Magnetization plateaux in a generalised ladder model

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Abstract

A spin-1/2 antiferromagnetic(AFM) generalised ladder model is constructed which consists of four-spin plaquettes, coupled through weaker exchange interactions, to two-spin rungs. In an extended parameter regime, the exact ground state of the ladder is determined. In this state, the four-spin plaquettes and the rungs are in their ground state spin configurations. In the presence of an external magnetic field, the magnetization/site has a plateau structure as a function of the magnetic field.

Key words: Magnetization plateaux, Spin ladder, RVB state
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1 Introduction

Recently, low-dimensional spin systems, particularly the Heisenberg spin ladders, have been the focus of several analytic, numerical and experimental studies[1,2]. Spin ladders can be considered as bridges between one-dimensional (1D) and two-dimensional(2D) systems. The 1D systems are more or less well understood whereas considerable gaps still exist in our understanding of 2D systems. The study of ladder models is expected to provide insight on how electronic and magnetic properties are modified from 1D to 2D. A large number of magnetic compounds with ladder structure have been discovered[1,2] which exhibit a variety of novel phenomena in the undoped as well as the doped states. Different types of spin-1/2 ladder models have been proposed including frustrated ladder models and models with modulated exchange interactions[3-14]. Ladder models have been studied in zero as well as finite magnetic fields. In this paper, we construct a spin-1/2 ladder model with modulated exchange interactions. The model consists of four-spin plaquettes connected to two-spin rungs. The dominant exchange interactions are within the plaquettes and the rungs. The coupling between the plaquettes and the rungs are through weaker

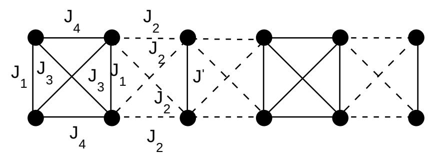


Fig. 1. Two-chain ladder model consisting of four-spin plaquettes coupled to two-spin rungs. The exchange interaction strengths are as shown in the Figure.

exchange interactions. Molecular magnets provide another example of spin systems consisting of weakly coupled spin clusters[15]. In the case of our ladder model, the spin clusters are the four-spin plaquettes and the two spin rungs.

We show that in a certain parameter regime, the exact ground and low-lying excited states of the full ladder model are of the product form, i.e., can be written in terms of the exact ground states of the four-spin plaquettes and the two-spin rungs. Also, in the presence of an external magnetic field, the magnetization/site exhibits the phenomenon of magnetization plateaux. The condition for the appearance of a plateau is given by[16]

$$S_u - m_u = integer \tag{1}$$

where S_u and m_u are the total spin and magnetization in unit period of the ground state.

2 Ground and excited states of the ladder model

The ladder model constructed by us is shown in Fig. 1. The solid lines represent the dominant exchange interactions within the four-spin plaquettes and along the two-spin rungs. The dotted lines describe the weaker exchange couplings between the plaquettes and the rungs. Within a plaquette, the horizontal, vertical and diagonal exchange interactions are of strengths J_4, J_1 and J_3 respectively. The exchange interaction along a two-spin rung is of strength J'. The ladder model consists of alternating four-spin plaquettes and two-spin rungs coupled via horizontal and diagonal exchange interactions (dotted lines) of strength J_2. Periodic boundary condition is assumed to hold true. The ladder model generalises a simpler model studied earlier[17] in which the horizontal and vertical exchange interactions in a four-spin plaquette are of equal strength. The spin Hamiltonian describing the ladder model is

$$H = \sum_{i=3j+1, j=0,1,\cdots} [J_4(\overrightarrow{S}_{1i}.\overrightarrow{S}_{1i+1} + \overrightarrow{S}_{2i}.\overrightarrow{S}_{2i+1}) + J_1(\overrightarrow{S}_{1i}.\overrightarrow{S}_{2i} + \overrightarrow{S}_{1i+1}.\overrightarrow{S}_{2i+1})$$

$$+J_3(\overrightarrow{S}_{1i}.\overrightarrow{S}_{2i+1}+\overrightarrow{S}_{2i}.\overrightarrow{S}_{1i+1})]+J'\sum_{i=3j,j=0,1,\cdots}\overrightarrow{S}_{1i}.\overrightarrow{S}_{2i}$$

