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REPRESENTABILITY OF POSITIVE FUNCTIONALS ON ABSTRACT STAR ALGEBRAS WITHOUT IDENTITY WITH APPLICATIONS TO LOCALLY CONVEX *ALGEBRAS

By

SUBHASH J. BHATT

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In the context of unbounded representation theory, representable functionals on a star algebra are introduced and are characterized as extendable functionals. This gives faithful representations of certain star algebras as Op^*-algebras. Every representation of a pseudo-complete hermitian locally convex *algebra is shown to be essentially selfadjoint. Also proved that a Generalized B^*-algebra (without identity) admits a bounded approximate identity which yields the representability of every positive functional.

R. T. Powers [13, Theorem 6.3] has proved that the well known *GNS* construction with a positive functional f on a star algebra A with identity yields a closed strongly cyclic star representation π of A on a Hilbert space H mapping the elements of A into closable linear operators all defined on a common domain $D(\pi)$ dense in H; and π represents f in a suitable sense. In this paper, we consider the case when A does not possess identity. In § 2, representable functionals on A are introduced in the setting of unbounded representations; and are characterized (as extendable functionals) as in the case of Banach star algebras [5, Theorem 37.11]. This gives a number of sufficient conditions for the representability of every positive functional which in turn are used to construct faithful representations of certain algebras as Op^*-algebras [11, Definition 2.1]. In § 3 we give a couple of applications to locally convex *algebras. It is shown that every representation of a hermitian pseudo-complete locally convex *algebra A is essentially selfadjoint. Further if A has a bounded approximate identity and the positive elements A^+ forms a closed convex cone, then A is *isomorphic to a selfadjoint Op^*-algebra. We also show that a locally convex Generalized B^*-algebra [8] admits a bounded approximate identity.

1. Our notations and terminology will follow [13] except for a suitable modification when the algebra does not possess identity. They are outlined bellow.

Notations and terminology. Let A be a complex star algebra. A *representation* $(\pi, D(\pi), H)$ of A on a Hilbert space H is a mapping π of A into linear operators all defined on a common domain $D(\pi)$ dense in H such that for all x, y in A; α, β in $\mathbb{C}$ and ζ, η in $D(\pi)$;

(a) $\pi(\alpha x+\beta y)\zeta=\alpha\pi(x)\zeta+\beta\pi(y)\zeta$

(b) $\pi(x)D(\pi)\subset D(\pi)$ and $\pi(x)\pi(y)\zeta=\pi(xy)\zeta$.

It is called *hermitian* or a *star representation* if

(c) $D(\pi)\subset D(\pi(x)^*)$ (here $D(\pi(x)^*)$ denote the domain of the operator adjoint $\pi(x)^*$ of $\pi(x)$) and $\pi(x^*)\subset\pi(x)^*$.

In case A possesses identity 1, then

(d) $\pi(1)=I$.

Throughout we shall only consider hermition representations that are *essential* in the sense that

$$\{\zeta\in D(\pi) \mid \pi(x)\zeta=0 \text{ for all } x \text{ in } A\}=\{0\}\,.$$

It is easily seen that, π is essential if and only if $\pi(A)D(\pi)$ is dense in H. By (c), each $\pi(x)$ is a closable operator whose closure is denoted by $\overline{\pi(x)}$. Let A_e be the star algebra obtained by adjoining identity to A. Then $\pi_e(x+\lambda 1)=\pi(x)+\lambda I$ ($x\in A$, $\lambda\in\mathbb{C}$) defines a representation of A_e with domain $D(\pi)$. The *induced topology* (or the A_e*-topology*) on $D(\pi)$ is the locally convex topology defined by the seminorms $\zeta\to\|\pi_e(x)\zeta\|$ for varying x in A_e. The completion of $D(\pi)$ in this topology [11, Lemma 3.2] is

$$\begin{aligned} D(\bar\pi)&=\bigcap\{D(\overline{\pi_e(x)}) \mid x\in A_e\}\\ &=\bigcap\{D(\overline{\pi(x)}) \mid x\in A\}\end{aligned}$$

(The latter equality follows from: If S and T are densely defined operators in a Hilbert space with $D(T)=D(S)$, then $(T+S)^-=\bar T+\bar S$ provided either of them is bounded). Now this defines a representation $\bar\pi$ of A with domain $D(\bar\pi)$ as $\bar\pi(x)=\overline{\pi(x)}\,|_{D(\bar\pi)}$, called the *closure* of π; and π is *closed* if $\pi=\bar\pi$.

