynamics.

In this paper we study a physically realistic GCM that is, in principle, experimentally realizable. This GCM is comprised of individual complex Ikeda mappings [5], which describe chaos in optical bistability. First, we discuss the model and the physical meaning of the global coupling. Then we examine phenomenologically the dynamics of the mean field, and study the behavior of the MSD with respect to the number of elements coupled. There we find evidence of violation of both the central limit theorem and the law of large numbers, and broad peaks in the power spectrum of the mean field. Finally, we investigate the influence of static fluctuations on the parameters of the system.

II. THE MODEL

Our system comprises a set of optical devices where the individual dynamics is described by an Ikeda map [5], plus a coupling term. Every device has arbitrary initial conditions and is driven weakly by a mean electric field, which is obtained as an average over the electric fields inside all the devices. The latter gives rise to the coupling between the maps. Our motivation in studying this sys-

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tem is twofold: this model is physically realistic and could be, in principle, subject to experimental realization, and is a higher-dimensional example of globally coupled maps, as the local mappings in our system are complex.

Let us describe more accurately one element of our system. An optical device is a unidirectional ring cavity containing an active medium of two-level atoms homogeneously broadened and interacting with a coherent electromagnetic wave. The plane-wave, slowly varying amplitude and rotating-wave approximations are assumed [6]. Ikeda analyzed the case when propagation effects become important, and showed that the amplitude of the slowly varying envelope of the electric field for successive round trips obeys the mapping

$$E_{n+1}=A+BE_n e^{i|E_n|^2} \tag{2}$$

when (i) the active medium has a longitudinal relaxation time much smaller than the cavity round-trip time, and (ii) the injected field is off resonance with the medium [5]. In Eq. (2) the parameter A is proportional to the coherent external field and B is an attenuation factor. The Ikeda instabilities were observed in a hybrid optically bistable device with a delayed feedback [7], and also in an all-optical bistable device using a single-mode optical fiber as a nonlinear medium in a ring cavity pumped by a train of mode-locked pulses [8].

The model we propose here consists of a large amount of optical devices coupled through a mean field (see Fig. 1). We extract a small fraction of the output of each device. These signals are then mixed and fed to an optical amplifier. The amplified mean field is subsequently redistributed and added to the input field of the different devices. We assume that the time delays for the individual mappings ($YZWX$ in Fig. 1) and for the mixing and amplifying part ($YQRSPX$) are the same. We also assume a perfectly linear amplifier. Including the new field in the mapping, we obtain the equation

$$\begin{aligned} E_{n+1}(j)&=A+Bf(E_n(j))+\epsilon h_n \ , \\ h_n &\equiv \frac{1}{N}\sum_{k=1}^{N} f(E_n(k)) \end{aligned} \tag{3}$$

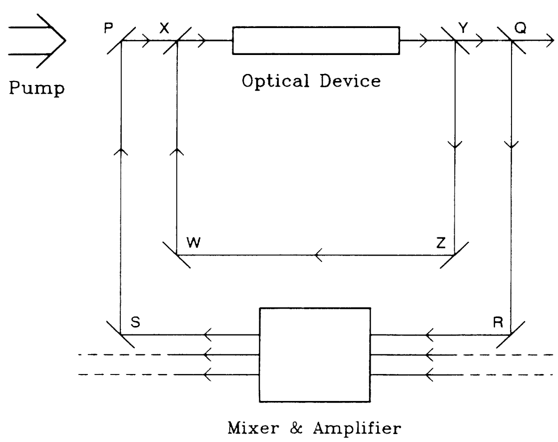

FIG. 1. Schematic diagram of the setup of the model. Here we show only one element of the lattice.

where we have defined $f(z)\equiv z\exp(i|z|^2)$, and h_n is the mean field at time n.

A note should be made about the introduction in our model of an amplifier to provide a modifiable mean field. In principle, we could consider a model where the mean field is obtained directly from a fraction of the output of each single device. In this model we would need to change B to $B-\epsilon$ in Eq. (3). We prefer not to do so since these changes of the nonlinear parameters of the individual maps obscure the analysis of the results [9]. Besides, in practice, it is simpler to alter the gain of one amplifier than the reflectance-transmittance ratio of the very large amount of mirrors we are considering here.