$$+J_2\sum_{i=3j+2,j=0,1,\cdots}(\overrightarrow{S}_{1i}+\overrightarrow{S}_{2i}+\overrightarrow{S}_{1i+2}+\overrightarrow{S}_{2i+2}).(\overrightarrow{S}_{1i+1}+\overrightarrow{S}_{2i+1}) \quad (2)$$

$$=H_C+H_R+H_{CR}$$

The spin operator $\overrightarrow{S}_{1i}$ ($\overrightarrow{S}_{2i}$) is associated with the i-th site of the lower (upper) chain of the ladder, the site indices are sequential in a chain. The sub-Hamiltonians H_C and H_R describe the four-spin plaquettes and the rungs, respectively, whereas H_{CR} contains the exchange couplings between the plaquettes and the rungs. The total spin of each rung is a conserved quantity due to the special structure of the Hamiltonian.

We now determine the ground state of the ladder model using the method of 'divide and conquer'[18]. It is easy to show that the state, in which the four-spin plaquettes and the two-spin rungs are in their ground state spin configurations, is an exact eigenstate of the full Hamiltonian H(Eq. 2). H is a sum of three sub-Hamiltonians H_C, H_R and H_{CR}. The Hamiltonian H_C+H_R acting on the specified state gives back the same state with the eigenvalue, E_{CR}, equal to the sum of the ground state energies of all the plaquettes and the rungs in the ladder. The sub-Hamiltonian H_{CR} acting on the same state gives zero. Thus, the state is an exact eigenstate of H with the eigenvalue $E_1=E_{CR}$. In an extended parameter regime, the exact eigenstate also turns out to be the exact ground state. The proof is as follows:-

Let E_g be the exact ground state energy of the full Hamiltonian H. Then $E_g \leq E_1$. Let $|\psi_g\rangle$ be the exact ground state wave function. Then from variational theory,

$$E_g=\sum_j \langle\psi_g|\,H_j\,|\psi_g\rangle+\sum_j \langle\psi_g|\,H'_j\,|\psi_g\rangle \geq \sum_j(E_{jo}+E'_{jo}) \quad (3)$$

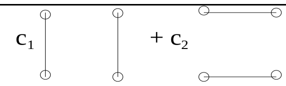
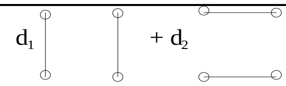
$$H=\sum_j(H_j+H'_j)$$

where H_j 's are the plaquette Hamiltonians with the ground state energy E_{jo} and H'_j's are the six-spin cluster Hamiltonians, each of which contains the rung exchange interaction Hamiltonian and the eight exchange couplings (four horizontal and four diagonal) which connect the rung to nearest-neighbour plaquettes. The ground state energy of H'_j is E'_{jo}. For $J_2 \leq \frac{J'}{4}$, E'_{jo} is the ground state energy of the rung Hamiltonian. In the ground state, the rung is in a singlet configuration. We can now write down the inequality,

$$\sum_i(E_{io}+E'_{io}) \leq E_g \leq E_1 \quad (4)$$

E_1 is, however, exactly equal to $\sum_i (E_{io} + E'_{io}) = E_{CR}$. Thus, $E_g = E_1$, i.e., the exact eigenstate of the ladder model is also the exact ground state. The exact ground state energy is given by $E_g = N(E_{i0} - 3\frac{J'}{4})$ where N is the total number of plaquettes as well as rungs in the ladder.