The *hermitian adjoint* of π is a (not necessarily hermitian) representation $(\pi^*, D(\pi^*), H)$ with

$$\begin{aligned} D(\pi^*)&=\bigcap\{D(\pi_e(x^*)^*) \mid x\in A_e\}\\ &=\bigcap\{D(\pi(x^*)^*) \mid x\in A\}\quad\text{(as above)}\end{aligned}$$

with $\pi^*(x)=\pi(x^*)^*\,|_{D(\pi^*)}$. π is *essentially selfadjoint* (respectively *selfadjoint*) if $\bar\pi=\pi^*$ (respectively, $\pi=\pi^*$).

A vector ζ in $D(\pi)$ is called *strongly cyclic* (respectively *cyclic*) if $\pi(A)\zeta$ is dense in $D(\pi)$ in the induced topology (respectively, $\pi(A)\zeta$ is norm dense in H.). It is called an *ultracyclic* vector if $\pi(A)\zeta=D(\pi)$.

2. Let f be a representable functional [5, Definition 37.10] on a Banach star algebra A. The *GNS* construction [5, Theorem 37.11] gives a cyclic representation π of A on a Hilbert space H such that $f(x)=(\pi(x)\xi, \xi)$ where ξ is a cyclic vector. Due to the admissibility of f [15, Ch. IV, §5] each $\pi(x)$ is a bounded operator with the result that π automatically turns out to be strongly cyclic. In the context of unbounded representations, this, together with Powers' work [13, Theorem 6.3], suggests the following definition.

Definition 2.1. A positive functional f on a star algebra A is called ***representable*** if there exists a closed strongly cyclic representation $(\pi, D(\pi), H)$ of A on a Hilbert space H with a strongly cyclic vector ξ such that $f(x)=(\pi(x)\xi, \xi)$ for all x in A. In this case, f is said to be represented by π.

As for Banach star algebras, the above mentioned result of Powers shows that every positive functional on a star algebra with identity is representable. It also follow, by the same arguments as there, that a representable functional is represented by a unique (up to unitary equivalence) closed strongly cyclic representation. The following analogue of [5, Theorem 37.11] shows that representable functionals on A are precisely those that extend positively to A_e.

Theorem 2.2. *A positive functional f on a star algebra A is representable if and only if $|f(x)|^2\leqq kf(x^*x)$ holds for all x in A and for some constant k depending on f only.*

Proof. Since $\pi(x^*)\subset\pi(x)^*$ for each x in A, one way implication is trivial. Conversely, let $|f(x)|^2\leqq kf(x^*x)$ for all $x\in A$. Then g on A_e defined as $g(x+\lambda 1)=f(x)+k\lambda$ $(x\in A,\ \lambda\in\mathbb{C})$ gives a positive linear extension of f. The standard *GNS* construction with g is as follows:

Let $N_g=\{a\in A_e \mid g(a^*a)=0\}$; let $X_g=A/N_g$, a vector space with the canonical inner product $(a+N_g, b+N_g)=g(b^*a)$. For each $a\in A_e$, $\pi_g'(a)$ on X_g is defined by $\pi_g'(a)(b+N_g)=ab+N_g$ $(b\in A_e)$. Let H_g be the *Hilbert* space completion of X_g. Then π_g' defines an ultracyclic representation of A_e on H_g with $D(\pi_g')=X_g$. Let $(\pi_g, D(\pi_g), H_g)$ be its closure; for convenience, being denoted by $(\pi, D(\pi), H)$. It is a closed strongly cyclic representation of A_e such that $g(a)=(\pi(a)\xi_0, \xi_0)$ $(a\in A_e)$; $\xi_0=1+N_g$ being a strongly cyclic vector.

Let

$$N=\{y \in D(\pi) \mid \pi(a)y=0 \text{ for all } a \in A\}$$
$$=\mathrm{Cl}\,\{y \in X_g \mid \pi(a)y=0 \text{ for all } a \in A\}$$

where Cl denotes the closure in $D(\pi)$ in the induced topology. Further, let

$$M=\{z \in D(\pi) \mid (z, \eta)=0 \text{ for all } \eta \in N\}$$
$$=D(\pi) \cap N^{\perp} \quad \text{where} \quad N^{\perp}=H \ominus N .$$

From the facts that π is closed and the induced topology on $D(\pi)$ is finer than the norm topology, it follows at once that N is a closed subspace of H. Thus $H=N \oplus N^{\perp}$ and $D(\pi)=N+M$ which too is a direct sum. As N and M are both π-invariant subspaces, $\pi=\pi_1+\pi_2$ where π_1 is $(\pi|_N, D(\pi_1)=N, H_1=N)$ and π_2 is $(\pi|_M, D(\pi_2)=M, H_2=N^{\perp})$, each a representation of A_e. Also $\xi_0=v+u$ with $v \in D(\pi_1)$, $u \in D(\pi_2)$, $u \neq 0$ and for each $x \in A$, $f(x)=(\pi_2(x)u, u)$. Let $K=(\pi(A)u)^{-}$, the closure in H of $\pi(A)u$. Define a representation π_3 of A in K with $D(\pi_3)=\pi(A)u$ as $\pi_3(a)=\pi_2(a)|_{D(\pi_3)}$. We show that its closure $(\bar{\pi}_3, D(\bar{\pi}_3), K)$ is the desired representation of A with u as a strongly cyclic vector such that $f(a)=(\bar{\pi}_3(a)u, u)$ $(a \in A)$, and for this, it only suffices to show that $u \in D(\bar{\pi}_3)$.

