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NONEXISTENCE OF NODAL SOLUTIONS OF ELLIPTIC EQUATIONS WITH CRITICAL GROWTH IN $\mathbb{R}^2$

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ABSTRACT. Let $f(t) = h(t)e^{bt^2}$ be a function of critical growth. Under a suitable assumption on h, we prove that

$$\begin{aligned} -\Delta u &= f(u) \quad \text{in } B(R) \subset \mathbb{R}^2, \\ u &= 0 \quad \text{on } \partial B(R), \end{aligned}$$

does not admit a radial solution which changes sign for sufficiently small R.

1. INTRODUCTION

Let $B(R)$ denote the ball of radius R in $\mathbb{R}^2$ with center at zero. Let $f(t) = h(t)e^{bt^2}$ be a function of critical growth (see Adimurthi-Yadava [1]). Consider the following problem

$$(1.1) \qquad \begin{cases} -\Delta u = f(u) & \text{in } B(R), \\ u = 0 & \text{on } \partial B(R). \end{cases}$$

If f satisfies the following condition

$$(1.2) \qquad \lim_{t\to\infty} \frac{\log h(t)}{t} = \infty,$$

then (1.1) admits an infinite number of radial solutions which change sign (see Adimurthi-Yadava [1]).

In this note we show that the condition (1.2) is optimal for existence of infinitely many radial solutions which change sign by proving the following:

Theorem 1. *Let* $f(t) = t|t|^m e^{bt^2+|t|^\beta}$, $m \geq 0$, $b > 0$ *and* $0 \leq \beta \leq 1$. *Then for every* β *there exists* $R^{(\beta)} > 0$ *such that for* $0 < R < R^{(\beta)}$, *the problem*

$$(1.3) \qquad \begin{cases} -\Delta u = f(u) & \text{in } B(R), \\ u = 0 & \text{on } \partial B(R), \end{cases}$$

does not admit any radial solution which changes sign.

If $1 < \beta < 2$, *then* f *satisfies* (1.2) *and hence* (1.2) *is optimal.*

In this connection similar results are available for critical exponent problems in $\mathbb{R}^n$, $n \geq 3$. There the dimension plays a role in the case of existence

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(see Cerami-Solomini-Struwe [5]) and nonexistence (see Atkinson-Brezis-Peletier [4]) of radial solutions which change sign.

2. Proof of Theorem 1

Since we are looking for radial solutions, (1.3) becomes

$$\begin{cases} -(u'' + \frac{1}{r}u') = f(u) & \text{in } (0, R), \\ u'(0) = u(R) = 0. \end{cases} \tag{2.1}$$

By studying the following initial value problem we will prove the nonexistence of nodal solutions of (2.1) as in Atkinson-Brezis-Peletier [4]

$$\begin{cases} -(u'' + \frac{1}{r}u') = f(u), \\ u'(0) = 0, \\ u(0) = \gamma > 0. \end{cases} \tag{2.2}$$

Let $R_k(\gamma)$, $k = 1, 2, \dots$, denote the kth zero of u. Then by the similar argument as in Atkinson-Peletier [3] we have

$$\lim_{\gamma \to 0} R_1(\gamma) = \begin{cases} \infty & \text{if } m > 0, \\ C & \text{if } m = 0, \end{cases} \tag{2.3}$$

where C is some positive constant. For the sake of completeness we will sketch the proof of (2.3) in Appendix 2. Now the proof of the theorem follows from the following:

Claim 1. For each $0 \le \beta \le 1$, there exists a constant $c(\beta) > 0$ such that

$$\underline{\lim}_{\gamma \to \infty} R_2(\gamma) > c(\beta). \tag{2.4}$$

In order to prove Claim 1, make the standard substitution (as in Atkinson-Peletier [2]) by $r = 2e^{-t/2}$ and $u(r) = y(t)$, then (2.2) becomes

$$\begin{cases} -y'' = e^{-t} f(y), \\ y(\infty) = \gamma, \\ y'(\infty) = 0. \end{cases} \tag{2.5}$$

Let $y(t, \gamma)$ be the corresponding solution and $T_k(\gamma)$ the kth zero of $y(t, \gamma)$. Then

$$R_k(\gamma) = 2e^{-T_k(\gamma)/2}. \tag{2.6}$$

Now we have the following estimates on $T_1(\gamma)$.

