Spatiotemporal Intermittency on the Sandpile

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The self-organized critical state exhibited by a sandpile model is shown to correspond to motion on an attractor characterized by an invariant distribution of the conserved variable. The largest Lyapunov exponent is equal to zero. Yet over time scales of the order of the linear size of the system, the model displays intermittent chaos. The divergence of local histories is found to exhibit intermittency in both time and space.

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The ubiquity in nature of phenomena exhibiting self-similarity over a wide range of spatial and temporal scales suggests that mechanisms giving rise to such behavior must be robust with respect to variations of system parameters or initial conditions, as well as being independent of the detailed physics in question. This has led Bak, Tang, and Wiesenfeld[1,2] to propose a very general scenario for "self-organized criticality" where an extended dissipative system spontaneously evolves into a "critical" state possessing spatial and temporal scale invariance.

The sandpile as an explicit, discrete realization of a model with nonlinear diffusive dynamics in the presence of noise has attracted a lot of attention.[1-10] In a finite system the "self-organized critical state" is reached and maintained under the addition of a small but finite density of grains of sand. This *dynamical* state corresponds to motion on an attractor in phase space, under this random driving force. It is the purpose of this Letter to show that the sandpile exhibits spatial and temporal intermittency by elucidating the multifractal scaling properties of this attractor.

Throughout this Letter we shall assume an "experimental" stance. Although the system is discrete in time and space and the variables assume discrete values, we find that there exist well-defined scaling laws in the limit of long times or spatial separations, or "small" distances in phase space, which is indeed the basis of determining dimensionalities and expansion coefficients in much of the experimental work, either in simulations or in the laboratory.

We begin by recalling the definition of the model.[1,8,9] In two dimensions, to which we will confine our attention, the "sandpile" is characterized by integer z_k at all sites k. The external driving force corresponds to adding a particle at a random site k such that $z_k(n+1) = z_k(n)+1$, where n is the discrete time variable. If at any time $z_k \geq 4$, then

$$z_k(n+1) = z_k(n) - 4\,,$$
$$z_{k+\delta}(n+1) = z_{k+\delta}(n) + 1\,, \tag{1}$$

where δ signifies the unit vector to nearest-neighbor sites. Clearly z is conserved. For simplicity we have taken "closed" boundary conditions[1] on two contiguous edges and "open" boundary conditions on the other two. The frequency of addition of grains is low enough to allow the system to come to rest before the next grain is added. In what follows, n will signify time measured in number of particles added.[8]

The stationarity condition is given by the requirement that the rate of flow into a particular microscopic state—a particular z value—should equal the rate of flow out of that state.[10] This implies that the steady state is characterized by an invariant distribution[11] $\rho(z)$, which is indeed what we have found. It should be remarked that the deterministic dynamics of relaxation from supercritical configurations is not sufficient to reach this invariant distribution, but that the annealing effect of the presence of noise is necessary.[12] The size of the relative fluctuations decreases with system size.

A more detailed description of the attractor is afforded by the hierarchy of exponents that have been introduced[13,14] to characterize the multifractal distribution of phase points on the attractor. We have computed the correlation integral[13]

$$C(l) = \lim_{M\to\infty} \frac{1}{M^2}[\text{number of pairs } (\{z\}_i,\{z\}_j) \text{ with } d(\{z\}_i,\{z\}_j) < l]\,, \tag{2}$$

where $\{z\}_i$ is the stable configuration reached after the ith grain is added, $i=1,\ldots,M$, and $d(.\,,.)$ is the distance[14] in phase space between the two configurations i and j. For $M \sim L^2$ and $1 \ll l < O(L)$, where L is the linear size of the system, we find that $C(l) \sim l^\nu$ with $\nu = 0.37 \pm 0.01 \ll L^2$, which is the signature of a nontrivial attractor. If D_{LB} is the number of effective positive Lyapunov exponents (LE's), the rigorous inequality[13] $D_{LB} \leq \nu$, implies $D_{LB}=0$ in this case. Thus all LE's are less than or equal to zero.[15] The eigendirections of the zero LE's correspond to conserved quantities,

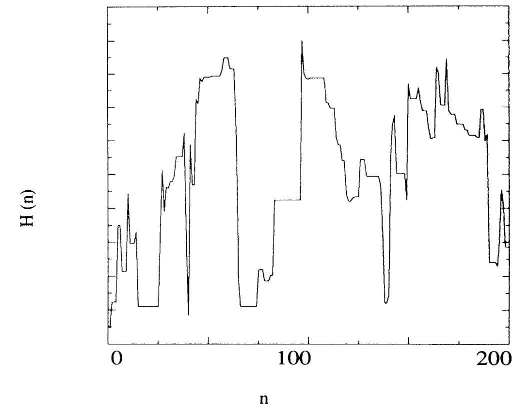

FIG. 1. Hamming distance in arbitrary units between two nearby configurations, as a function of n. In Eq. (3) we set $z_k'(0)=z_k(0)$, except at four randomly chosen sites where $z_k'(0)=z_k(0)+1$.

the existence of which is claimed[5] to underlie spatiotemporal scale invariance.

