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Stochastic evolution equations in locally convex space

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Abstract. Ito's stochastic integral is defined with respect to a Wiener process taking values in a locally convex space and Ito's formula is proved. Existence and uniqueness theorem is proved in a locally convex space for a class of stochastic evolution equations with white noise as a stochastic forcing term. The stochastic forcing term is modelled by a locally convex space valued stochastic integral.

Keywords. Locally convex space; Wiener process; stochastic integral; Ito's formula; stochastic evolution equation.

List of symbols

Let E and F be two sequentially complete locally convex space and $(\Omega,\mathcal{B},P)$ be a complete finite measure space. $T \in \mathbf{R}$, $T < \infty$ and $r \geq 1$.

$\mathcal{P}_E \equiv$ family of all continuous semi-norms on E.

$L(E, F) \equiv$ space of all continuous linear operators from E into F.

$L(E, E)$ will be denoted by $L(E)$.

$E^* \equiv$ space of all continuous linear functionals on E.

$L^r(\Omega,E,P) \equiv$ space of all Bochner integrable functions X from Ω into E such that $\int_\Omega \{p(X)\}^r \,\mathrm{d}P < \infty$ for every $p \in \mathcal{P}_E$.

$C([0,T], E) \equiv$ space of all continuous functions from $[0,T]$ into E.

$\mu \equiv$ Lebesgue measure on $[0,T]$.

$\mu(\mathrm{d}t)$ will be denoted by $\mathrm{d}t$.

Let A be a densely defined linear operator from E into E.

$\rho(A) \equiv$ resolvent set of A.

$R(\lambda,A) \equiv (I-\lambda A)^{-1}$ (inverse of the operator $(I-\lambda A)$) for $\frac{1}{\lambda} \in \rho(A)$.

1. Introduction

Stochastic analysis in infinite dimensions appears in several fields e.g. random vibrations, random particle system, etc. An abstract theory of stochastic evolution equation has been established by Curtain [2], Curtain and Pritchard [3], Ichikawa [5] and others in Hilbert space, Kuo [8] in abstract Wiener space. However it is also interesting from the viewpoint of applications to discuss stochastic evolution equation in some locally convex space. In this paper, we consider the following class of stochastic evolution equation

$$\mathrm{d}u(t) = A(t)\, u(t)\, \mathrm{d}t + \phi(t)\mathrm{d}\omega(t) + f(t)\, \mathrm{d}t, \quad 0 < t \le T \tag{1}$$

$$u(0) = u_0$$

in a locally convex space F, where $A(t)$ for each $t \in [0, T]$ is a linear unbounded operator on F, $\{\omega(t), 0 \le t \le T\}$ a Wiener process in a locally convex space E, $\phi(t)$ and $f(t)$ are stochastic processes with values in $L(E,F)$ and F respectively. In §2 of this paper, we will study Bochner integrals, evolution operators and Wiener process in a locally convex space. In §3, we will define stochastic integrals in F with respect to a Wiener process in E and will describe some of its properties. We will also prove Ito's formula in §3. In the last section, we will prove the existence and uniqueness theorem for the stochastic evolution equation (1).

2. Notations and preliminary results

2.1 *Bochner integration in locally convex space*

Let $(\Omega, \mathcal{B}, P)$ be a complete finite measure space and E be a real sequentially complete locally convex space. Let $\mathcal{P}_E$ be the collection of all continuous semi-norms on E. A function X from Ω into E is said to be Bochner measurable if there exists a sequence $(X_n)_{n \in N}$ of simple functions from Ω to E such that X_n converges to X in E. If X is Bochner measurable then $p(X)$ is measurable for all $p \in \mathcal{P}_E$. Let

$$X = \sum_{i=1}^{n} \alpha_i | E_i$$

be a simple function. For $B \in \mathcal{B}$, define

$$\int_B X\, \mathrm{d}P = \sum_{i=1}^{n} \alpha_i\, P(B \cap E_i).$$

A function X from Ω into E is said to be Bochner integrable if there exists a sequence $(X_n)_{n \in N}$ of simple functions converging to X in E a.s. such that

$$\lim_{n \to \infty} \int_\Omega p(X_n - X)\, \mathrm{d}P = 0$$

for every $p \in \mathcal{P}_E$. For a Bochner integrable function X and $B \in \mathcal{B}$, define

$$\int_B X\, \mathrm{d}P = s - \lim_{n \to \infty} \int_B X_n\, \mathrm{d}P.$$

For a Bochner integrable function X, we have

(a) $$p\left(\int_B X \, dP\right) \leq \int_B p(X) \, dP$$

for every $p \in \mathcal{P}_E$.

