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INTERNATIONAL CENTRE FOR
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ABSTRACT

The very surprising broad peaks seen in the power spectra of the mean field in a globally coupled map system [1], indicating subtle coherences between the elements even in the "turbulent" phase, are investigated in detail with respect to number of elements coupled, nonlinearity and global coupling strength. We find that the peaks are determined by two distinct components: effective renormalization of the nonlinearity parameter in the local mapping and the strength of the mean field iteration term. We also demonstrate the influence of background noise on the peaks – which is quite counterintuitive, as the peaks become *sharper* with increase in strength of noise, upto a certain critical noise strength.

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1. INTRODUCTION

Global coupling in dynamical systems yields a host of very novel features. This class of complex systems is of considerable importance in modelling phenomena as diverse as Josephson junction array, vortex dynamics in fluids and even evolutionary dynamics, biological information processing and neurodynamics. The ubiquity of globally coupled phenomena has thus made it a focus of much recent research activity[1-8].

In this paper we study the globally coupled map (GCM) introduced by Kaneko[1]. It is a dynamical system of N elements consisting of local mappings as well as a "mean-field" type iteration term, through which the global information influences the individual elements. It is thus analogous to a mean field version of coupled map lattices. The explicit form of the GCM is:

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j)) \quad (1)$$

where n is a discrete time step and i is the index of the elements ($i = 1,2..N$). The function $f(x)$ was chosen to be the well known dissipative chaotic logistic map.

$$f(x) = 1 - ax^2 \quad (2)$$

This choice also helps us make contact with the results of Kaneko.

The above GCM model has two conflicting trends: destruction of coherence due to the chaotic divergences of the individual elements, and a synchronizing force through the global averaging. The very surprising result Kaneko found[1] was the following: in the fully "turbulent" phase where coherence is completely destroyed by chaos in the individual maps and there is no explicit manifestation of correlation among the elements, a subtle collective behaviour emerged. Kaneko showed that in this regime there is no temporal intermittency and all Lyapunov exponents are positive[1]. Since all $x_n(i)$ almost independently take random values, one may expect that the mean field h_n defined as

$$h_n = \frac{1}{N} \sum_j f(x_n(j)) \tag{3}$$

will obey the central limit theorem and the law of large numbers. If this were true the mean square deviation (MSD) ($\equiv < h^2 > - < h >^2$) would decrease as N^{-1} and the mean field would converge to a fixed value as $N \to \infty$. Kaneko examined the above expectation and found that the mean field respected the central limit theorem, *but violated the law of large numbers*. Infact the MSD stopped decreasing after a critical value of N. Further, it was observed that the power spectrum of h had broad peaks! This result indicates the emergence of some order, a partial coherence in the dynamics.

We examine phenomenologically the dynamics of the mean field. In Section 1, the "transition" between the power spectrum of a single $x_n(i)$

(which is flat) and the spectrum of the collective quantity h_n which displays significant peaks, indicating "collective beats" so to say, is investigated. In Section 3 we examine the power spectrum of another "global" quantity, namely the probability distribution for x to take value y at different times n, and there too we find evidence of violation of the law of large numbers. In Section 4 we attempt an analysis of this emergence of order in terms of two distinct effects: one, due to the renormalization of the nonlinearity parameter in the map and the other due the strength of the mean field acting on the individual elements. Finally, we investigate the influence of noise on the partial coherences. The suprising result here is that the peaks in the power spectrum get *sharper* as the strength of noise increases, upto a certain critical noise strength. This counterintuitive phenomena is demonstrated through numerical experiments in Section 5.

2. EMERGENCE OF PEAKS IN THE POWER SPECTRUM

We want to trace the development of the peaks in the power spectrum of the mean field. Clearly, when $N = 1$, i.e. when there is a single logistic map we have a completely flat (aperiodic) spectrum. But even as we put in another element ($N = 2$) we find a "ghost" of the peak making its presence felt. So, the appearence of one broad peak in the power spectrum is almost immediate, as is evident from the power spectra for very low lattice sizes in Fig. 1. It takes larger lattices to resolve this peak into its various components. We can evaluate the autocorrelation function for these spectra, which is defined as

$$C = \frac{1}{M}\sum_{i=1}^{M} \frac{\sum_{j=1}^{M} P(j + i \bmod M)P(j)}{\sum_{j=1}^{M} P(j)P(j)} \tag{4}$$

where $P(j)$ is the value of the power at the jth frequency index, and M is the number of discrete points in the spectrum. This provides a good measure of the 'flatness' of the spectrum, and C takes value 1 when the spectrum is completely flat and value 0 when there are δ peaks. We now introduce a quantity which can serve as a sensitive indicator of the sharpness of the peaks. It is given by

$$S = -\log_{10} C \tag{5}$$

$S = 0$ is a signature of a flat spectrum (no peaks), and $S = \infty$ is the signature of (very sharp) δ peaks. We find that S increases very fast with increasing lattice size N (see Fig. 2), indicating that the peaks emerge rapidly, on addition of elements, from the flat spectrum corresponding to that of a system with a single element.

We also investigate the power spectra of partial sums, given as

$$S_m(n) = \frac{1}{m}\sum_{i=1}^{m} x_n(i) \tag{6}$$

where the $x_n(i)$ evolve under the effect of the full mean field h_n, as given by Eqn. 1. It is interesting to note that while the power spectra of $S_1(n)$ is quite flat, the increasing partial sums seem to add constructively and

conspire to give the large peak observed finally in the spectra of the mean field h_n, which is nothing but $S_N(n)$. This phenomena is displayed in Fig. 3.

3. PROBABILITY DISTRIBUTIONS

We investigate now the dynamics of the probability distribution, defined as

$$P_\delta(y;n) = \frac{1}{2\delta N}\sum_{i=1}^{N} \Theta[\delta - |x_n(i) - y|] \tag{7}$$

for small δ and large N. This quantity is invariant (within statistical fluctuations) for a logistic map in the chaotic regime. Here the individual local mapping is chaotic (nonlinearity parameter $a = 1.99$). But the mean field interaction may introduce a kind of "beating" in $P(y)$, and there is a possibility that the distribution violates the law of large numbers and is not invariant. What numerical experiments show is that, indeed the distribution violates the law of large numbers. This can be seen in Fig. 4, where the MSD of $P(y)$, at three different y's are plotted. (It should be noted that we consider well populated bins here). It is clear from the plot, that after a critical N, the MSD does not fall as N^{-1}, but instead saturates. This means that the fluctuations do not die with increasing system size, and the distribution is *not invariant*.