Claim 2. For every β, $0 \le \beta \le 1$, there exist constants $C_\beta > 0$ and $\gamma_0 > 0$ such that for all $\gamma \ge \gamma_0$,

$$\gamma y'(T_1(\gamma), \gamma) \le C_\beta, \tag{2.7}$$

$$\frac{T_1(\gamma)}{\gamma} \le C_\beta, \tag{2.8}$$

$$\lim_{\gamma \to \infty} T_1(\gamma) = \infty. \tag{2.8$'$}$$

Proof of Claim 1. Assuming Claim 2 we will complete the proof of Claim 1. Without loss of generality we may assume

$$\varliminf_{\gamma\to\infty} T_2(\gamma) \geq 1. \tag{2.9}$$

By using the convexity of y on $[T_2(\gamma), T_1(\gamma)]$ together with (2.7) and (2.8) we have for all $\gamma \geq \gamma_0$ and $t \in [T_2(\gamma), T_1(\gamma)]$,

$$|y(t, \gamma)| \leq |T_1(\gamma)y'(T_1(\gamma), \gamma)| \leq \left|\frac{T_1(\gamma)}{\gamma}\gamma y'(T_1(\gamma), \gamma)\right| \leq C_\beta^2. \tag{2.10}$$

Let

$$K(\beta) = \sup\left\{\frac{f(y)}{y} : 0 \leq y \leq C_\beta^2\right\} \tag{2.11}$$

and choose $t_0(\beta) > 0$ such that for $t \geq t_0(\beta)$,

$$4t^2e^{-t}K(\beta) < 1. \tag{2.12}$$

From (2.8)$'$, we can choose a $\gamma_1 > \gamma_0$ such that for all $\gamma \geq \gamma_1$,

$$t_0(\beta) < T_1(\gamma). \tag{2.13}$$

Hence from (2.10), (2.11) and (2.12) for all $t \geq t_0(\beta)$, $t \in [T_2(\gamma), T_1(\gamma)]$, $\gamma \geq \gamma_1$, we have

$$4t^2e^{-t}\frac{f(y(t, \gamma))}{y(t, \gamma)} < 1. \tag{2.14}$$

Let $Z = t^{1/2}$, then Z satisfies

$$Z'' + \frac{1}{4t^2}Z = 0 \tag{2.15}$$

and

$$y'' + \frac{1}{4t^2}\left(4t^2e^{-t}\frac{f(y)}{y}\right)y = 0. \tag{2.16}$$

Hence from (2.14) and by Sturm's Comparison Theorem we have for all $\gamma \geq \gamma_1$,

$$T_2(\gamma) < t_0(\beta). \tag{2.17}$$

Now (2.4) follows from (2.6) and (2.17). This completes the proof of Claim 1 and hence Theorem 1.

In order to prove Claim 2 we need the following proposition.

Let $F : \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a locally Lipschitz continuous function and $s_0 \geq 0$ such that

$$F(s) \text{ is strictly increasing for } s \geq s_0. \tag{2.18}$$

$$\text{Let } G(s) = \log F(s) \text{ be } C^2 \text{ and convex for } s \geq s_0. \tag{2.19}$$

$$(\gamma G'(\gamma))^2 e^{-\{G(\gamma) - \frac{1}{2}(\gamma - s_0)G'(\gamma)\}} = O(1) \quad \text{as } \gamma \to \infty. \tag{2.20}$$

$$\lim_{\gamma\to\infty} \frac{\gamma G^{(p+1)}(\gamma)}{G^{(p)}(\gamma)} = L_p \neq 0 \quad \text{for } p = 0, 1, \tag{2.21}$$

where $G^{(p)}$ denotes the pth derivative of G.