(b) Let A be a closed linear operator with domain $\mathcal{D}(A) \subset E$ (E is a Fréchet space) and range in a Fréchet space Y. Let X from Ω into E be a Bochner integrable function such that $X(\Omega) \subset \mathcal{D}(A)$ and AX is also Bochner integrable, then

$$\int_B X \, dP \in \mathcal{D}(A) \tag{2}$$

and

$$A \int_B X \, dP = \int_B AX dP \tag{3}$$

for every $B \in \mathcal{B}$. We will say that the two Bochner integrable function are equal if they are equal a.s. For $1 \leq r < \infty$, define

$$L^r(\Omega, E, P) = \{X:\Omega \to E, X \text{ Bochner integrable and}$$

$$\int_\Omega \{p(X)^r \, dp < \infty \ \forall \ p \in \mathcal{P}_E\}.$$

$L^r(\Omega, E, P)$ becomes sequentially complete locally convex space under the family of semi-norms

$$q(X) = q(X, p) = [\int_\Omega \{p(X)\}^r \, dP]^{1/r},$$

where $p \in \mathcal{P}_E$. Let $(\Omega, \mathcal{B}, P)$ be a complete probability space. A Bochner integrable function X from Ω into E is called a random variable.

For a random variable X, $E(X)$ will denote the $\int X \, dP$ and will be called expectation of X. For every x^* belongs to E^* (space of all continuous linear functionals on E) we have

$$E\langle X, x^*\rangle = \langle E(X), x^*\rangle.$$

Let $\mathcal{B}'$ be a sub σ-algebra of $\mathcal{B}$ and X be an E-valued simple random variable defined on $(\Omega, \mathcal{B}, P)$ such that $X = \sum_{i=1}^{n} \alpha_i | E_i$. Define

$$E(X/\mathcal{B}') = \sum_{i=1}^{n} \alpha_i \, P(E_i/\mathcal{B}). \tag{4}$$

Obviously $E(X/\mathcal{B}')$ is a $\mathcal{B}'$-measurable E-valued random variable. $E(X/\mathcal{B}')$ will be called the conditional expectation of X given $\mathcal{B}'$. The following properties of $E(X/\mathcal{B}')$ can be easily obtained.

(a) for every $p \in \mathcal{P}_E$,

$$p(E(X/\mathcal{B}')) \leq E(p(X)/\mathcal{B}'),$$

(b) $E(E(X/\mathcal{B}')) = E(X)$,

(c) for every $x^* \in E^*$,

$$\langle E(X/\mathcal{B}'), x^*\rangle = E(\langle X, x^*\rangle/\mathcal{B}').$$

Let X belong to $L^1(\Omega, E, P)$ then there exists a sequence $\{X_n\}$ of E-valued simple random variables such that X_n converges to X in E a.s. and

$$\lim_{n\to\infty} \int_\Omega p(X_n - X)\, dp = 0$$

for every $p \in \mathcal{P}_E$. Define

$$E(X/\mathcal{B}') = \lim_{n\to\infty} E(X_n/\mathcal{B}') \tag{5}$$

in $L^1(\Omega, E, P)$. Since $L^1(\Omega, E, P)$ is complete, we have $E(X/\mathcal{B}')$ is an E-valued $\mathcal{B}'$-measurable, unique upto equivalence class, random variable. $E(X/\mathcal{B}')$ will be called conditional expectation of X given $\mathcal{B}'$. Properties (a), (b), (c) of conditional expectation stated earlier for simple random variable are also obviously true for any X in $L^1(\Omega, E, P)$. For analogous result in Banach spaces see [9].

2.2 *Evolution operator in locally convex space*

Definition (2.1).

Let F be a locally convex space and $T = [0, T]$ a real finite interval and denote $\Delta(T) = \{(t, s),\ 0 \leq s \leq t \leq T\}$. A function $U(.,.)$ from $\Delta(T)$ into $L(F)$ (space of all continuous linear operator from F into F) is said to be almost strong evolution operator if

(a) $U(t, r)\ U(r, s) = U(t, s)$ for $0 \leq s \leq r \leq t \leq T$,

(b) $U(t, s)$ is strongly continuous in s on $[0, t]$ and in t on $[s, T]$.

(c) for each $t \in T$ there exists a densely-defined closed linear operator $A(t)$ on F such that

$$U(t, s) : \mathcal{D}(A(s)) \to \mathcal{D}(A(t)) \text{ for } t > s$$

(d) $\int_s^t A(r)\ U(r, s)x\ ds = (U(t, s) - I)x$

for $x \in \mathfrak{D}_{s,t}(A) = \{x \in F : U(r, s)x \in \mathfrak{D}(A(r)) \text{ for } s \leq r \leq t\}$.
Condition (d) implies that

(d′) $\dfrac{\partial}{\partial t} U(t, s)x = A(t)\, U(t, s)x$ a.s. (Lebesgue measure in $[0, T]$) for every $x \in \mathfrak{D}_{s,t}(A)$.

$\{A(t), t \,\varepsilon\, [0, T]$ is called generator associated to almost strong evolution operator $U(.\,,.)$.

Regarding the existence of almost strong evolution operator, we have the following theorem from Yosida [12].

Theorem 2.1.
Let F be a sequentially complete locally convex space and $A(t)$ for each t, be a closed linear operator with domain $\mathfrak{D}(A(t))$ and range $R(A(t))$ both in F. Suppose that $\{A(t),\ t \in [0, T]\}$ satisfies the following conditions:

(1) $\mathfrak{D}(A(t))$ is independent of t and it is dense in F.