Further, we noticed that the power spectrum of $P(y;n)$ showed the same broad peaks as the mean field h. This can be seen by taking the first few moments of this distribution and doing a spectral analysis. (We have done the analysis for the first four moments of the probability distribution.) Fig. 5 shows the power spectrum of a representative bin where the peaks are clearly discernable. (There exist bins, though, where the peaks are less pronounced). The other interesting fact is that the position of the peaks of the "beating" bins match with that of the mean field. This suggests that the underlying rough periodicities of the mean field term, acting on the individual elements, imparts a certain rough periodicity to other "global" quantities like the probability distribution.

4. DEPENDENCE ON THE GLOBAL COUPLING PARAMETER

We now focus on the effect the global coupling parameter, ϵ, has on the deviation of the mean field h_n from its mean value. This parameter gives a measure of the strength of global averaging manifested in each individual local map. In that sense it is the source of the "synchronization" effect. It is thus instructive to study the functional dependence of the MSD on this parameter.

We have checked the value of MSD of the mean field h as a function of ϵ. At first viewing the functional dependence seems very erratic. Moreover,

in the explored range of ϵ (0.0 - 0.2), the maximum value of the MSD was found to be one order of magnitude larger than the value at $\epsilon = 0.1$, where most of the work has been concentrated up to now[1].

We now attempt an explanation for this apparently non-systematic behavior, and in particular the surprisingly large values of MSD found in certain small ranges of ϵ. This can partially be accounted for if we consider the effects of the coupling as divided roughly into two components. The first one is the renormalization of the nonlinear paramerter a by the introduction of the multiplicative $(1-\epsilon)$ term in the individual maps. The other is through the mean field, whose effective strength in the dynamics of the individual elements is determined by ϵ. To check this hypothesis we have explored the behavior of a set of uncoupled logistic maps with an effective nonlinear parameter given by

$$a_{eff} = a(1-\epsilon)^2 \qquad (8)$$

We have computed the MSD for such a system, and find that its profile is similar to that of the fully coupled map (see Fig. 6). What is striking here is the appearance of a plateau of large values for the MSD close to a similar plateau in the fully coupled problem. This plateau occurs around $a_{eff} \sim 1.75$ and corresponds to the 3-window of the logistic map[9]. The width of the plateau is related to the width of the periodic window. Further, the second smaller and narrower sharp peak appears at $a_{eff} \sim 1.94$, which corresponds to a 4-window. This suggests that there may be an influence of periodic windows in the value of the MSD, even for the fully coupled problem, through the renormalization of the nonlinearity parameter in the local mappings. This hypothesis is further sustained by the fact that the power spectrum of h_n in the fully coupled map, for values of ϵ corresponding to the plateau, shows a clearly dominant 1/3 frequency (see Fig. 7), and the power spectrum at ϵ corresponding to the smaller peak shows a clear 1/4 frequency (see Fig. 8).

So the skeleton of the functional dependence of the MSD on coupling comes from the effects of renormalizing the nonlinearity parameter in the individual maps, which may push the local maps into periodic motion, leading to strong synchronization. Consequently, there are these narrow ranges of ϵ where the deviation is an order of magnitude larger than elsewhere. But, this is clearly not a full explanation of the almost periodic fluctuations of h_n. The MSD for the uncoupled case is much too small compared to that of the fully coupled case. So, the "flesh" of the MSD comes from the effects of the mean field which lead to synchronization by global averaging[10]. For a full characterization of the broad "collective beating" of the mean field one must then take into account both effects.

5. EFFECTS OF NOISE

We now examine the effects of noise on the dynamics of the mean field. We simulate the system

$$x_{n+1}(i) = (1-\epsilon)f(x_n(i)) + \frac{\epsilon}{N}\sum_{j=1}^{N} f(x_n(j)) + \sigma\eta_n^i \qquad (9)$$

where η_n^i is a random number uniformly distributed in the interval [−0.5 : 0.5]. As described in Ref. 1, adding noise to the system reduces the value of the MSD to its expected values, according to the Law of Large Numbers. We have found, however, that this does not mean that the mean field h_n stops being almost-periodic. On the contrary, we have found that for values of the added noise, up to a critical noise strength, ***the sharpness of the power spectrum actually increases***. This counterintuitive behavior can be clearly seen in Figs. 9a, b and c, where we have plotted the power spectra for a few values of σ, and in Fig. 10, which shows the value of S, the measure of sharpness defined in Eqn. 5, vs. σ. Clearly the sharpness increases with increasing noise, upto $\sigma = 0.09$. We do not have an explanation for this very surprising phenomena as yet.

6. CONCLUSIONS

Here we have investigated various aspects of the dynamics of the mean field in a globally coupled chaotic system. The mean field shows evidence of a rough periodicity as is suggested through the broad, significant peaks in its power spectrum[11]. We trace the development of these peaks with respect to the number of elements coupled. Further, we examine another important "global" quantity, the probability distribution, and find that it is *not invariant*, and like the mean field, violates the law of large numbers. Moreover, there is evidence of a similar "beating" pattern in its power spectra, with the frequencies of this roughly-periodic behaviour matching with that of the mean field.

Next we find the functional dependence of the mean square deviation of the mean field on the global coupling parameter. We then attempt to decompose the effect we observe as coming from two distinct sources: one, the renormalization of the nonlinearity parameter in the local maps and another the effective strength of the mean field term in the local maps—which is indicative of the "synchronization" induced by the global averaging. This way of looking at the system helps us account for the extrememly large deviations found in certain ranges of the coupling paramter. We can infact identify the largest plateau in the MSD values with the period 3 window, into which the individual local maps are pushed, due to the effective renormalization of a.