III. RESULTS

We have simulated Eq. (3) with the (real) parameters $A=3.0$, $B=0.3$, and different values of N and ϵ. For the single Ikeda map these parameters are located in the region of completely chaotic behavior. In all cases considered, we have checked to see that the coupled dynamics is not synchronized.

First we have checked how close to a Gaussian shape the distribution found for the h's is. For this, we have compared our numerical results for two different values of ϵ, with a bivariate Gaussian distribution a with means $\bar{x},\bar{y}$, dispersions σ_x,σ_y, and correlation index ρ. Here x and y are the real and imaginary parts of h. In each case we have calculated numerically the five parameters of the distribution, and then have used some higher moments to compare the actual data with the proposed distribution. First we shift the origin of coordinates to the mean, then, for a given direction in space [i.e., for a variable $z(\theta)\equiv\cos\theta(x-\bar{x})+\sin\theta(y-\bar{y})$], odd moments should be zero, and for n even, the ratio between the nth moment and the $(n/2)$th power of the second moment should be $R_n=(n-1)(n-3)\cdots(1)$. We have calculated numerically the angular average of the deviation from these values

$$\mathcal{E}_n=\frac{1}{\pi}\int_0^{\pi}d\theta\left|\frac{\langle z^n(\theta)\rangle}{\sqrt{\langle z^2(\theta)\rangle^n}}-R_n\right| . \tag{4}$$

TABLE I. Values for the parameters of the distribution of h in the coupled Ikeda maps. The "Test" case was produced by simulating two Gaussian distributions, in x and y, with the same number of samples as the other two cases, and is used to compare the normal levels of error for this number of samples.

Parameter	$\epsilon=0.03$	$\epsilon=0.04$	Test
$\bar{x}$	0.0190	0.0208	−0.000866
$\bar{y}$	0.0832	0.0756	−0.000362
σ_x	0.0220	0.0360	1.000
σ_y	0.0264	0.0520	1.000
ρ	0.438	0.682	0.00354
$\mathcal{E}_3$	0.223	0.231	0.00490
$\mathcal{E}_4$	0.236	0.600	0.0109
$\mathcal{E}_5$	2.05	1.43	0.0433
$\mathcal{E}_6$	3.37	6.36	0.176

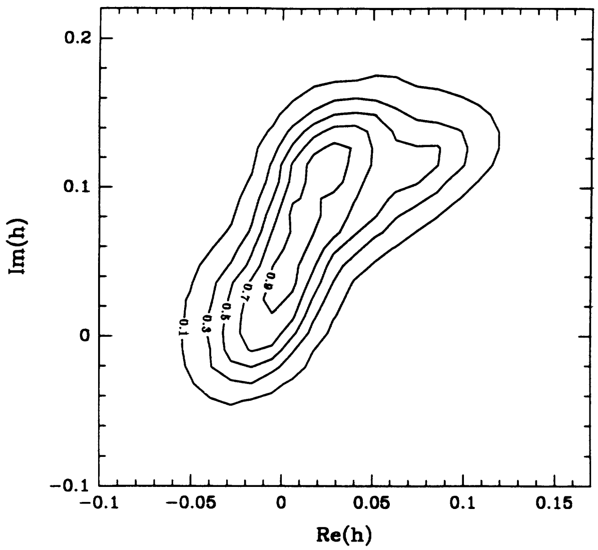

FIG. 2. Contour plot of the distribution of h for $\epsilon=0.04$, $N=40\,000$. Here and in all other simulations we have used $A=3.0$, $B=0.3$, and there is a transient of 5000 iterations. The distribution was built using 100 runs of 1024 iterations each, and is normalized with its maximum height equal to 1.

Here $R_n=0$ for n odd and as defined above for n even. The results for two trials are given in Table I. These results show clearly that the distribution is far from a Gaussian [10] shape after the saturation of the MSD. A contour plot of the histogram of h for $\epsilon=0.04$ is given in Fig. 2, showing the asymmetric spreading of the tails of the distribution.