(2) $(0, \infty) \subset \rho(A(t))$ (resolvent set of $A(t)$) for each t. There exists a fundamental family $\mathcal{P}_F$ of continuous semi-norms on F satisfying the following condition:

for every $\lambda > 0$, $p \in \mathcal{P}_F$, there exists a positive constant M such that

$$p\{R(\lambda, A(t_n))\, R(\lambda, A(t_{n-1}))\ldots R(\lambda, A(t_1))x\} \leq M\, p(x).$$

Here M is independent of λ, $t_i (1 \leq i \leq n)$, n and x; $R(\lambda, A(t)) = (I - \lambda\, A(t))^{-1}$ and $0 \leq t_1 \leq \ldots \leq t_n = T$ is a partition of $[0, T]$.

(3) $A(s)^{-1} \in L(F)$, $A(t)\, A^{-1}(s) \in L(F)$, $0 \leq s,\ t \leq T$,

(4) For every $x \in F$,

$$\frac{1}{(t-s)}\ c(t, s)x = \frac{1}{(t-s)}\ (A(t)\, A^{-1}(s) - I)x$$

is bounded and uniformly continuous in t and s, $t \neq s$ and

$$\lim_{k \to \infty} k\, c\left(t, t - \frac{1}{k}\right)x = c(t)x$$

exists uniformly in t, where $c(t) \in L(F)$. We have moreover,

$$p\left(\frac{1}{(t-s)}\ c(t, s)x\right) \leq Np(x)$$

for some constant $N > 0$ independent of $x \in F$, t and s. Then $\{A(t), 0 \leq t \leq T\}$ is generator of an almost strong evolution operator.

Let μ denote Lebesgue measure on $[0, T]$ and as usual, we will denote $\mu(\mathrm{d}t)$ by $\mathrm{d}t$. For the existence and uniqueness of the solution of deterministic non-homogeneous evolution equation, we have

Proposition 2.2

Let F be a sequentially complete locally convex space and $\{A(t), 0 \leq t \leq T\}$ be the generator of almost strong evolution operator $U(.,.)$. Assume that

(1) $U(t,.) f(.) \in \mathscr{D}(A(t))$ for almost all t,

(2) $f \in L^1([0, T], F, \mu)$ and for each t, $A(t)\, U(t,.) f(.) \in L^1([0, T], F, \mu)$

Then the abstract evolution equation

$$\begin{aligned} \dot{u}(t) &= A(t)\, u(t) + f(t), \\ u(s) &= u_0 \in \mathscr{D}(A(s)), \end{aligned} \tag{6}$$

in F has a unique strongly continuous solution

$$u(t) = U(t, s) u_0 + \int_s^t U(t, r) f(r)\, \mathrm{d}r. \tag{7}$$

2.3 *Wiener process in locally convex spaces*

Let $(\Omega, \mathscr{B}, P)$ be a complete probability space and E be a reflexive real Fréchet space with a Schauder basis $\{e_n\}_{n \in N}$ such that for every $p \in \mathscr{P}_E$ there exists a constant M_p satisfying $p(e_n) \leq M_p$ for all n. Thus for every $x \in E$ we have

$$x = \sum_{n=1}^{\infty} \langle x, e_n^* \rangle e_n,$$

where $e_n^* \in E^*$ (strong dual of E). Further assume that $\{e_n^*\}$ be a Schauder basis of E^*. For $x_1, x_2 \in E$, define a linear operator $x_1 o x_2$ from E^* into E by

$$x_1 o x_2(x^*) = \langle x_2, x^* \rangle x_1$$

for all $x^* \in E^*$. Obviously $x_1 o x_2$ defines a continuous linear operator from E^* into E.

Definition 2.2.

Let u be an E-valued random variable defined on $(\Omega, \mathscr{B}, P)$ and $u \in L^2(\Omega, E, P)$. The covariance of u denoted by $\mathrm{cov}(u)$ is a linear operator from E^* into E, defined by

$$\mathrm{cov}(u) = E\{(u - E(u))\, o\, (u - E(u))\}. \tag{8}$$

Since $u \in L^2(\Omega, E, P)$, $\mathrm{cov}(u)$ exists as a bounded linear operator from E^* into E. We have following representation for the covariance operator of u,

$$(\mathrm{Cov}(u))\,(x^*) = \int_E \langle x, x^*\rangle\, x\, \mathcal{P}_u(\mathrm{d}x) - \int_E \langle x, x^*\rangle\, x_1\, \mathcal{P}_u(\mathrm{d}x)$$

$$- \int_E \langle x_1, x^*\rangle\, x\, \mathcal{P}_u(\mathrm{d}x) + \langle x_1, x^*\rangle\, x_1,$$

where $x_1 = E\{u\}$ and P_u is the probability distribution of the random variable u. Let Q be any probability measure on E such that $\int_E (p(x))^2\, Q(\mathrm{d}x) < \infty\ \forall\, p \in \mathcal{P}_E$ and x_1 be any fixed element of E. Then the linear operator T: $E^* \to E$, given by

$$Tx^* = \int_E \langle x, x^*\rangle\, xQ(\mathrm{d}x) - \int_E \langle x, x^*\rangle\, x_1\, Q(\mathrm{d}x)$$

$$- \int_E \langle x_1, x^*\rangle\, xQ(\mathrm{d}x) + \langle x_1, x^*\rangle\, x_1$$

is obviously a covariance operator of an E-valued random variable u on the probability space $(E, \mathcal{B}_E, Q)$ with $u(x) = x\ \forall x\ E$ and $E\{u\} = x_1$.