Lastly, we explore the effects of noise on the rough periodicities observed in the mean field. We find that the periodicities do infact persist upto a reasonably large strength of noise. Futhermore, the peaks actually get sharper with increasing strength of noise, upto a critical noise strength.

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10. We have also computed the MSD for a system where the local map is not multiplied by the $(1-\epsilon)$ term, and so there is no renormalization of

the nonlinearity parameter. For such a system the effect comes solely from the interaction of the mean field, and we find that the MSD, as expected, increases monotonically with ϵ. In order to recover the full dependence of MSD on ϵ we have to couple this effect with that from renormalizing a, described in Section 4.

11. We are currently investigating more realistic physical models that may display the same kind of phenomena.

FIGURE CAPTIONS

1. Power spectra of the mean field for lattice size $N = 1, 8, 64$ and 512 (from top to bottom). Here $a = 1.99$, $\epsilon = 0.1$ and we average over 100 runs of length 1024 each.

2. The measure of the sharpness of peaks in the power spectra, as defined in the text, vs. lattice size N. ($a = 1.99$, $\epsilon = 0.1$)

3. Power spectra of partial sums S_m, as defined in text, for $m = 1, 8, 64$ and the full lattice (from top to bottom). Here $a = 1.99$, $\epsilon = 0.1$, $N = 10000$ and we average over 100 runs of length 1024 each.

4. Mean Square Deviation of $P_\delta(y; n)$, ($\delta = 0.01$), vs lattice size N, at three different values of $y =: 0.0$ (■), 0.5 (□) and 0.9 (▲). ($a = 1.99$, $\epsilon = 0.01$, $N = 10000$, number of time steps considered $= 10^4$)

5. Power spectrum of $P_\delta(y; n)$ ($\delta = 0.01$) at $y = 0.9$. Here we average over 100 runs of length 1024 each. ($a = 1.99$, $\epsilon = 0.1$, $N = 10000$).

6. Mean square deviation vs global coupling parameter ϵ ($a = 1.99$, $N = 10000$) for a) the full map, as given by Eqn. 1 in the text, and b) the set of uncoupled logistic maps with $a_{eff} = a(1-\epsilon)^2$.

7. Power spectrum of h_n at $\epsilon = 0.075$. ($a = 1.99$, $N = 10000$). Here we average over 100 runs of length 1024 each.

8. Power spectrum of h_n at $\epsilon = 0.0125$. ($a = 1.99$, $N = 10000$). Here we average over 100 runs of length 1024 each.

9. Power Spectra of the mean field in the presence of noise of strength $\sigma =$ a) 0.0, b) 0.004, c) 0.009. ($a = 1.99$, $\epsilon = 0.1$, $N = 10000$). Here we average over 100 runs of length 1024 each.

10. The measure of the sharpness of peaks in the power spectra, as defined in the text, vs. strength of noise σ. ($a = 1.99$, $\epsilon = 0.1$, $N = 10000$)

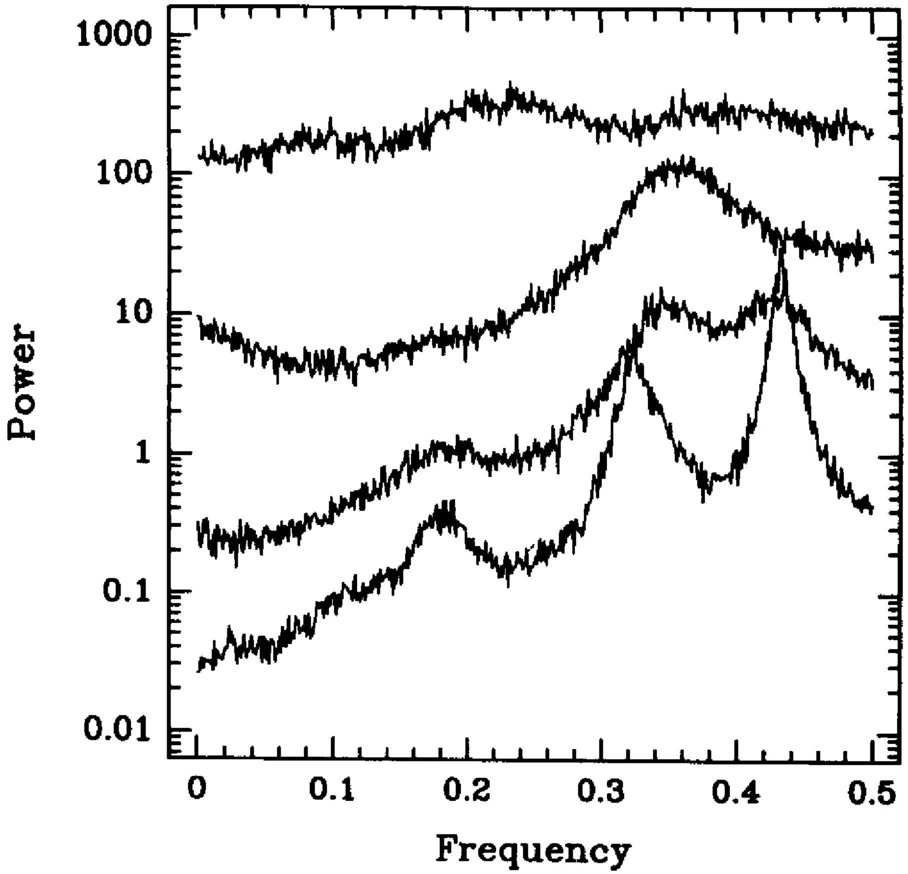
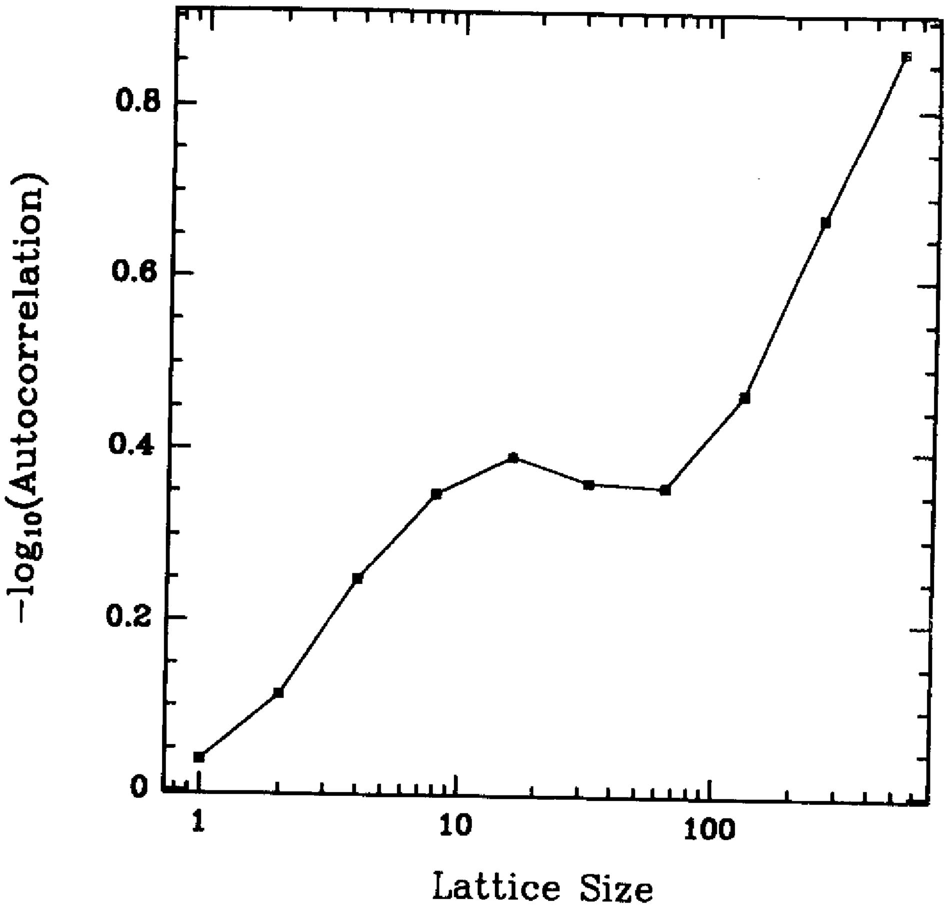