TABLE I

S	Eigenvalues	S^z	Eigenstates
0	$-\frac{J_1+J_4+J_3}{2} - X$	0	$\|\psi_1\rangle=$ 
0	$-\frac{J_1+J_4+J_3}{2} + X$	0	$\|\psi_2\rangle=$ 
1	$\frac{-J_1-J_4+J_3}{2}$	0	$\|\psi_3\rangle= \uparrow\downarrow\downarrow\uparrow - \downarrow\uparrow\uparrow\downarrow$
1	$\frac{-J_1+J_4-J_3}{2}$	0	$\|\psi_4\rangle= \uparrow\downarrow\uparrow\downarrow - \downarrow\uparrow\downarrow\uparrow$
1	$\frac{J_1-J_4-J_3}{2}$	0	$\|\psi_5\rangle= \uparrow\uparrow\downarrow\downarrow - \downarrow\downarrow\uparrow\uparrow$
2	$\frac{J_1+J_4+J_3}{2}$	0	$\|\psi_6\rangle= \uparrow\uparrow\downarrow\downarrow + \downarrow\downarrow\uparrow\uparrow + \uparrow\downarrow\uparrow\downarrow + \downarrow\uparrow\downarrow\uparrow + \uparrow\downarrow\downarrow\uparrow + \downarrow\uparrow\uparrow\downarrow$
1	$\frac{-J_1-J_4+J_3}{2}$	1	$\|\psi_7\rangle= \uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\uparrow$
1	$\frac{-J_1+J_4-J_3}{2}$	1	$\|\psi_8\rangle= \uparrow\uparrow\uparrow\downarrow - \uparrow\uparrow\downarrow\uparrow + \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow$
1	$\frac{J_1-J_4-J_3}{2}$	1	$\|\psi_9\rangle= \uparrow\uparrow\uparrow\downarrow + \uparrow\uparrow\downarrow\uparrow - \uparrow\downarrow\uparrow\uparrow - \downarrow\uparrow\uparrow\uparrow$
2	$\frac{J_1+J_4+J_3}{2}$	1	$\|\psi_{10}\rangle= \uparrow\uparrow\uparrow\downarrow + \uparrow\uparrow\downarrow\uparrow + \uparrow\downarrow\uparrow\uparrow + \downarrow\uparrow\uparrow\uparrow$
2	$\frac{J_1+J_4+J_3}{2}$	2	$\|\psi_{11}\rangle= \uparrow\uparrow\uparrow\uparrow$

$$X = \sqrt{J_1^2 + J_4^2 + J_3^2 - J_1 J_4 - J_1 J_3 - J_3 J_4}$$

Table I: The energy eigenvalues and eigenvectors of a four-spin plaquettes with exchange interactions of strengths J_1(vertical), J_4(horizontal) and J_3(diagonal). $\frac{c_1}{c_2}$ and $\frac{d_1}{d_2}$ are functions of J_1, J_4 and J_3.

The eigenvalues and the eigenstates of the four-spin plaquette Hamiltonian are shown in Table 1. For the most generalised case, the ground state energy E_{jo} is $\frac{-(J_1+J_4+J_3)}{2} - X$. The ground state wave function is of the RVB(resonating valence bond)-type. It is a linear combination of two valance bond(VB) states with coefficients c_1 and c_2 depending on the exchange interaction strengths. Note that the other singlet-state is also of the RVB-type with the coefficients d_1, d_2 depending on the exchange interaction strengths. Some of the special cases of interest are

(i) $J_1 = J_4,\ J_3 = 0$

The ground state energy $E_{j0} = -2J_1$ and $\frac{c_1}{c_2} = 1$. The other singlet state has energy zero and $\frac{d_1}{d_2} = -1$.
(ii) $J_1 = J_4,\ J_3 \neq 0$

For $J_3 < J_1$, $E_{j0} = -2J_1 + \frac{J_3}{2}$, $\frac{c_1}{c_2} = 1$. For $J_3 > J_1$, $E_{j0} = -\frac{3J_3}{2}$, i.e., the other RVB state becomes the ground state with $\frac{d_1}{d_2} = -1$.
(iii) $J_1 = J_4 = J_3$

The ground state becomes doubly degenerate. The two states have a pair of VBs along either the horizontal or the vertical bonds, with $E_{j0} = -\frac{3J_1}{2}$.
(iv) $J_1 = J_3 \leq \frac{J_4}{2}$

The ground state has a pair of singlets along the horizontal bonds with $E_{j0} = -\frac{3J_4}{2}$.
(v) $J_4 = J_3 \leq \frac{J_1}{2}$

The ground state has a pair of singlets along the vertical bonds with $E_{j0} = -\frac{3J_1}{2}$.
In the generalised as well as the special cases and for $J_2 \leq \frac{J'}{4}$, the exact ground state of the full ladder model is of the product form. The plaquettes are in their ground state spin configurations and the rungs are occupied by singlets.