(2.22) There exist positive constants C_1, C_2, l and γ_1 such that for all $\gamma \geq \gamma_1$,

$$C_1\gamma^l \leq G(\gamma) \leq C_2\gamma^l.$$

Let $Y(t, \gamma)$ denote the solution of

$$\begin{cases} -Y'' = e^{-t}F(Y), \\ Y(\infty) = \gamma, \\ Y'(\infty) = 0, \end{cases} \tag{2.23}$$

and $S(\gamma)$ the first zero of $Y(t, \gamma)$. Let $S_0(\gamma)$ be such that $Y(S_0(\gamma), \gamma) = s_0$. Note that $S(\gamma) \leq S_0(\gamma)$. Then we have the following:

Proposition 2. *We have, as* $\gamma \to \infty$,

$$\begin{aligned} &Y'(S_0(\gamma), \gamma) \\ &\quad = \frac{2}{G'(\gamma)}\left[1 + O\left(\frac{(\log\gamma)^2}{G(\gamma)}\right) + O(\gamma G'(\gamma)e^{-\{G(\gamma)-\frac{1}{2}(\gamma-s_0)G'(\gamma)\}})\right], \end{aligned} \tag{2.24}$$

$$\begin{aligned} S_0(\gamma) = {} & \left(G(\gamma) - \frac{1}{2}\gamma G'(\gamma)\right) + s_0\left(\frac{G'(\gamma)}{2}\right) + \log\frac{G'(\gamma)}{2} \\ & + O((\log\gamma))^2 + O[(\gamma G'(\gamma))^2 e^{-\{G(\gamma)-\frac{1}{2}(\gamma-s_0)G'(\gamma)\}}], \end{aligned} \tag{2.25}$$

$$S(\gamma) \geq \left(G(\gamma) - \frac{1}{2}\gamma G'(\gamma)\right) + \log\frac{G'(\gamma)}{2} + O(1). \tag{2.26}$$

Proof of this proposition follows exactly as in Atkinson-Peletier [2] (see Lemma 10 and Theorem 4). Since the hypotheses here on G are little bit different from those in Atkinson-Peletier [2] we shall for completeness sketch the proof in Appendix 1.

Proof of Claim 2. Let $F(s) = s|s|^m e^{bs^2+|s|^\beta}$, then for $s \geq 0$, we have

$$G(s) = bs^2 + s^\beta + (m+1)\log s, \tag{2.27}$$

$$G'(s) = 2bs + \beta s^{\beta-1} + \frac{m+1}{s}, \tag{2.28}$$

$$G''(s) = 2b + \beta(\beta-1)s^{\beta-2} - \frac{m+1}{s^2}, \tag{2.29}$$

$$G(s) - \frac{1}{2}sG'(s) = \left(1 - \frac{\beta}{2}\right)s^\beta + (m+1)\log s - \frac{m+1}{2}, \tag{2.30}$$

$$\lim_{s\to\infty}\frac{sG'(s)}{G(s)} = 2, \quad \lim_{s\to\infty}\frac{sG''(s)}{G'(s)} = 1, \tag{2.31}$$

$$bs^2 \leq G(s) \leq \left(b + 1 + \frac{m+1}{2e}\right)s^2 \quad \text{for } s \geq 1. \tag{2.32}$$

Since $\beta \leq 1$, from (2.29) we can choose an $s_0 > 0$ such that for all $s > s_0$, $G''(s) \geq 0$. Combining this with (2.27) to (2.32), F satisfies all the assumptions from (2.18) to (2.22). Hence from Proposition 2, (2.28) and (2.30) we have as $\gamma \to \infty$,

$$Y'(S_0(\gamma), \gamma) = O(1/\gamma), \tag{2.33}$$

$$S_0(\gamma) = s_0 b\gamma + O(\gamma^\beta), \tag{2.34}$$

$$T_1(\gamma) = S(\gamma) \geq (1 - \beta/2)\gamma^\beta + O(\log \gamma). \tag{2.35}$$

Hence $T_1(\gamma) \to \infty$ as $\gamma \to \infty$. This proves (2.8)$'$. Now from (2.34) and using $0 \leq \beta \leq 1$ we have for γ large,

$$\frac{T_1(\gamma)}{\gamma} \leq \frac{S_0(\gamma)}{\gamma} \leq s_0 b + O(\gamma^{\beta-1}) \leq C_3 \tag{2.36}$$

for some constant $C_3 > 0$. This proves (2.8).