Second, we have calculated the mean-square deviation $\mathcal{D}$ of the absolute value of the fluctuations of the mean field

$$\mathcal{D}=\frac{1}{T}\sum_{j=1}^{T}(h_j-\bar{h})^*(h_j-\bar{h}) \tag{5}$$

as a function of N. Here h_j is the mean field obtained at iteration j and $\bar{h}$ is the average obtained over the (very large) number of iterations T. This MSD does decrease as N grows up to a critical value N_c, and then saturates, as can be observed in Fig. 3. For large values of the coupling parameter ϵ this decrease differs consistently from the $1/N$ behavior predicted by the law of large numbers. This suggests that the nonstatistical behavior of the process manifests itself well before the saturation value N_c. We can then say that the MSD for the mean field h decays as a power α, with $\alpha\le 1$, up to a critical value N_c after which it stabilizes.

We have also checked the behavior of the MSD with respect to the value of the coupling ϵ. This behavior is very different in the case of models where the local maps are also modified by the introduction of the coupling [3,9]. In the model we are considering, the MSD seems to grow monotonically with the coupling, with very slow growth in the region of saturation. The data shown in Fig. 4 give some support to the hypothesis [4] of a scaling of the MSD with ϵ, as

$$\mathcal{D}(\epsilon)\approx\mathcal{D}(0)\epsilon^{\alpha}\ , \tag{6}$$

with $\alpha\approx 2$, but the data at the moment do not allow us to give a conclusive answer.

The Fourier transform of the mean field also reveals the emergence of order as the number of sites in the lattice is increased. In Fig. 5 we have plotted the power spectrum for five different lattice sizes. It is clear that this power spectrum develops some very prominent peaks as N is increased, peaks that do not correspond to the position of the mild humps that can be seen for small values of N. We have repeated the calculation of the Fourier spectrum for $\epsilon=0.03$, and the results are quite similar in

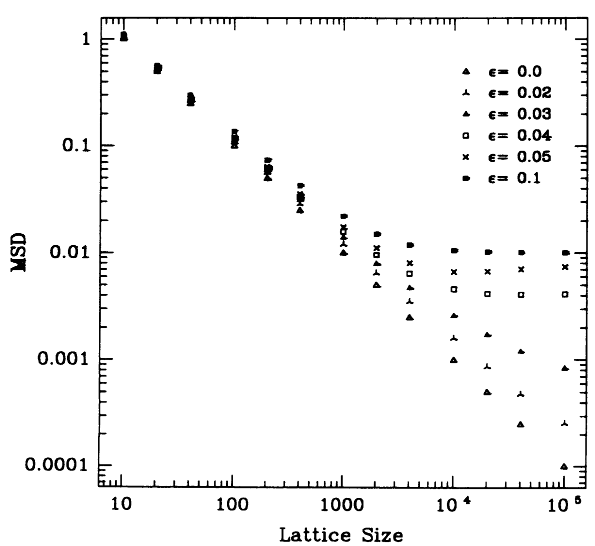

FIG. 3. Mean-square deviation of the mean field vs lattice size N, as six different values of ϵ. In all cases we have used 50 runs of 1024 iterations each.

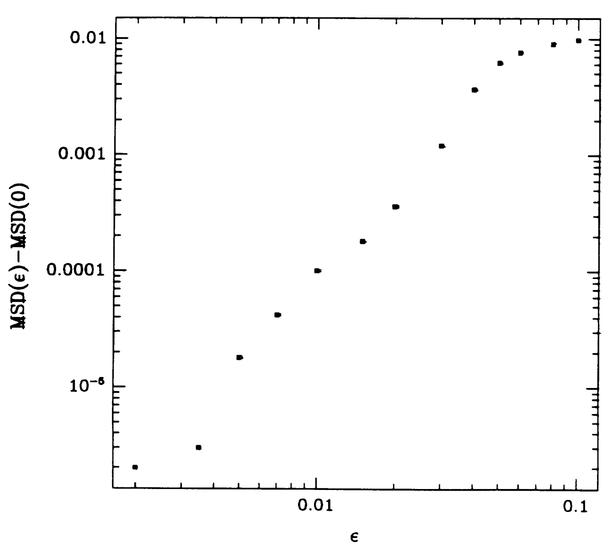

FIG. 4. Mean-square deviation vs global coupling parameter ϵ. Here $N=20\,000$ and we are using 50 runs of 1024 iterations each.