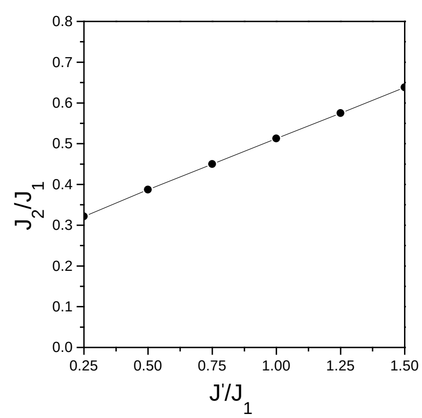


Fig. 2. Phase diagram of the ladder model (fig.1) in the parameter space of $\frac{J_2}{J_1}$ and $\frac{J'}{J_1}$ for $\frac{J_3}{J_1} = 0.75$ and $\frac{J_4}{J_1} = 0.5$. The parameter space below the solid line corresponds to the phase in which the exact ground state is a product over the ground states of the rungs and the plaquettes.

We next checked whether the exact ground state retains its product form when J_2 is made larger than $\frac{J'}{4}$. For this, we write the total Hamiltonian H (Eq. (2)) as a sum over six-spin sub-Hamiltonians, h_i's, i.e., $H = \sum_i h_i$. Each sub-Hamiltonian describes a plaquette coupled to a rung. The six-spin sub-Hamiltonian can be diagonalised exactly to obtain the ground state energy. Again, we use the method of 'divide and conquer'. When the six-spin sub-Hamiltonians are added together to obtain the full Hamiltonian, the J_1, J_3, J' bonds are counted twice and the J_2 bonds only once. We identify the region of parameter space in which the full ladder ground state has the product form. Fig. 2 shows the phase boundaries, in the parameter space of $\frac{J_2}{J_1}$ and $\frac{J'}{J_1}$ for fixed values of $\frac{J_4}{J_1}$(=0.5) and $\frac{J_3}{J_1}$(=0.75). In the parameter regime below the phase boundary, the exact ground state has the product form. One finds that even for $J_2 > \frac{J'}{4}$, the exact ground state has the product structure.

3 Magnetization plateaux

We next include an external magnetic field term $-h\sum_{i=1}^{6N} S_i^z$ in the Hamiltonian H (Eq. (2)), where 6N is the total number of sites in the ladder. Let us first consider the case of a single 4-spin plaquette in a magnetic field. The magnetic field couples to the z-component of the total spin of the plaquette, S^z_{tot},

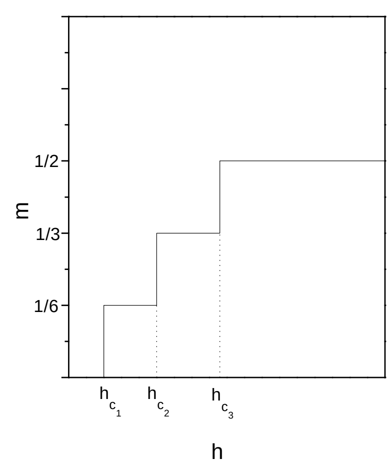


Fig. 3. Plot of magnetization/site m versus external magnetic field h for the two-chain ladder model shown in fig. 1. The plot is obtained in the parameter region in which the exact ground states in different S^z_{tot} subspaces have the product form. Two non-trivial magnetization plateaux occur at $m = \frac{1}{6}$ and $m = \frac{1}{3}$.