1000
100
10
1
0.1
0.01
Power
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.1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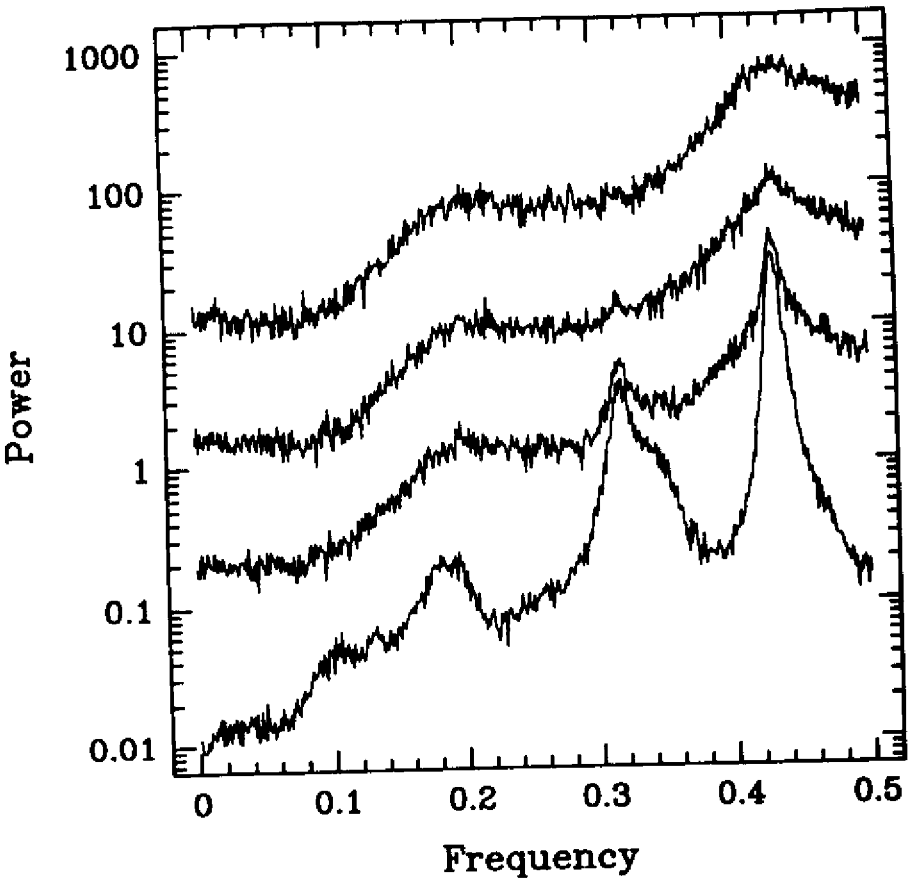
0.6
0.4
0.2
0
$-\log_{10}$(Autocorrelation)
1
10
100
Lattice Size

Fig.2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100
10
1
0.1
0.01
Power
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.3