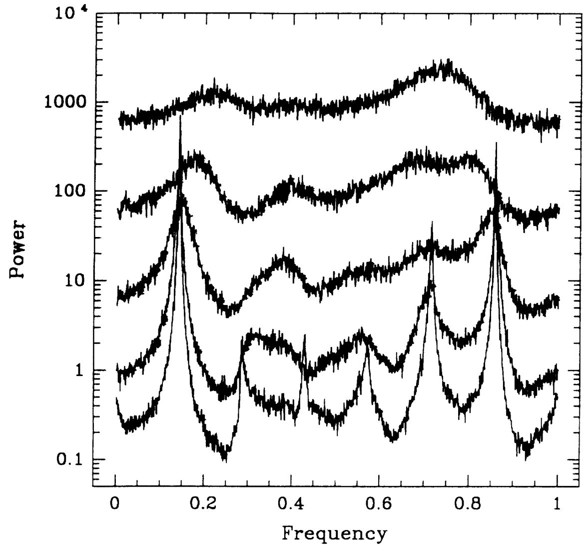

FIG. 5. Power spectra of the mean field for lattice size $N=10$, 100, 1000, 10^4, and 10^5 (from top to bottom). Here we average over 50 runs of length 1024 each, and ϵ is fixed at 0.04. Notice that the peaks of the spectrum fall in the harmonics of $\omega=\frac{1}{7}$.

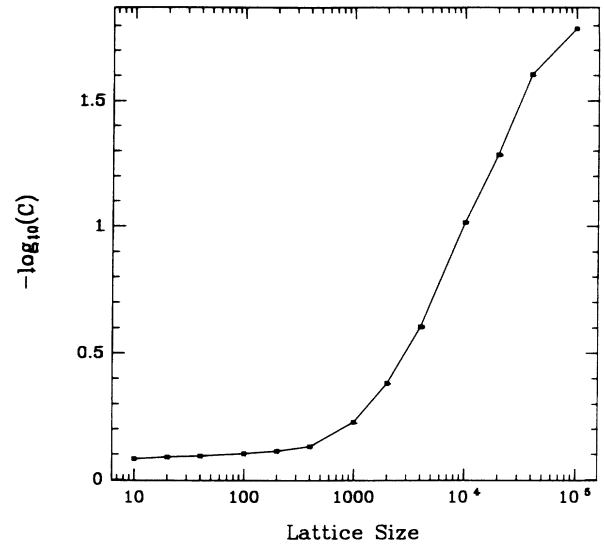

FIG. 6. Measure of the sharpness of peaks in the power spectra, as defined in the text. Here $\epsilon=0.04$, and we are averaging over 50 runs of 1024 iterations each.

shape and in the position of the peaks. This similarity does not occur [3,9] when the local maps depend on ϵ. We can quantify the sharpness of the peaks in these power spectra by using their autocorrelation, which is defined by

$$C\equiv\frac{1}{M}\sum_{i=1}^{M}\frac{\sum_{j=1}^{M}P(i+j \bmod M)P(j)}{\sum_{j=1}^{M}P(j)P(j)}\ , \tag{7}$$

where $P(j)$ is the power at the jth frequency index, and M is the number of discrete points in the spectrum. In practice, we use as a measure $S\equiv-\log_{10}C$. This quantity goes to zero for flat spectra and diverges when the spectrum contains only δ spikes. This measure is displayed in Fig. 6. Notice that the sharpness of the spectrum keeps growing even after the corresponding MSD has saturated. We have not been able to reach the saturation region for the power spectrum, even using lattices of 10^5 elements. This situation is very different from the one encountered in the one-dimensional cases, where saturation of the power spectrum and of the MSD were concurrent. We do expect the sharpness of the power spectrum to saturate at some larger value of N, but the verification of this falls outside our computational resources.