Let $C_4 = \sup_{0 \leq s \leq s_0} F(s)$, then from (2.23) we have for $t \in [S(\gamma), S_0(\gamma)]$,

$$-Y'' \leq C_4 e^{-t}. \tag{2.37}$$

Integrating (2.37) from $T_1(\gamma)$ $(= S(\gamma))$ to $S_0(\gamma)$ we have

$$Y'(T_1(\gamma), \gamma) \leq Y'(S_0(\gamma), \gamma) + C_4(e^{-T_1(\gamma)} - e^{-S_0(\gamma)}). \tag{2.38}$$

Now from (2.33), (2.34) and (2.35), we can choose a constant $C_5 > 0$ such that for all γ large, (2.38) implies $\gamma Y'(T_1(\gamma), \gamma) \leq C_5$. This proves (2.7) and hence Claim 2.

Remark. The above proof shows that Theorem 1 can be stated in a more general form as follows.

Let $f: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a C^1 function and let $s_0 \geq 0$ be such that

$$f \text{ is strictly increasing for } s \geq s_0, \tag{2.39}$$

$$g(s) = \log f(s) \text{ is } C^2 \text{ convex for } s \geq s_0, \tag{2.40}$$

$$g(s) = bs^2 + g_1(s) \text{ with } b > 0 \text{ such that}$$

$$\lim_{s\to\infty} \frac{g_1(s)}{s^2} = 0, \qquad \overline{\lim_{s\to\infty}} |g_1'(s)| < \infty, \tag{2.41}$$

$$\lim_{s\to\infty} g_1''(s) = 0, \qquad \overline{\lim_{s\to\infty}} \frac{|g_1(s) - \frac{1}{2}sg_1'(s)|}{s} < \infty.$$

Then we have the following

Theorem 1$'$. *Let f satisfy* (2.39), (2.40) *and* (2.41). *Futher assume that $f(0) = 0$ and extend f as an odd function on $\mathbb{R}$. Then there exists an $R_0 > 0$ such that for $0 < R < R_0$,* (1.1) *does not admit any radial solution which changes sign.*

3. Appendix

Appendix 1. Let $F: \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a locally Lipschitz continuous function and $G(s) = \log F(s)$ satisfies (2.18) to (2.22). Following the same notations as in Proposition 2 and denoting $G(\gamma) = G$, $G'(\gamma) = G'$, $Y(t, \gamma) = Y(t)$, we have the following

Lemma 3.1. *For* $S_0 \le t < \infty$ *we have*

$$Y(t) \le \gamma - \frac{2}{G'} \log\left(1 + \frac{1}{2} G' e^{G-t}\right), \tag{3.1}$$

$$G(Y(t)) \ge G - 2\log\left(1 + \frac{1}{2} G' e^{G-t}\right), \tag{3.2}$$

$$t > G - \frac{1}{2}(\gamma - Y(t))G' + \log\frac{G'}{2}, \tag{3.3}$$

$$t \le \frac{1}{2}\{G + G(Y(t))\} + \log\frac{G'}{2} - \log[1 - e^{\{G(Y(t))-G\}/2}], \tag{3.4}$$

$$Y'(t) \le e^{\{(G+G(Y(t)))/2-t\}}, \tag{3.5}$$

$$Y'(t) \ge e^{\{G-(\gamma-Y(t))G'/2-t\}}, \tag{3.6}$$

$$S(\gamma) \ge G - \frac{1}{2}\gamma G' + \log\frac{G'}{2} + O(1) \quad \textit{as } \gamma \to \infty. \tag{3.7}$$

For the proof of this lemma we refer to Atkinson-Peletier [2]. In fact (3.1), (3.2), (3.3), (3.4), (3.5), (3.6) and (3.7) of the above lemma correspond to (4.4), (4.5), (4.16), (4.18), (4.21), (4.22) and (3.5) of Atkinson-Peletier [2].