By the definition, $D(\bar{\pi}_3)=\bigcap \{D(\overline{\pi_3(a)}^K) \mid a \in A\}$ where now $\overline{\pi_3(a)}^K$ is the closure of $\pi_3(a)$ as an operator in K with domain $D(\pi_3)$. Then

$$u \in D(\pi)$$
$$=D(\bar{\pi}) \quad \text{as } \pi \text{ is closed}$$
$$=\bigcap \{D(\overline{\pi(a)}) \mid a \in A\}$$

and so $u \in D(\overline{\pi(a)})$ for each $a \in A$. Further, the graph of the operator $\overline{\pi(a)}$ being

$$G(\overline{\pi(a)})=\{\{\pi_e(x+\lambda 1)\xi_0,\ \pi(a)\pi_e(x+\lambda 1)\xi_0\} \mid x \in A,\ \lambda \in \mathbb{C}\}^{-} \quad (\text{the closure in } H \times H)$$
$$=\{\{\pi(x)u+\lambda v,\ \pi(a)(\pi(x)u+\lambda v)\} \mid x \in A,\ \lambda \in \mathbb{C}\}^{-}$$

there are sequences (x_n) in A, (λ_n) in $\mathbb{C}$ such that $\pi(x_n)u+\lambda_n v \to u$ and $\pi(a)(\pi(x_n)u+\lambda_n v) \to \pi(a)u$ both in H. As $D(\pi)=N \oplus M$, also a direct sum in the norm topology, it follows that $\pi(x_n)u \to u$ and $\lambda_n \to 0$; and so $\pi(a)\pi(x_n)u \to \overline{\pi(a)u}$. Thus $u \in D(\overline{\pi_3(a)}^K)$ for each $a \in A$ and so $u \in D(\bar{\pi}_3)$. This completes the proof.

The following corollary contains the varients of some of the results known [5, § 37] in the context of automatic continuity of positive functionals on Banach star algebras. Note that a representable functional on a Banach star algebra is continuous.

Corollary 2.3.

(a) *Let A be a star algebra and f a positive functional on A. For each b in*

A, define $f_b(x)=f(b^*xb)$ $(x \in A)$. Then each f_b is a representable positive functional on A.

(b) *Let $(\pi, D(\pi), H)$ be a representation of a star algebra A. For each $\zeta \in D(\pi)$, let $F_\zeta(x)=(\pi(x)\zeta, \zeta)$. Then each F_ζ is a representable positive functional on A.*

(c) *Let A be a star algebra such that* (i) $A^2=A$ *and* (ii) *each nonzero positive functional on A dominates a nonzero representable functional. (Then each positive functional on A is representable.)*

Proof. (a) is a immediate from the Cauchy-Schwartz inequality [5, Lemma 37.6 (ii)], whereas (b) is obvious. (c) is a varient of a continuity theorem of Murphy [5, Theorem 37.13] and is proved thus: Let f be positive; g be representable with $f \geqq g$. Let $\tilde{g}$ be the positive extension of g to A_e. As finite linear combinations of elements of type $a^*a (a \in A)$ span A, a simple verification shows that $\tilde{f}(x+\lambda 1)=f(x)+\lambda\tilde{g}(1)$ defines a positive linear extension of f on A_e. The assertion follows from Theorem 2.2.

Op^*-algebras [11, Definition 2.1] provides a fairly general class of unbounded operator algebras. The next result gives construction of a faithful representation of a certain star algebra as an Op^*-algebra (without identity).

Corollary 2.4. *Let A be a star algebra and $R(A)$ be the set of all representable functionals on A. The following are equivalent.*

(a) *For each nonzero x in A, there exists an f in $R(A)$ such that $f(x^*x)>0$.*

(b) *There exists a closed faithful representation of A as an Op^*-algebra.*

Proof. For f in $R(A)$, let $(\pi_f, D(\pi_f), H_f)$ be the closed representation of A with a strongly cyclic vector ξ_f that represents f. Let $(\pi, D(\pi), H)$ be the direct-sum [13, remark following Theorem 7.5] of the π_f's. Then π is closed. Assume (a). Let $x \in A$, $x \neq 0$. Let $f \in R(A)$ be such that $f(x^*x)>0$. Let $\xi'=(\xi_g' \mid g \in R(A))$ be the vector $\xi_g'=0$, $(g \neq f)$; $\xi_g'=\xi_f$, $(g=f)$. Then $\|\pi(x)\xi'\|^2=\|\pi_f(x)\xi_f\|^2=f(x^*x)>0$. Hence π is faithful.