To picture the motion in phase space with the largest LE $\lambda_1=0$, it is instructive to consider the behavior of the Hamming distance $H(n)$,

$$H(n)\equiv \sum_{\text{all sites } k} [z_k(n)-z_k'(n)]^2 , \qquad (3)$$

between two slightly different configurations as a function of time.[16] Figure 1 suggests a cyclic pattern with a characteristic time scale of the order of L over which the trajectories are intermittently diverging,[17] after which there is an abrupt collapse to nearby configurations.

Clearly, the Lyapunov exponents, which are global quantities, do not suffice to fully describe this situation. Although over long times $N\gg L$ the trajectories do not diverge, it is of interest to investigate over time scales $\sim O(L)$, i.e., over one "cycle," the inhomogeneous way in which segments of the trajectory "attract" or "repel" each other. We define $(N\times L^2)$-dimensional vectors, or "histories"[18] via

$$\mathbf{X}_i^{(N)}=(\{z\}_i,\{z\}_{i+1},\ldots\{z\}_{i+N-1}) \qquad (4)$$

starting at the phase point $\{z\}_i$. The probability of encountering another history in a neighborhood of size l around $\mathbf{X}_0^{(N)}$ in history space is

$$P_l^{(N)}(\mathbf{X}_0^{(N)})=\frac{1}{M}\sum_{j=1}^{M}\Theta(l-\Delta(\mathbf{X}_0^{(N)}-\mathbf{X}_j^{(N)})) , \qquad (5)$$

where $M\sim NL^2$ is the number of histories, Θ is the step function, and

$$\Delta(\mathbf{X}_0^{(N)}-\mathbf{X}_j^{(N)})=\frac{1}{NL^2}\sum_{i=1}^{N}\sum_{k=1}^{L^2}|z_k(i)-z_k(i+j)| \qquad (6)$$

is the distance between the pair of histories, normalized so that it takes continuous values between zero and z_c. After Paladin, Peliti, and Vulpiani[18] one may define local expansion parameters λ_l via $P_l^{(N)}(l)\sim e^{-\lambda_l N}$, which provide a heuristic measure of the rate at which a trajectory is "expanding" in a particular region of history space. Over time scales $1<N<L$, $l\to 0$, the partition function scales like

$$\Gamma_N(q)\equiv\sum_l P_l^{(N)}(l)^q\sim e^{-N(q-1)K(q)} , \qquad (7)$$

where $K(q)$ is a nonlinear function of q (see Fig. 2). This points to a multifractal distribution for the λ_l, and signals intermittency[17,18] in history space. The $K(q)$ measure the relative weight of "laminar" and "turbulent" regions in phase space within a given cycle. The sum in Eq. (7) is dominated by those terms with the smallest (largest) λ_l for large (small) q, respectively.

In the limit $N\to\infty$, where one effectively averages over many cycles, we cross over to a qualitatively different behavior, with $K(q)=0$ for all q. In this limit the $K(q)$, heuristically defined in Eq. (7), go over to the Renyi entropies,[18,19] and the Pesin relation,[18] $K(1)=\sum$(all non-negative LE's), is fulfilled, with $K(1)=0$.

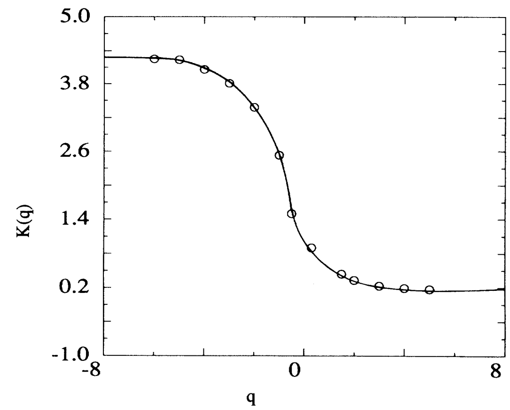

FIG. 2. Nonlinear q dependence of $K(q)$ over time scales of the order of one cycle.