Let k be a large positive (but fixed) number and define

$$\delta = k \log \gamma, \tag{3.8}$$

$$S_1 = G + \log\frac{G'}{2} - \delta. \tag{3.9}$$

Then we have the following

Lemma 3.2. *As* $\gamma \to \infty$, *we have*

$$Y(S_1) = \gamma - \frac{2}{G'}\delta + O\left(\frac{\delta^2}{G}\right), \tag{3.10}$$

$$G(Y(S_1)) = G - 2\delta + O\left(\frac{\delta^2}{G}\right), \tag{3.11}$$

$$Y'(S_1) = \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right)\right]. \tag{3.12}$$

Proof. Taking $t = S_0$ in (3.4) we have for large γ,

$$\begin{aligned} S_0 &\le G + \log\frac{G'}{2} - \frac{1}{2}G + O(1) \\ &= S_1 - \left(\frac{1}{2}G - \delta\right) + O(1) < S_1. \end{aligned} \tag{3.13}$$

Hence from (3.1) and (3.2), we have

$$Y(S_1) \le \gamma - \frac{2}{G'}\log(1 + e^{\delta}) \le \gamma - \frac{2}{G'}\delta + O\left(\frac{2}{G'}e^{-\delta}\right) \tag{3.14}$$

and

$$G(Y(S_1)) \geq G - 2\delta + O(e^{-\delta}). \tag{3.15}$$

Since G is an increasing function, we have from (3.14)

$$\begin{aligned} G(Y(S_1)) &\leq G\left(\gamma - \frac{2}{G'}\delta + O\left(\frac{2}{G'}e^{-\delta}\right)\right) \\ &= G - \left[\frac{2}{G'}\delta + O\left(\frac{2}{G'}e^{-\delta}\right)\right] G' + \left[\frac{2\delta}{G'} + O\left(\frac{2}{G'}e^{-\delta}\right)\right]^2 \frac{G''(\xi)}{2} \end{aligned} \tag{3.16}$$

for some ξ in the interval $[\gamma - 2\delta/G' + O(2e^{-\delta}/G'), \gamma]$. Now from (2.21), we have $G''(\xi)/(G')^2 = O(1/G)$ and hence (3.16) implies

$$G(Y(S_1)) \leq G - 2\delta + O(\delta^2/G).$$

Therefore from (3.15) we have

$$G(Y(S_1)) = G - 2\delta + O(\delta^2/G). \tag{3.17}$$

This proves (3.11).

From (3.15) we have

$$\begin{aligned} Y(S_1) &\geq G^{-1}(G - 2\delta + O(e^{-\delta})) \\ &= \gamma - \frac{(2\delta + O(e^{-\delta}))}{G'} - \frac{(2\delta + O(e^{-\delta}))^2}{2}\frac{G''(\eta)}{(G'(\eta))^3} \end{aligned} \tag{3.18}$$

for some η such that

$$G - 2\delta + O(e^{-\delta}) \leq G(\eta) \leq G.$$

Now from (2.22) it follows that there exists a constant $C_1 > 0$ such that $C_1\gamma \leq \eta \leq \gamma$. Therefore from (3.18) and (3.14) we have

$$Y(S_1) = \gamma - \frac{2\delta}{G'} + O\left(\frac{\delta^2}{G}\right). \tag{3.19}$$

This proves (3.10).

Let $t = S_1$ in (3.5). Then using (3.11) we have

$$Y'(S_1) \leq e^{\{-\log G'/2 + O(\delta^2/G)\}} = \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right)\right]. \tag{3.20}$$

Similarly from (3.6) we obtain

$$Y'(S_1) \geq \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right)\right]. \tag{3.21}$$

Combining (3.20) and (3.21) we get (3.12). This completes the proof of the lemma.