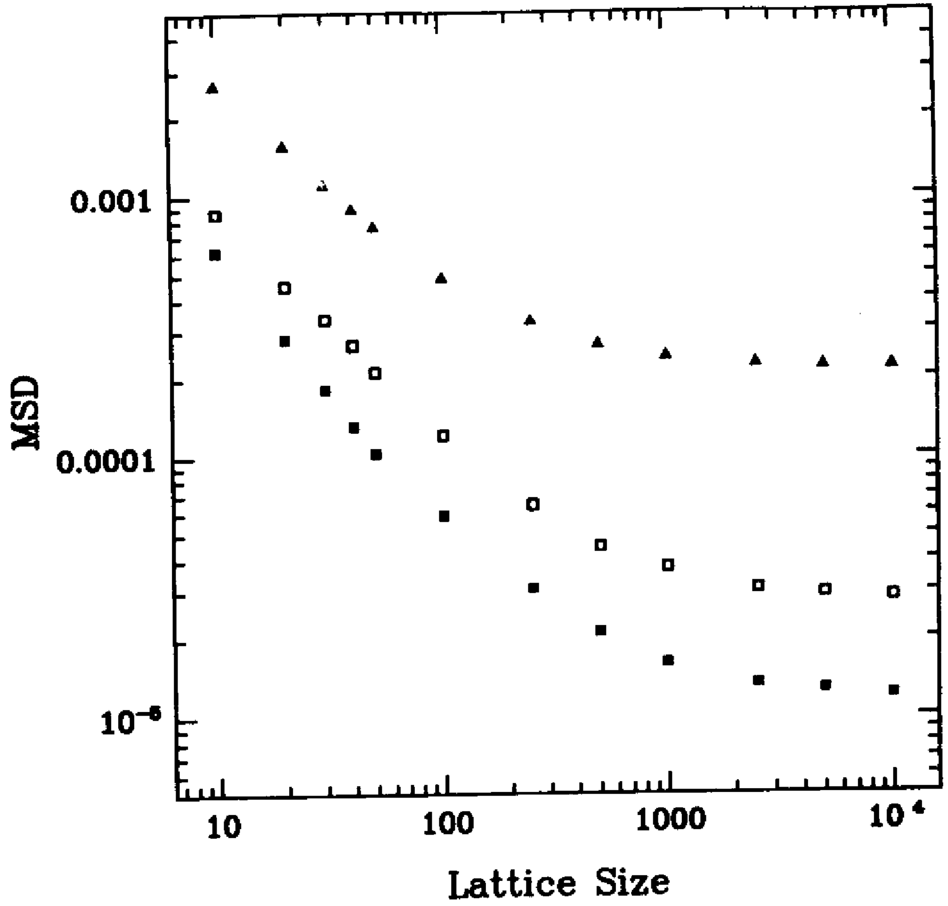
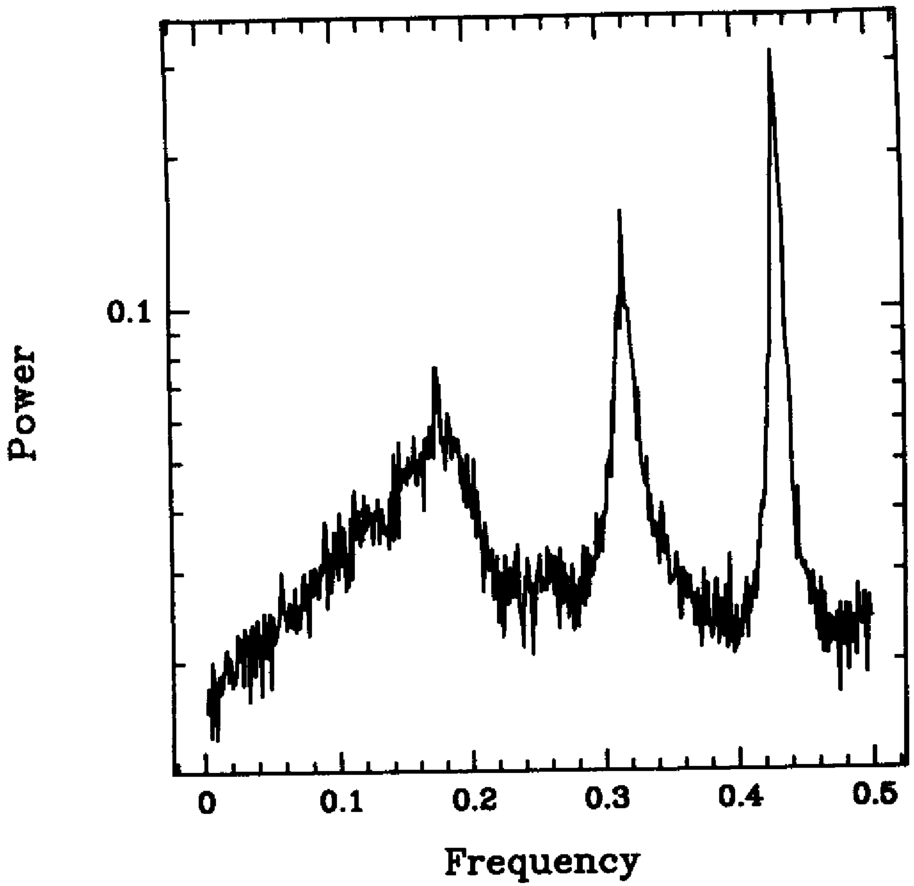
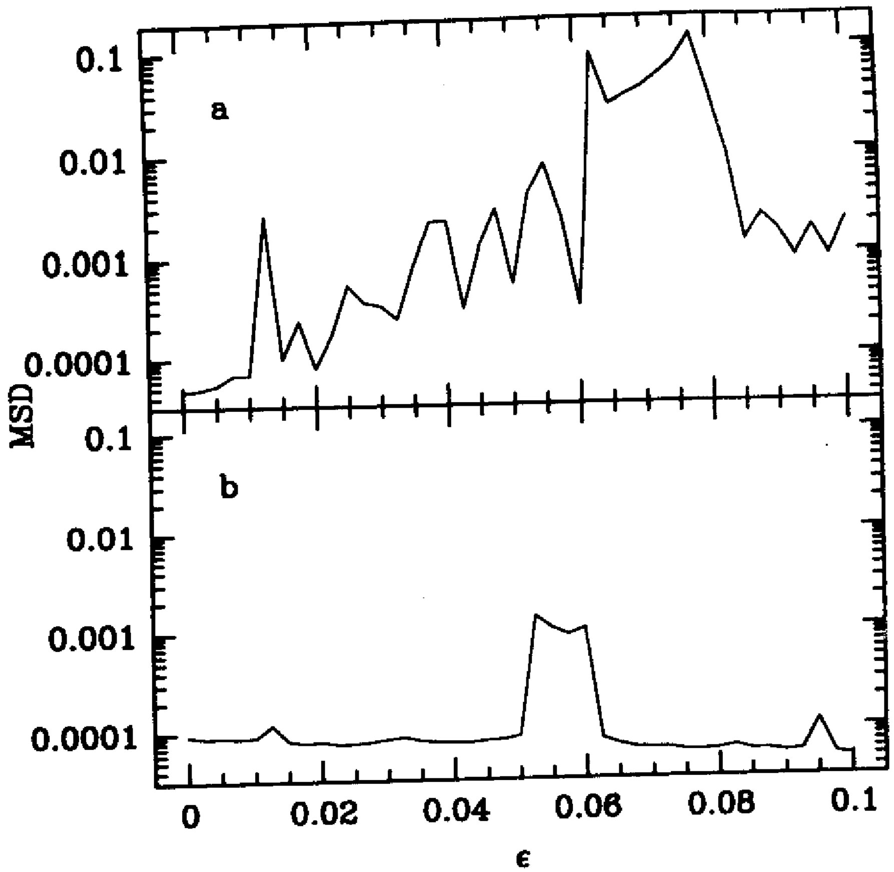