Definition 2.3.

An E-valued stochastic process $\{\omega(t), 0 \le t \le T\}$ of Gaussian[†] random variables is called a Wiener process if

(a) $\omega(t) \in L^2(\Omega, E, P)$ for all $t \in [0, T]$, $\omega(0) = 0$ a.s.,
(b) $E\{\omega(t) - \omega(s)\} = 0$,
(c) $\mathrm{cov}\{\omega(t) - \omega(s)\} = (t-s)\,W$,

where W: $E^* \to E$ is a continuous linear operator defined by

$$W e_n^* = \lambda_n\, e_n \tag{9}$$

such that $\lambda_n \ge 0$ and $\sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$.

(d) $\{\omega(t), 0 \le t \le T\}$ is a process of independent increments and almost all paths of $\omega(t)$ are continuous.
Note that W has the following form

[†] An E-valued random variable u is called Gaussian if $\langle u, e_n^*\rangle$ is a real Gaussian random variable for all n.

$$Wx^* = \sum_{n=1}^{\infty} \langle x^*, e_n \rangle \lambda_n e_n .$$

Since E^* is barreled and E is quasi-complete, therefore W is a nuclear operator [10].

Lemma 2.3.

Let $\{\omega(t),\ 0 \le t \le T\}$ be an E-valued Wiener process. Then
(a) There exists a sequence $\{\beta_n(t)\}$ of mutually independent real Wiener processes such that

(i) $E\{\beta_n(t) - \beta_n(s)\}^2 = (t-s)\lambda_n$,

(ii) $\sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$,

(iii) $\omega(t) = \sum_{n=1}^{\infty} \beta_n(t) e_n$.

(b) for every $p \in \mathcal{P}_E$ there exists a positive constant C such that

$$E[p\{\omega(t) - \omega(s)\}^2] \le C(t-s) \tag{10}$$

for $t \ge s$.

Proof: (a) Define $\beta_n(t) = \langle \omega(t), e_n^* \rangle$. It is easy to see that $\{\beta_n(t)\}$ is the required sequence of mutually independent real Wiener processes.

(b) Define $\omega_n(t) = \sum_{m=1}^{n} \beta_m(t) e_m$. For any $p \in \mathcal{P}_E$, $p(\omega_n(t))$ converges to $p(\omega(t))$ a.s. and $p(\omega_n(t)) \le M_p \sum_{m=1}^{\infty} |\beta_m(t)|$. Since $\sum_{m=1}^{\infty} \sqrt{\lambda_m} < \infty$, we have $\sum_{m=t}^{\infty} |\beta_m(t)| \in L^2(\Omega, \mathbf{R}, P)$. Rest of the proof follows by a simple application of Lebesgue dominated convergence theorem.

We have the following converse of the lemma 2.3 (a).

Lemma 2.4. Let $\{\beta_n(t)\}$ be a sequence of real valued mutually independent Wiener processes such that $E\{\beta_n(t) - \beta_n(s)\}^2 = (t-s)\lambda_n$ and $\sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$. Then the process $\omega(t) = \sum_{n=1}^{\infty} \beta_n(t)\, e_n$ is an E-valued Wiener process.

Proof. Since $\sum_{n=1}^{\infty} \sqrt{\lambda_n} < \infty$, $\omega(t) \in L^2(\Omega, E, P)$. Since $\langle \omega(t), e_n^* \rangle = \beta_n(t)$, $\{\omega(t)\}$ is a Gaussian process. It is easy to verify that $\{\omega(t)\}$ is an E-valued Wiener process with $\operatorname{cov}\{\omega(t)-\omega(s)\} = (t-s)W$, where $W e_n^* = \lambda_n e_n$.

3. Stochastic integration

Let F be a countably Hilbert space, i.e., F is a complete locally convex space which topology is given by countable family of compatible Hilbertain norms $|\,,\,|_n$, $n \in N$ such that $|\,,\,|_{n_1} \leq |\,,\,|_{n_2}$ for all $n_1, n_2 \in N$ with $n_1 < n_2$. Since F is complete, we have $F = \bigcap_n F_n$, where F_n be the completion of F under the norm $|\,,\,|_n$. Let E be a locally convex space of §2.3 and $\{\omega(t), 0 \leq t \leq T$ be a Wiener process in E. Let $L(E, F)$ be the space of all continuous linear operators from E into F. By Banach Steinhaus theorem, $L(E, F)$ be a sequentially complete locally convex space under the topology of simple convergence [11]. Let $\mathcal{B}_t = \sigma(\omega(s), s \leq t)$. Define

$M(E, F) = \{\phi(.,.)$: ϕ is an $L(E, F)$-valued measurable process adapted to $\mathcal{B}_t\}$,

$M_r^0(E, F) = \{\phi(.,.) \in M(E, F)$: ϕ is a t-step function on $[0, T]$ and $\phi \in L^r([0, T] \times \Omega, L(E, F), \mu \times p)$,

$M_r(E, F) = \{\phi(.,.) \in M(E, F) : \phi \in L^r([0, T] \times \Omega, L(E, F), \mu \times P)\}$

where $r \geq 1$ is an integer. Define a locally convex topology on $M_r(E, F)$ by family of semi-norms given by

$$q(\phi) = q(\phi, p) = \left[\int_0^T E\{p(\phi(t))\}^r \, dt\right]^{1/r}$$

for some $p \in \mathcal{P}_{L(E,F)}$ It is easy to see that $M_2^0(E, F)$ is dense in $M_2(E, F)$.