Lemma 3.3. *For $S_0 \leq t \leq S_1$, we have*

$$Y'(t) = \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right) + O(\gamma G' e^{-\{G - \frac{1}{2}(\gamma - s_0)G'\}})\right]. \tag{3.22}$$

Proof. From (3.3) we have

$$G(Y(t)) - t \leq \left\{G(Y(t)) - \frac{1}{2}Y(t)G'\right\} - \left\{G - \frac{1}{2}\gamma G'\right\} - \log\frac{G'}{2} \equiv \psi(Y).$$

Since $\psi''(Y) \geq 0$, it follows that

$$G(Y(t)) - t \leq \max\{\psi(Y(S_1)), \psi(Y(S_0))\}. \tag{3.23}$$

Using (3.10) and (3.11), the above implies that

$$G(Y(t)) - t \leq -\log\frac{G'}{2} + \max\left\{-\delta, -\left(G - \frac{1}{2}(\gamma - s_0)G'\right)\right\} + O(1). \tag{3.24}$$

Hence from (3.12) and (3.24) we have for any $t \in [S_0, S_1]$

$$\begin{aligned} Y'(t) &= Y'(S_1) + \int_t^{S_1} e^{G(Y(s))-s}\,ds \\ &= \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right)\right] + O\left[\frac{S_1 - S_0}{G'}\max\{e^{-\delta}, e^{-(G-\frac{1}{2}(\gamma-s_0)G')}\}\right]. \end{aligned} \tag{3.25}$$

From (2.20), $G - \frac{1}{2}(\gamma - s_0)G' > 0$ for γ large, hence we have from (3.3), $S_0 \geq \log G'/2$ which implies that $S_1 - S_0 \leq G$. Hence (3.25) implies

$$Y'(t) = \frac{2}{G'}\left[1 + O\left(\frac{\delta^2}{G}\right) + O(\gamma G' e^{-\{G-\frac{1}{2}(\gamma-s_0)G'\}})\right].$$

This proves the lemma.

Proof of Proposition 2. (2.24) follows from Lemma 3.3. (2.26) follows from (3.7) of Lemma 3.1. Now from the mean value theorem, there exists a $t \in [S_0, S_1]$ such that

$$S_0 = S_1 - \frac{Y(S_1) - s_0}{Y'(t)}. \tag{3.26}$$

From (3.10) and (3.22), (3.26) implies that

$$\begin{aligned} S_0 &= S_1 - \frac{\gamma - 2\delta/G' + O(\delta^2/G) - s_0}{(2/G')[1 + O(\delta^2/G) + O(\gamma G' e^{-\{G-\frac{1}{2}(\gamma-s_0)G'\}})]} \\ &= G - \frac{1}{2}\gamma G' + \frac{s_0 G'}{2} + \log\frac{G'}{2} + O\left(\frac{\gamma G'}{G}\delta^2\right) \\ &\quad + O((\gamma G')^2 e^{-\{G-\frac{1}{2}(\gamma-s_0)G'\}}). \end{aligned} \tag{3.27}$$

Since $O(\gamma G'\delta^2/G) = O((\log\gamma)^2)$, (3.27) implies (2.25). This proves the proposition.

Appendix 2. Let $f\colon \overline{\mathbb{R}}_+ \to \overline{\mathbb{R}}_+$ be a C^1 function such that $f(0) = 0$. Let $y(t, \gamma)$ be the solution of

$$\begin{cases} -y'' = e^{-t}f(y), \\ y(\infty) = \gamma > 0, \\ y'(\infty) = 0, \end{cases} \tag{3.28}$$

and $T_1(\gamma)$ the first zero of $y(t, \gamma)$. Then there exists a real number C such that

$$\lim_{\gamma\to 0} T_1(\gamma) = \begin{cases} -\infty & \text{if } f'(0) = 0, \\ C & \text{if } f'(0) \neq 0. \end{cases} \tag{3.29}$$

Proof. (i) Let $f'(0) = 0$. Then integrating (3.28) from $T_1(\gamma)$ to ∞, we obtain

$$\begin{aligned}\gamma &= \int_{T_1(\gamma)}^{\infty} (s - T_1(\gamma))e^{-s} f(y(s))\,ds \\ &\le \gamma e^{-T_1(\gamma)} \sup_{0\le y\le\gamma} \frac{f(y)}{y}.\end{aligned}$$

This implies that

$$e^{T_1(\gamma)} \le \sup_{0\le y\le\gamma} \frac{f(y)}{y} \to 0 \quad \text{as } \gamma \to 0.$$

Hence $T_1(\gamma) \to -\infty$ as $\gamma \to 0$.