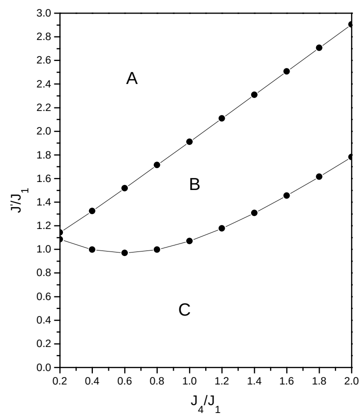


Fig. 4. Phase diagram of the ladder model (fig. 1) in a finite magnetic field and in the parameter space of $\frac{J'}{J_1}$ and $\frac{J_4}{J_1}$ with $\frac{J_2}{J_1} = 0.1$ and $\frac{J_3}{J_1} = 0$. The regions A, B and C are explained in the text.

which is a conserved quantity. The ground state energy $E_g(S^z_{tot})$ at $h = 0$ for $S^z_{tot} = 0, 1$ and 2 can be obtained from Table I. When the external field $h \neq 0$, the ground state in each S^z_{tot} subspace is $E_g(S^z_{tot}, h) = E_g(S^z_{tot}, 0) - hS^z_{tot}$. The ground state magnetization curve can be easily obtained. The magnetization per site m is zero from $h = 0$ upto a critical field h_{c_1}. Below $h = h_{c_1}$, S^z_{tot} of the ground state is zero. For $h_{c_1} < h < h_{c_2}$, S^z_{tot} of the ground state is 1, so that $m = \frac{1}{4}$ and beyond $h = h_{c_2}$, S^z_{tot} of the ground state is 2, i.e., the saturation magnetization $m = \frac{1}{2}$ is obtained. Let us first consider the case $J_3 < J_1$. For $J_3 < J_4$, the ground state in the $S^z_{tot} = 1$ subspace is $|\psi_7\rangle$ and the critical fields are $h_{c_1} = J_3 + X$ and $h_{c_2} = J_1 + J_4$. For $J_3 > J_4$, the ground state in the $S^z_{tot} = 1$ subspace is $|\psi_8\rangle$ and $h_{c_1} = J_4 + X$ and $h_{c_2} = J_1 + J_3$. At $J_3 = J_4$ these two states are degenerate. For $J_3 > J_1$, as long as $J_1 < J_4$, the ground state in the $S^z_{tot} = 1$ subspace is $|\psi_9\rangle$ with $h_{c_1} = J_1 + X$ and $h_{c_2} = J_4 + J_3$. For $J_1 > J_4$, the ground state in the $S^z_{tot} = 1$ subspace is $|\psi_8\rangle$ with $h_{c_1} = J_4 + X$ and $h_{c_2} = J_1 + J_3$. At $J_1 = J_4$, $|\psi_8\rangle$ and $|\psi_9\rangle$ are degenerate ground states. Hence depending on the exchange interaction strengths, we get different values for h_{c_1} and h_{c_2}.

For the external field $h = 0$, we have already seen that there is an extended parameter regime in which the exact ground state of the full ladder is a product of the ground states of the rungs and the plaquettes. We now investigate whether the same holds true in the presence of a finite magnetic field. Again, we use the method of 'divide and conquer' and the six-spin sub-Hamiltonian consists of a plaquette coupled to a rung. For the full ladder, one can identify

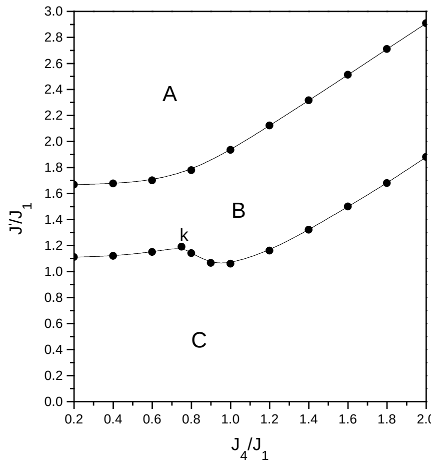


Fig. 5. Phase diagram of the ladder model (fig. 1) in a finite magnetic field and in the parameter space of $\frac{J'}{J_1}$ and $\frac{J_4}{J_1}$ with $\frac{J_2}{J_1} = 0.1$ and $\frac{J_3}{J_1} = 0.75$. The regions A, B and C are explained in the text.