Definition 3.1

Let $\phi \in M_2^0(E, F)$ such that

$$\phi(t, \omega) = \sum_{j=1}^{N} \phi(t_j, \omega)\, \chi^{(t)}_{[t_j, t_{j+1}]}, \tag{11}$$

where $0 = t_1 < t_2 < \ldots < t_N = T$. The stochastic integral of ϕ with respect to the Wiener process $\{\omega(t), 0 \leq t \leq T\}$ is an F-valued random variable denoted by $\int_0^T \phi(t)\, d\omega(t)$ and is defined by

$$\int_0^T \phi(t)\, d\omega(t) = \sum_{j=1}^{N} \phi(t_j)\, \{\omega(t_{j+1}) - \omega(t_j)\}. \tag{12}$$

Lemma 3.2

If $\phi \in M_2^0(E, F)$, then

(a) $$E\left\{\int_0^T \phi(t)\, d\omega(t)\right\} = 0$$

(b) Given any $n \in N$ there exist a constant $C > 0$ and $q \in \mathcal{P}_{L(E,F)}$ such that

$$E\left\| \int_0^T \phi(t)\, d\omega(t) \right\|_n^2 \leq C \int_0^T E\{q(\phi(t))\}^2\, dt.$$

Proof: (a) is an easy consequence of the fact that $\{\omega(t), 0 \leq t \leq T\}$ is a process of independent increments. The proof of (b) follows from the fact that if $A|\in|L(E, F)$ then $A \in L(E, F_n)$ for every n. Define a map $I : M_2^0(E, F) \to L^2(\Omega, F, P)$, by

$$I(\phi) = \int_0^T \phi(t)\, d\omega(t).$$

For lemma 3.2b, we conclude that I is a continuous linear map from $M_2(E, F)$ into $L^2(\Omega, F, P)$ defined on $M_2^0(E, F)$. Since $L^2(\Omega, F, P)$ is complete and $M_2^0(E, F)$ is dense in $M_2(E, F)$, map I has unique extension $\bar{I}$ as a continuous linear map from $M_2(E, F)$ into $L^2(\Omega, F, P)$. For any $\phi \in M_2(E, F)$, $\bar{I}(\phi)$ will still be denoted by

$$\int_0^T \phi(t)\, d\omega(t)$$

and is called the stochastic integral of ϕ with respect to Wiener process $\{\omega(t), 0 \leq t \leq T\}$. From lemma 3:2, we have

Theorem 3.3

If $\phi \in M_2(E, F)$, then

(a) $$\int_0^T \phi(t)\, d\omega(t) = \sum_{i=1}^{\infty} \int_0^T \phi(t)\, e_i\, d\beta_i(t), \tag{13}$$

(b) $$E\left\{\int_0^T \phi(t)\, d\omega(t)\right\} = 0, \tag{14}$$

(c) Given any $n \in N$ there exist a constant $C > 0$ and $q \in \mathcal{P}_{L(E,F)}$ such that

$$E\left\|\int_0^T \phi(t)\, d\omega(t)\right\|_n^2 \leq C\int_0^T E\{q(\phi(t))\}^2\, dt. \tag{15}$$

Proof. (a) If $\phi \in M_2^0(E, F)$, then equation (13) is obvious. For $\phi \in M_2(E, F)$, equation (13) holds by the usual limiting arguments.

(b) Let $\phi_n \in M_2^0(E, F)$ such that ϕ_n converges to ϕ in $M_2(E, F)$ and $I(\phi_n)$ converges to $I(\phi)$. By Holder's inequality, we have

$$\lim_{n\to\infty} E\int_0^T \phi_n(t)\, d\omega(t) \doteq E\int_0^T \phi(t)\, d\omega(t)$$

in F, which completes the proof.

(c) Let $\{\phi_m\}_{m\in N}$ be same as in (b). Then for any $n \in N$, we have

$$E\left\|\int_0^T \phi(t)\, d\omega(t)\right\|_n^2 \leq E\left\{\left\|\int_0^T (\phi_m-\phi)(t)\, d\omega(t)\,\right\|_n \left\|\int_0^T \phi(t)\, d\omega(t)\right\|_n\right\} + E\left\{\left\|\int_0^T (\phi_m-\phi)(t)\, d\omega(t)\,\right\|_n \left\|\int_0^T \phi_m(t)\, d\omega(t)\right\|_n\right\}$$
$$+E\left\|\int_0^T \phi_m(t)\, d\omega(t)\right\|_n^2$$

By Holder's inequality

$$\begin{aligned} E\left\|\int_0^T \phi(t)\, d\omega(t)\right\|_n^2 &\leq \lim_{m\to\infty} E\left\|\int_0^T \phi_m(t)\, d\omega(t)\right\|_n^2, \\ &\leq C \lim_{m\to\infty} \int_0^T E\{q(\phi_m(t))\}^2\, dt, \\ &= C\int_0^T E\{q(\phi(t))\}^2\, dt \end{aligned}$$

for some positive constant C and for some $q \in \mathcal{P}_{L(E,F)}$.
Analogous to (2) and (3) we have

Lemma 3.4.