(ii) Let $f'(0) > 0$. Let φ be the solution of

$$\begin{cases} -\varphi'' = f'(0)e^{-t}\varphi\,, \\ \varphi(\infty) = 1\,, \\ \varphi'(\infty) = 0\,, \end{cases} \tag{3.30}$$

and T the first zero of φ.

For any two nonnegative continuous functions ρ_1 and ρ_2 defined on $\mathbb{R}$ with $\rho_1 \ge \rho_2$, consider the following problem π_i $(i = 1, 2)$.

$$(\pi_i){:}\quad \begin{cases} -\psi'' = \rho_i e^{-t}\psi \quad \text{in } \mathbb{R}\,, \\ \psi(\infty) > 0\,, \\ \psi'(\infty) = 0. \end{cases} \tag{3.31}$$

Let ψ_i be a solution of (π_i) and T_i the first zero of ψ_i. Then we claim that

$$T_2 \le T_1. \tag{3.32}$$

Suppose not, then $T_1 < T_2$ and let $W(t) = \psi_1\psi_2' - \psi_2\psi_1'$. Then $W'(t) = \psi_1\psi_2 e^{-t}(\rho_1 - \rho_2)$ and hence integrating $W'(t)$ from T_2 to ∞, we have

$$-\psi_1(T_2)\psi_2'(T_2) = \int_{T_2}^{\infty} \psi_1\psi_2 e^{-t}(\rho_1 - \rho_2)\,dt \ge 0$$

which is a contradiction. This proves (3.32).

Now for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for $0 < y < \delta$,

$$(1-\varepsilon)f'(0) \le f(y)/y \le (1+\varepsilon)f'(0)\,, \tag{3.33}$$

Let $0 < \gamma < \delta$ and by taking

$$\rho_2 = (1-\varepsilon)f'(0)\,, \quad \rho_1 = \frac{f(y)}{y}\,, \quad \psi_2(t) = \varphi\left(t + \log\frac{1}{1-\varepsilon}\right)\,,$$

we obtain from (3.32) and (3.33),

$$T + \log(1-\varepsilon) \le T(\gamma). \tag{3.34}$$

Similarly by taking

$$\rho_1 = (1+\varepsilon)f'(0)\,, \quad \psi_1 = \varphi\left(t + \log\frac{1}{1+\varepsilon}\right)\,, \quad \rho_2 = \frac{f(y)}{y}\,,$$

we obtain

$$T(\gamma) \le T + \log(1+\varepsilon). \tag{3.35}$$

Since ε is arbitrary, from (3.34) and (3.35) we obtain $\lim_{\gamma\to 0} T(\gamma) = T$. This proves (3.29).

REFERENCES

1. Adimurthi and S. L. Yadava, *Multiplicity results for semilinear elliptic equations in a bounded domain of* $\mathbb{R}^2$ *involving critical exponents*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **27** (1990), 481–504.
2. F. V. Atkinson and L. A. Peletier, *Ground states and Dirichlet problem for* $-\Delta u = f(u)$ *in* $\mathbb{R}^2$, Arch. Rational Mech. Anal. **96** (1986), 147–165.
3. ——, *Emden-Fowler equations involving critical exponents*, Nonlinear Anal. TMA **10** (1986), 755–776.
4. F. V. Atkinson, H. Brezis and L. A. Peletier, *Nodal solutions of elliptic equations with critical Sobolev exponents*, C. R. Acad. Sci. Paris **306** (1988), 711–714.
5. G. Cerami, S. Solomini and M. Struwe, *Some existence results for super linear elliptic boundary value problems involving critical exponents*, J. Functional Anal. **69** (1986), 289–306.

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