Finally, we have considered the effects of static random fluctuations in the values of the parameters of the model (Fig. 7). To do this we have simulated a map of the form

$$E_{n+1}(i)=A(i)+B(i)f(E_n(i))+\epsilon(i)h_n\ , \tag{8}$$

with f and h defined as before, and where the now local parameters are defined by

$$A(i)=[1+\kappa_A\xi(i)]A\ , \tag{9}$$

and similar expressions for $B(i)$ and $\epsilon(i)$. Here $\xi(i)$ is a random complex number whose real and imaginary parts are normal deviates, and κ_A is the amplitude of the noise. It has been found [4] that, for one-dimensional maps, the introduction of such small static fluctuations does reduce the value of the MSD after the critical point N_c, but that after some larger value of N, the MSD not only saturates but actually grows until it reaches the value where it sat-

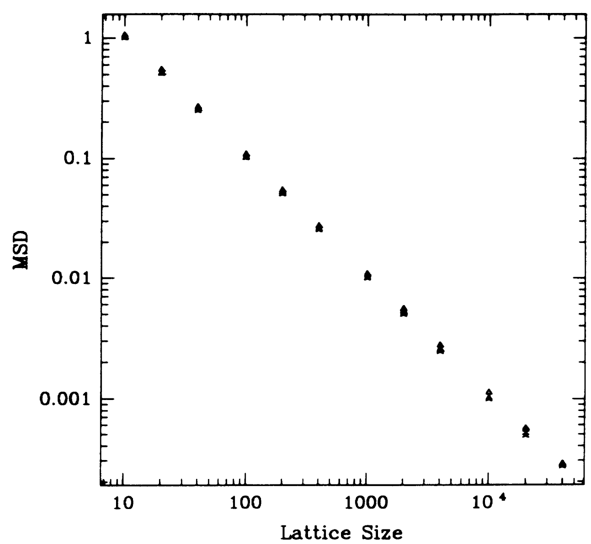

FIG. 7. MSD in the presence of static fluctuations in the parameters. We have performed two trials with $\epsilon=0.03$ and two with $\epsilon=0.04$, with $\kappa_A=\kappa_B=\kappa_\epsilon=0.01$ in one trial and 0.04 in another. For all four cases considered the noise restores statistical behavior, at least up to $N=40\,000$. The results are indistinguishable in the graph.

urates in the absence of parametric fluctuations. Due to computer limitations, we have been unable to observe this phenomenon, but we cannot completely exclude it. This, of course, represents a serious problem for the possible actual verification of the dynamics of the model.

IV. CONCLUSIONS

Here we have investigated various aspects of the dynamics of the mean field in a system of optical devices weakly coupled via a linear amplifier. This system can be modeled by a large set of Ikeda mappings, coupled through a mean field. These mappings are complex, and present a simple but physically realistic example for the study of global coupling on systems with higher-dimensional local dynamics.

As in previously studied one-dimensional cases, we have found that the mean field shows evidence of violation of the law of large numbers and of the central-limit theorem. These violations are very clear after the size of the lattice has reached a critical value N_c. In this regime the mean-square deviation stops decreasing with N, and instead saturates to a fixed value. At the same time, the distribution of the mean field becomes clearly non-Gaussian. There is some evidence of violation of the statistical laws even before N reaches N_c, since, for large values of the coupling parameter, the MSD decays as $N^{-\alpha}$, with $\alpha < 1$. Another evidence of this anomalous behavior is given by the emergence of several peaks in the power spectrum for the time sequence of the mean field. This indicates the emergence of a subtle coherence in the system, even though the individual mappings look completely unsynchronized. These peaks do not correspond to the mild humps observed in the power spectrum for smaller lattices. Finally, we found that, within the limits of our study, the introduction of very small fluctuations in the parameters of the local maps restores completely the regular statistical behavior of the MSD. Although it is clear that we cannot be sure of what happens for even larger lattices (say, 10^6 elements or more), this result suggests that the behavior in our model is different from that of the one-dimensional models studied previously. In those cases, it was found that the nonstatistical behavior of the globally coupled system was robust with respect to small parametric fluctuations. This result also suggests that it will be very difficult to observe any nonstatistical behavior in a real experiment.

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