Let A be a closed linear operator on F and $\phi \in M_2(E, F)$ such that
(i) for all $t \in [0, T]$ and for all i

$\phi(t)\, e_i \in \mathfrak{D}(A)$ a.s.,

(ii) for every $n \in N$,

$$\sum_{i=1}^{\infty} E \int_0^T \| A\phi(s) e_i \|_n \, ds < \infty.$$

Then

(a) $$\int_0^T \phi(s) \, d\omega(s) \in \mathfrak{D}(A),$$

(b) $$A \int_0^T \phi(s) \, d\omega(s) = \int_0^T A\phi(s) \, d\omega(s).$$

Lemma 3.5.

Let $U(t, s)$ be an almost strong evolution operator and $\phi \in M_2(E, F)$. Then

$$y_t = \int_0^t U(t, s) \, \phi(s) \, d\omega(s)$$

is strong continuous in mean square, i.e. $y_t \in C([0, T], L^2(\Omega, F, P))$.

Proof: Define

$$y_t^i = \int_0^t U(t, s) \, \phi(s) e_i \, d\beta_i(s).$$

For any $n \in N$ and $\delta > 0$, we have

$$E \| y_{t+\delta}^i - y_t^i \|_n^2 \leq 2E \| \, (U(t+\delta, t) - I) y_t^i \, \|_n^2$$
$$+ C \int_t^{t+\delta} E(q_1(\phi(s))\}^2 \, ds$$

for some $q_1 \in \mathcal{P}_{L(E,F)}$ and for some constant $C > 0$. Thus we have

$$\lim_{\delta \to 0} E \| y_{t+\delta}^i - y_t^i \|_n^2 = 0.$$

Similarly, we have for every $n \in N$ and $\delta > 0$ sufficiently small

$$\lim_{\delta \to 0} E\|y_t^i - y_{t-\delta}^i\|_n^2 = 0.$$

Using mutual independence of Wiener processes $\beta_i(t)$, it is easy to see that

$$\sum_{i=1}^{N} \int_0^t U(t, s)\, \phi(s) e_i\, \mathrm{d}\beta_i(s) \text{ converges to}$$

$$\int_0^T U(t, s)\, \phi(s)\, \mathrm{d}\omega(s)$$

in $C([0, T], L^2(\Omega, F, P))$, which completes the proof of lemma. The stochastic version of Fubini's type of theorem is not difficult to prove. We have

Lemma 3.6

Let $\psi(t, s, \omega) : [0, T] \times [0, T] \times \Omega \to L(E, F)$ be such that for each $x \in E$, $\psi(t, s, \omega)x$ is measurable on $[0, T] \times [0, T] \times \Omega$ and $\psi(t, \cdot, \cdot)x$ and $\psi(\cdot, t, \cdot)x$ are measurable relative to $\mathcal{B}_t$ for almost all $t \in [0, T]$. Suppose that for every $q \in \mathcal{P}_{L(E,F)}$,

$$\int_0^T \int_0^T E\{q(\phi)\}^2\, \mathrm{d}s\, \mathrm{d}t < \infty.$$

If we define

$$y_1(\omega) = \sum_{i=1}^{\infty} \int_0^T \left\{ \int_0^T \psi(t, s, \omega) e_i\, \mathrm{d}s \right\} \mathrm{d}\beta_i(t),$$

$$y_2(\omega) = \int_0^T \left\{ \sum_{i=1}^{\infty} \int_0^T \psi(t, s, \omega) e_i\, \mathrm{d}\beta_i(t) \right\} \mathrm{d}s.$$

Then $y_1 = y_2$ a.s. and $y_1, y_2 \in L^2(\Omega, F, P)$.
The following lemma is crucial in the proof of Ito's formula.

Lemma 3.7. Let G be a countably Hilbert space and $\{X(t),\ 0 \le t \le T\}$ be a $L(F, L(F, G))$-valued stochastic process which is adapted to $\mathcal{B}_t$ and is such that $X(t) \in L^2(\Omega, L(F, G)), P)$ for all $t \in [0, T]$. Let Y_0 be a $L(E, F)$-valued random variable measurable with respect to $\mathcal{B}_{t_0} (0 \le t_0 < T)$ and $Y_0 \in L^4(\Omega, L(E, F), P)$. Then

$$E\{X(Y_0\, \Delta\omega)\ [Y_0\, \Delta\omega]/\mathcal{B}_s\} = (t-s) \sum_{i=1}^{\infty} \lambda_i\, X(Y_0\, e_i)\ [Y_0\, e_i] \tag{16}$$

for all s, t with $t_0 \le s \le t$, where $X = X(s)$ and $\Delta\omega = \omega(t) - \omega(s)$. The proof of the following theorem is similar to the proof of Curtain and Falb [1] of Ito's formula in Hilbert space.