0.001
0.0001
10^{-5}
MSD
10
100
1000
10^4
Lattice Size

Fig.4



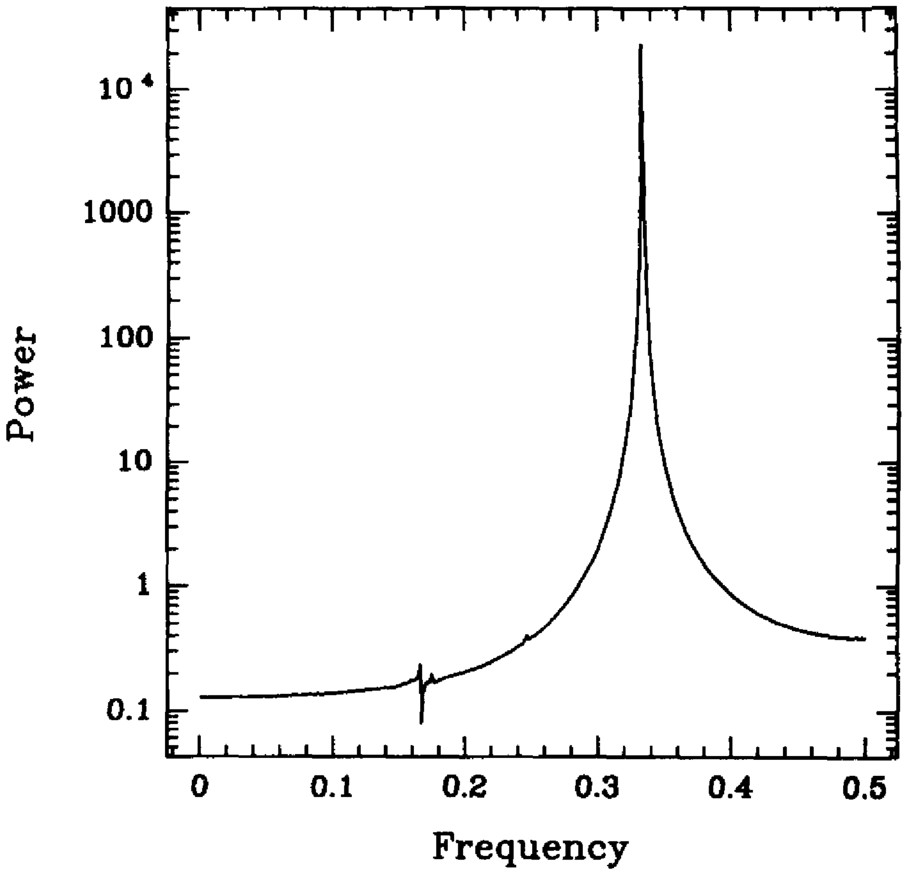
Power
0.1
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.5



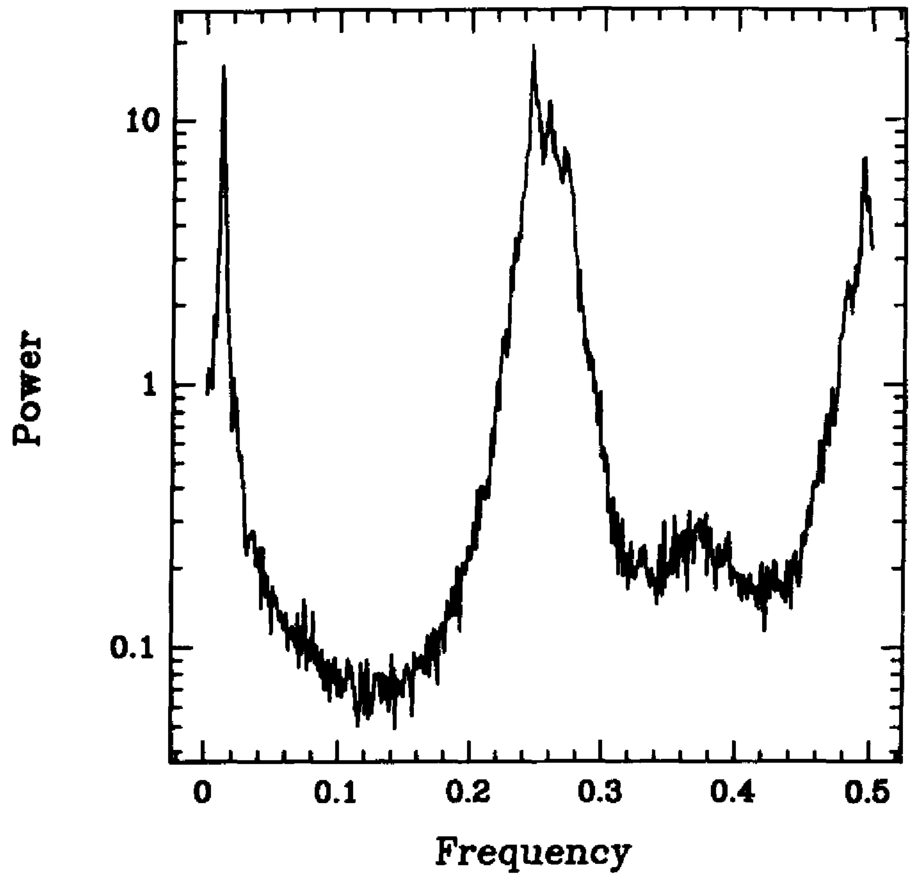
MSD
a
0.1
0.01
0.001
0.0001
b
0.1
0.01
0.001
0.0001
0
0.02
0.04
0.06
0.08
0.1
ϵ