a region (region A in Fig. 4 and Fig. 5) in parameter space in which for $0 < h < h_{c_1}$, m is zero. At h_{c_1}, there is a jump in the value of m to $m = \frac{1}{6}$ and a plateau is obtained for h upto h_{c_2} (Fig. 3). When $h_{c_1} < h < h_{c_2}$, the exact ground state has the plaquettes in their $S^z = 1$ ground states and the rungs in singlet spin configurations. Since, the number of plaquette is N and the total number of sites is 6N, the magnetization/site m in the ground state is $\frac{1}{6}$. The quantization condition in Eq. (1) is obeyed as unit period of the ground state contains six spins so that $S_u = 3$ and the magnetization m_u in the unit period is 1. At h_{c_2}, the second jump in m from $\frac{1}{6}$ to $\frac{1}{3}$is obtained. When $h_{c_2} < h < h_{c_3}$, the exact ground state has the plaquettes in their $S^z = 2$ ground states and the rungs in singlet spin configurations. In this case, S_u and m_u in Eq. (1) are 3 and 2 respectively. At $h = h_{c_3}$, there is a jump in m from $\frac{1}{3}$ to the saturation magnetization $\frac{1}{2}$. The value of h_{c_3}, the critical field for which the full ladder reaches saturation magnetization is $J' + J_2$. There are other parameter regions (regions B and C in Fig. 4 and Fig. 5) in the parameter space in which the full plateau structure in the m versus h plot, as shown in Fig. 3, is not obtained. Fig. 4 shows the phase diagram for the full ladder in a magnetic field in the $\frac{J'}{J_1}$ vs. $\frac{J_4}{J_1}$ parameter space when $J_3 = 0$ and $\frac{J_2}{J_1} = 0.1$. The region A exhibits the full plateau structure in m vs. h as shown in Fig. 3. In region B, the jump in m from 0 to $\frac{1}{6}$ occurs at $h = h_{c_1}$ (Fig. 3) but beyond h_{c_2}, the ground state is no longer of the product form. In region C, the ground state loses its simple product structure beyond $h = h_{c_1}$. A similar phase diagram is shown in Fig. 5, for $\frac{J_3}{J_1} = 0.75$, and with all other parameters the same as in the case of Fig. 4. We observe a kink at the point k, where

$J_4 = J_3$ and a transition from the phase B to phase C occurs. At this point, the ground state in $S^z_{tot} = 1$ subspace changes from $|\psi_8\rangle$ to $|\psi_9\rangle$. The kink is indicative of phase reentrance. When $\frac{J'}{J_1}$ is in the range 1.11¡$\frac{J'}{J_1}$¡1.19, one gets the phases B-C-B-C as $\frac{J_4}{J_1}$ is varied ($\frac{J_4}{J_1} \geq 0.2$). Similarly, for 1.06¡$\frac{J'}{J_1}$ $\leq$1.11, the phases C-B-C are obtained.

4 Concluding remarks

In this paper, we have introduced a generalised ladder model with modulated exchange interactions. The model consists of four-spin plaquettes coupled to two-spin rungs. In a wide parameter regime, the exact ground state of the model is a product over the ground states of the individual plaquettes and rungs. In the presence of an external magnetic field, magnetization plateaux are obtained when magnetization/site m is plotted as a function of the external field h. In an extended parameter regime, the exact ground states in the different magnetization subspaces are of the product form. The generalised model includes several other ladder models as special cases. For $J_2 = J_3 = J_4 = J$ and $J' = J_1$, the model reduces to the frustrated ladder model introduced by Bose and Gayen[19]. As Xian[20] has shown, for $\frac{J'}{J} > (\frac{J'}{J})_c \simeq 1.40148$, the exact ground state consists of singlets along the rungs of the ladder. At $\frac{J'}{J} = (\frac{J'}{J})_c$, there is a transition from the rung dimer state to the Haldane phase of the S=1 chain. Similar arguments show that for $J_4 = J_3$ in our generalised ladder model, the ground state in a certain parameter regime is that of a spin one chain with modulated exchange interactions. The sequence of exchange couplings along the chain has the structure $J_4 - J_2 - J_2 - J_4 - J_2 - J_2 \cdots\cdots$. Other examples of ladder models which are special cases of the generalised model, have been given in Section 2. Further studies are needed to obtain the phase diagram of the generalised model in the full parameter space.

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