Theorem 3.8 (Ito's formula)

Let E, F and G are as before and $\{\omega(t), 0 \le t \le T\}$ be an E-valued Wiener process. Let $\xi(t)$ be a F-valued stochastic process with stochastic differential $d\xi(t) = b(t)dt + \sigma(t)d\omega(t)$ and $g: F \to G$ be a continuous function such that

(a) g is twice weakly differentiable,†

(b) Dg and D^2g are bounded continuous functions from F into $L(F, G)$ and $L(F, L(F, G))$ respectively,

(c) $b(t)$ is a F-valued stochastic process adapted to $\mathcal{B}_t$ and $\sigma \in M_2(E, F)$ such that

$$\int_0^T q(b(s))\, ds < \infty \text{ a.s. } \forall\, q \in \mathcal{P}_F$$

and $\qquad \sigma \in L^4([0, T] \times \Omega, L(E, F), \mu \times P)$.

Then the process $z(t) = g(\xi(t))$ has G-valued stochastic differential

$$dz(t) = \{Dg(\xi(t))b(t) + \frac{1}{2} \sum_{i=1} \lambda_i D^2 g(\xi(t))$$

$$(\sigma(t)e_i)\ [\sigma(t)e_i]\}\, dt + [Dg(\xi(t))\ (\sigma(t))]\, d\omega(t). \tag{17}$$

4. Stochastic evolution equation

Let $\{A(t), 0 \le t \le T\}$ be generator of almost strong evolution operator $U(.,.)$ in a countably Hilbert space F and $\{\omega(t), 0 \le t \le T\}$ be a Wiener process in a locally convex space E. Consider the following stochastic evolution equation

$$\begin{aligned} du(t) &= A(t)\, u(t)\, dt + f(t)\, dt + \phi(t)\, d\omega(t), \\ u(0) &= u_0 \end{aligned} \tag{18}$$

in F, where $\phi \in M_2(E, F)$, $u_0 \in F$ and $f \in L^2([0, T] \times \Omega, F, \mu \times P)$. By (18), we mean that

$$u(t) = u_0 + \int_0^t A(r)\, u(r)\, dt + \int_0^t f(r)\, dr + \int_0^t \phi(r)\, d\omega(r).$$

†For definition and other details about weak differentiability in locally convex space we refer [7].

Definition 4.1.

A solution $u(t)$ of (18) is said to be strong solution if $u(t) \in \mathfrak{D}(A(t))$ a.s., $u(t) \in C([0, T], L^2(\Omega, F, P))$ and $u(t)$ satisfies (18) a.s. on $T \times \Omega$. The strong solution $u(t)$ is said to be unique solution if whenever $v(t)$ is another solution

$$P\left\{\sup_{0 \leq t \leq T} \|u(t) - v(t)\|_n \neq 0\right\} = 0$$

for every $n \in N$.

Theorem 4.1

Assume that

(i) $U(t, 0)u_0 \in \mathfrak{D}(A(t))$ for every t,

(ii) $U(t, s)\ \phi(s)e_i \in \mathfrak{D}(A(t))$ a.s. for all i and for almost all $t > s$, and

$$\sum_{i=1}^{\infty} \lambda_i E\left\{\int_0^t \|A(t)\ U(t, r)\ \phi(r)e_i\|_n^2\, dr\right\} < \infty$$

for every n, $E\|A(t)\ U(t, r)\ \phi(r)e_i\|_n^2 \in L^1([0, T] \times [0, T])$ for every n and every i,

(iii) $U(t, s)\, f(s) \in \mathfrak{D}(A(t))$ a.s. for almost all t, and

$$\int_0^t \|A(t)\ U(t, r)\, f(r)\|_n\, ds < \infty$$

a.s. for every $n \in N$.

Then (18) has a unique strong solution with continuous sample path given by

$$u(t) = U(t, 0)u_0 + \int_0^t U(t, r)\ \phi(r)\, d\omega(r)$$

$$+ \int_0^t U(t, r)\, f(r)\, dr. \tag{19}$$

Proof. Uniqueness of solution follows from the uniqueness of solution of deterministic homogeneous evolution equation $\dot{x}(t) = A(t)\, x(t)$. From lemma 3.5 and strong continuity of $U(t, s)$ in t we have

$$\int_0^t U(t, r)\ \phi(r)\, d\omega(r) \in C([0, T],\ L^2(\Omega, F, P))$$

and

$$\int_0^t U(t, r) f(r)\, \mathrm{d}r \in C([0, T], L^2(\Omega, F, P)).$$

Thus we have $u(t) \in C([0, T], L^2(\Omega, F, P))$. It remains to show that $u(t)$ given by (19) satisfies (18). From lemma 3.4, we have

$$A(t) \int_0^t U(t, r)\, \phi(r)\, \mathrm{d}\omega(r) = \int_0^t A(t)\, U(t, r)\, \phi(r)\, \mathrm{d}\omega(t) \text{ a.s.} \tag{20}$$