Fig.6



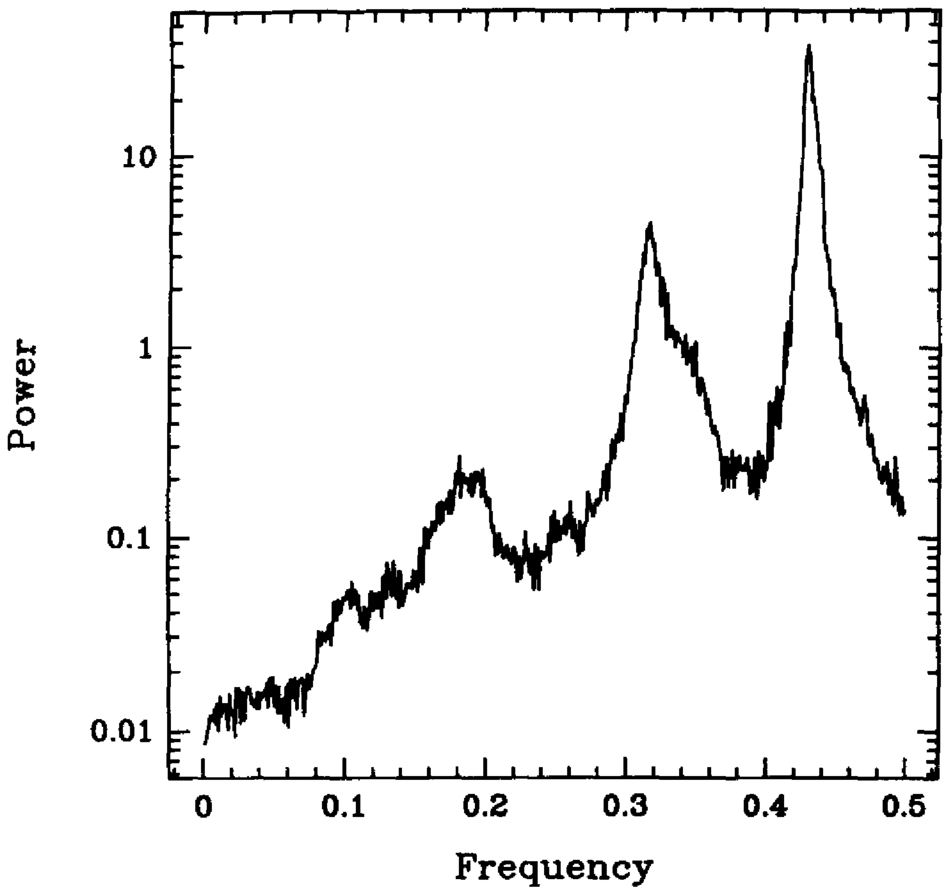
10⁴
1000
100
10
1
0.1
Power
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.7



10
1
0.1
Power
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.8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Power
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0.1
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0.3
0.4
0.5
Frequency

Fig.9a

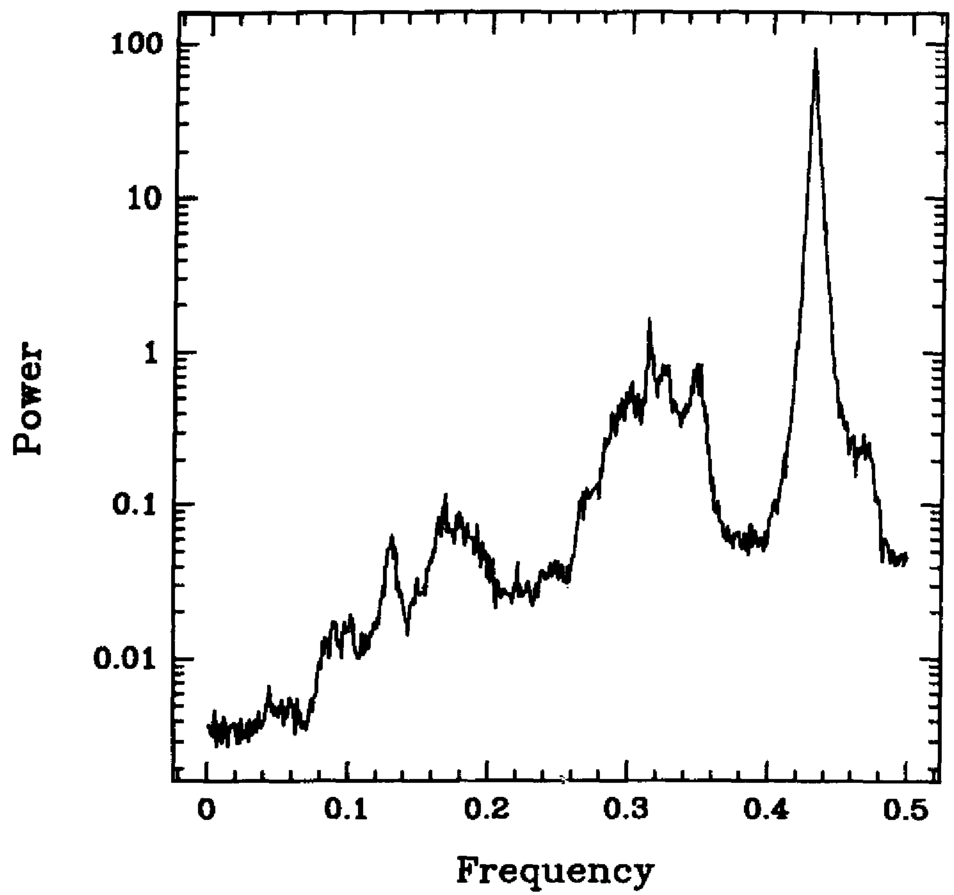
100
10
1
0.1
0.01
Power
0
0.1
0.2
0.3
0.4
0.5
Frequency