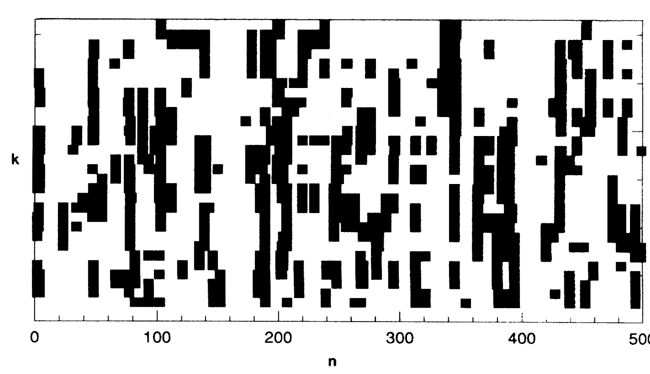

FIG. 3. Evolution in time of active sites (shown in black) on a one-dimensional cut through the sandpile. The vertical axis corresponds to the position along the cut and the horizontal axis to time over several avalanches ($L=30$).

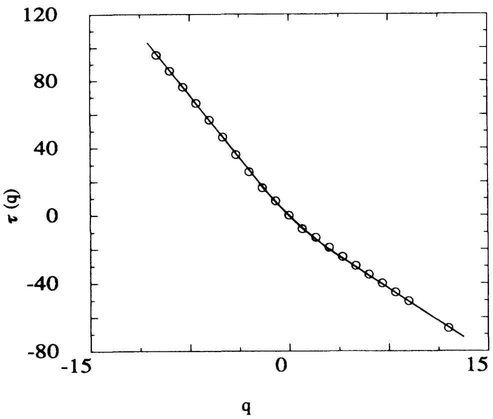

FIG. 4. Intermittency reflected as multifractality in the distribution of expansion parameters of local histories, for $L=30$, $r=1$. [See Eq. (9) in text.]

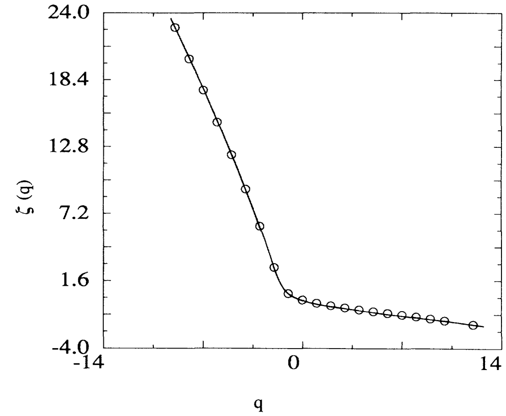

FIG. 5. Nonlinear q dependence of the qth moments of the local histories as a function of spatial separation.

We would now like to examine how the turbulent and laminar regions are distributed in the 2-space+1-time dimensions. For ease of visualization, we take a one-dimensional cut through the sandpile and plot, as a function of n, all points along this cut for which $z_k(n-1) - z_k(n) \neq 0$ (see Fig. 3).

What we get is a series of snapshots of the active regions falling on this particular intersection, in the course of several avalanches. Since some z_k may flip back to their previous value during an avalanche, the active clusters are found to be noncompact, in contrast to the usual approach.[1,2,9] For systems with linear size $L=30$ and with frequency of addition of grains low enough to allow the system to equilibrate before the next grain is added, we compute, for the fractal dimension of the set of active sites on a typical cut, $D_\perp = 0.6 \pm 0.05$. Assuming sufficient invariance under translation, the usual argument of additivity of codimensions[20] leads to $D_f = 1.6 \pm 0.05$ for the fractal dimension of the total set of active regions at any moment in time, and $D_f + 1 = 2.6 > 2$ for the dimension of the same set translated in time in a statistically self-similar fashion and embedded in $2+1$ dimensions.

Finally, we would like to introduce a new quantity, which we shall call a "local history," as a measure of how differences in local configurations scale with time and spatial separation. We define

$$h_k(r,n) = \sum_{m=1}^{n} [z_k(m) - z_{k+r}(m)]^2 \tag{8}$$

and

$$R(q;n,r) = \sum_{\text{all sites } k} h_k(r,n)^q . \tag{9}$$

In analogy with the multifractal scaling in history space, we now have, for the local histories, for fixed r and $1 < n < L$, $R(q;n,1) \sim e^{n\tau(q)}$, and for fixed $n \sim O(L)$ and $1 < r < L$, $R(q;r) \sim r^{\zeta(q)}$. We have plotted $\tau(q)$ and $\zeta(q)$ in Figs. 4 and 5, where one clearly observes the nonlinear dependence on q. For the case of $\zeta(q)$, the close similarity with multifractal scaling in turbulent media[14,21,22] is obvious. We have shown that concepts like correlation integrals and intermittency, both in configuration space and history space, can be extended to cellular automata in such a way as to complement their investigation via the "damage spreading" type of approaches.[16]

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