From lemma (3.6) and equation (20),

$$\int_0^t A(r) \left\{ \int_0^r U(r, \alpha)\, \phi(\alpha)\, \mathrm{d}\omega(\alpha) \right\} \mathrm{d}r$$

$$= \int_0^t \left\{ \sum_{i=1}^{\infty} \int_0^r A(r)\, U(r, \alpha)\, \phi(\alpha) e_i\, \mathrm{d}\beta_i(\alpha) \right\} \mathrm{d}r,$$

$$= \sum_{i=1}^{\infty} \int_0^t \left\{ \int_\alpha^t A(r)\, U(r, \alpha)\, \phi(\alpha) e_i\, \mathrm{d}r \right\} \mathrm{d}\beta_i(\alpha),$$

$$= \sum_{i=1}^{\infty} \int_0^t \{ U(t, \alpha)\, \phi(\alpha) e_i - \phi(\alpha) e_i \}\, d\beta_i(\alpha),$$

$$= \int_0^t U(t, \alpha)\, \phi(\alpha)\, \mathrm{d}\omega(\alpha) - \int_0^t \phi(\alpha)\, \mathrm{d}\omega(\alpha). \tag{21}$$

Similarly, we can prove

$$\int_0^t A(r) \left\{ \int_0^r U(r, \alpha) f(\alpha)\, \mathrm{d}\alpha \right\} \mathrm{d}r$$

$$= \int_0^t U(t, \alpha) f(\alpha)\, \mathrm{d}\alpha - \int_0^t f(\alpha)\, \mathrm{d}\alpha. \tag{22}$$

From (21) and (22), we see that $u(t)$ satisfies (18)

Example. (Stochastic transport equation).

Let $F = L^2[0, 1]$ and A be a linear operator on $L^2[0, 1]$ with domain $\mathfrak{D}(A) = \{f \in L^2[0, 1],\ f$ is absolutely continuous and $f' \in L^2[0, 1],\ f(1) = 0\}$ defined by

$$Af = f'$$

Obviously A is the generator of a strongly continuous semi-group on $L^2[0, 1]$ and $\sigma(A) = \phi$. For each $t \in [0, T]$, let $A(t) = A$. It is easy to see that the family $\{A(t), t \in [0, T]\}$ of linear operators on $L^2[0, 1]$ satisfies conditions (1), (2), (3), (4) of theorem 2.1 and hence a generator of an almost strong evolution operator.

Let $E = L^p[0, 1]$ (for any $2 \leq p < \infty$) with $\{e_n\}$ as its Haar basis, namely,

$$e_1(t) = 1$$

$$e_{2^k+1}(t) = \begin{cases} \sqrt{2^k}, & t \in \left[\dfrac{2l-z}{2^{k+1}}, \dfrac{2l-1}{2^{k+1}}\right] \\ -\sqrt{2^k}, & t \in \left[\dfrac{2l-1}{2^{k+1}}, \dfrac{2l}{2^{k+1}}\right] \\ 0, \text{ for other } t. \end{cases}$$

$$(l = 1, 2, \ldots, 2^k, \; k = 0, 1, 2, \ldots).$$

Consider the following stochastic process in $L^p[0, 1]$:

$$\omega(t) = \sum_{n=1}^{\infty} \beta_n(t) e_n,$$

where $\{\beta_n(t)\}$ is a sequence of mutually independent real Wiener processes such that $E\{\beta_n(t) - \beta_n(s)\}^2 = (1/n^4)\,(t-s)$.

By lemma 2.4, $\{\omega(t)\}$ is an E-valued Wiener process.

Define a bounded linear operator $\mathfrak{B} : L^p[0, 1] \to L^2[0, 1]$ by

$$(B\psi)\,(x) = \int_x^1 \psi(y)\,dy.$$

Let $\phi(t) = \mathfrak{B}$(non-random) for every $t \in [0, T]$. Obviously $\phi \in M_2(E, F)$. Let $\{T(t),\ t \geq 0\}$ be the strongly continuous semigroup generated by A and let $U(t, s) = T(t-s), t \geq s$. $\{U(t, s)\}$ is an almost strong evolution operator generated by $\{A(t)\}$. Note that the semi-group $\{T(t),\ t \geq 0\}$ is obviously given by

$$T(t)f = g, \quad g(s) = \begin{matrix} f(s+t) & \text{for } s+t \leq 1 \\ 0 & \text{for } s+t > 1. \end{matrix}$$

We would like to solve the following stochastic transport equation:

$$\begin{aligned} &du(t) = u'(t)\,dt + B\,d\omega(t), \\ &u_0 \in \mathfrak{D}(A). \end{aligned} \tag{23}$$

Since $u_0 \in \mathfrak{D}(A)$, $Be_n \in \mathfrak{D}(A)$ for all n and $||AU(t, r)\, Be_n||_{L^2} \leq$ constant (independent of n), conditions (i) and (ii) of theorem 4.1 are satisfied. Therefore by theorem 4.1, equation (23) has a unique solution given by

$$u(t) = T(t) + \sum_{n=1}^{\infty} \int_0^t T(t-s)\, Be_n \, \mathrm{d}\beta_n \, .$$

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