Fig.9b

Stampato in proprio nella tipografia
del Centro Internazionale di Fisica Teorica

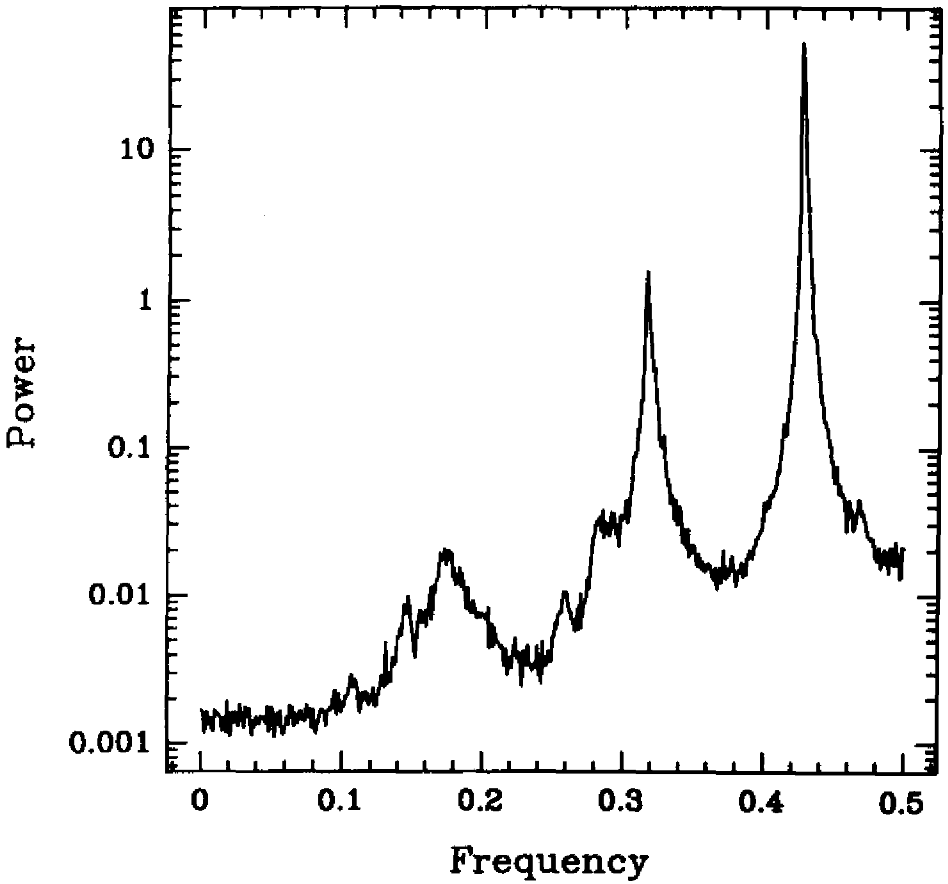

Fig.9c

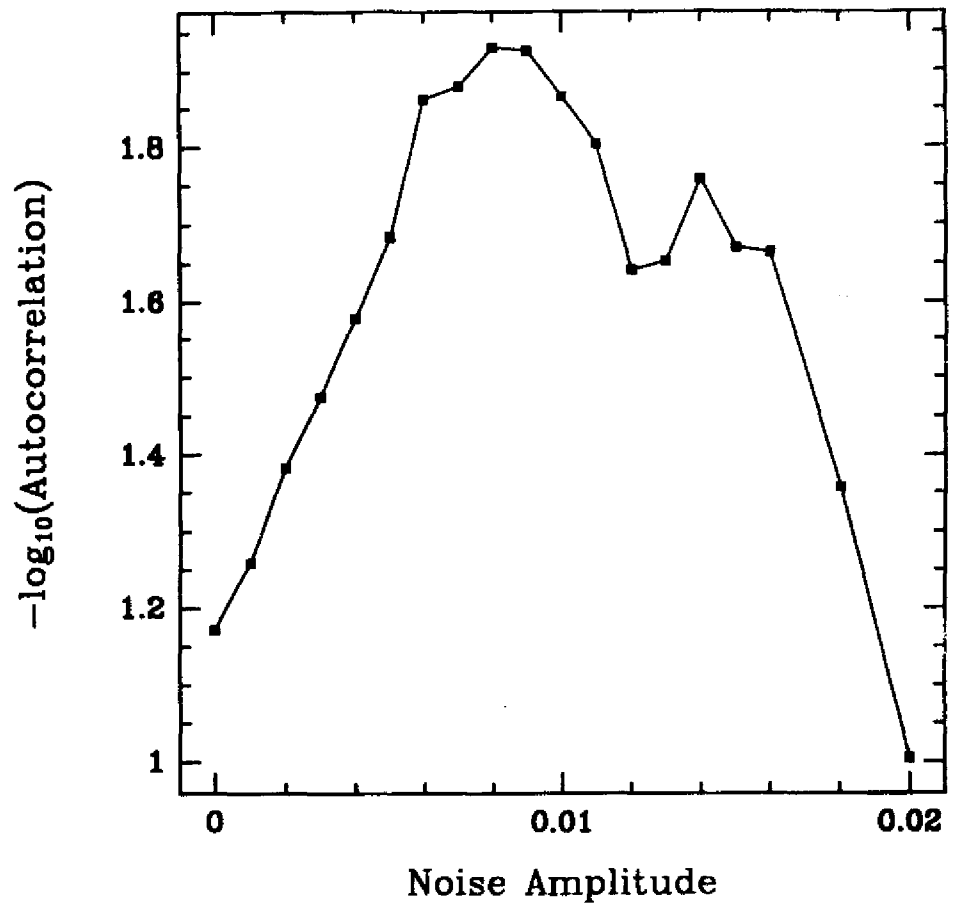

Fig.10