Conversely, if π is any faithful representation of A, then given $x \neq 0$ in A, $\pi(x)\eta \neq 0$ for some $\eta \in D(\pi)$. The functional $f(x)=(\pi(x)\eta, \eta)$ is in $R(A)$ and $f(x^*x)=\|\pi(x)\eta\|^2>0$.

3. Now we consider locally convex *algebras [2, § 2]. Throughout the continuity of the involution is assumed. It is immediate from [15, Chapter IV, § 5] that if f is a representable functional on a star algebra A and if π represents f, then each $\pi(x)$ is a bounded operator if and only if f is admissible. In this case, π is called a bounded representation. A result due to Powell [12, Theorem 2] (where too admissible functionals are considered, but in a different sense) leads to the following.

Propnsition 3.1. *Let A be a pseudo-complete locally convex *algebra such that $A=A_0$, the set of all bounded elements of A. Let f be a representable functional on A. Then each representation of A that represents f is a bounded cyclic representation.*

Proof. Let $f(x)=(\pi(x)\xi_0, \xi_0)$ where ξ_0 is a strongly cyclic vector. Let $K=\pi(A)\xi_0$ and let $\xi=\pi(y)\xi_0$ for $y\in A$. Then $\|\pi(x)\xi\|^2=f(y^*x^*xy)\leqq\beta(x^*x)f(y^*y)$ by [12, Theorem 2]. Here $\beta(x^*x)$ is the radius of boundedness [1, Definition] of x^*x which is finite as $x\in A=A_0$. Hence $\|\pi(x)\xi\|\leqq\beta(x^*x)^{1/2}\|\xi\|$ for all $\xi\in K$. Let $\xi\in D(\pi)$. As K is dense in $D(\pi)$ in the induced topology and since the induced topology is finer than the norm topology, there exists a net (x_α) in A such that $\pi(x_\alpha)\xi_0\to\xi$ and $\pi(x)\pi(x_\alpha)\xi_0\to\pi(x)\xi$ $(x\in A)$ both in the norm. This gives, for each x in A,

$$\begin{aligned}\|\pi(x)\xi\| &= \lim \|\pi(x)\pi(x_\alpha)\xi_0\| \\ &\leqq \beta(x^*x)^{1/2}\lim_{\alpha}\|\pi(x_\alpha)\xi_0\| \\ &= \beta(x^*x)^{1/2}\,\|\xi\|\ .\end{aligned}$$

Thus each $\pi(x)$ is a norm bounded operator on $D(\pi)$. Hence $x\to w(x)=\overline{\pi(x)}$ defines a *homomorphism of A into $\beta(H)$ (The C^*-algebra of all bounded operators on H) which is a cyclic representation. Hence the result.

It follows that each representable functional on such an algebra is admissible. The next result gives an important case where $A\neq A_0$ in general, but certain representable functionals are admissible.

Proposition 3 2. *Let A be a complete locally-m-convex *algebra and let f be a continuous representable functional on A. Then each representation of A that represents f is a bounded cyclic representation.*

The functional f is extendable, and continuity implies that it is admissible [6, Theorem 6.1]. Thus, in particular, each representable functional on a Frechet *algebra is admissible.

We say that a locally convex *algebra A has a bounded approximate identity if there exists a net (u_α) in A such that

(i) (u_α) is bounded;

(ii) for each x in A, $u_\alpha x \to x$ and $x u_\alpha \to x$

and

(iii) $(u_\alpha^* u_\alpha)$ is also bounded.

Note that in A, if the product of two bounded sets is bounded, in particular, if A is lmc, then (i) implies (iii). The next result which corresponds to [5, Theorem 37.15] is an immediate corollary of Theorem 2.2.

Proposition 3.3. *Let A be a locally convex *algebra with a bounded approximate identity. Then each continuous positive functional on A is representable.*

From this stems the following modification of corollary 2.4. In analogy with [2, §§ 2, 3] a locally convex *algebra A is hermitian if for each $h=h^*$ in A, the Allan spectrum [1, Definition 3.1] $\sigma_A(h) \subset \boldsymbol{R} \cup \{\infty\}$, or equivalently, for such h, $-h^2$ has a bounded adverse.

Corollary 3.4. *Let A be a locally convex *algebra with a bounded approximate identity. If the set $A^+=\{x^*x \mid x \in A\}$ forms a closed convex cone in A, then A admits a faithful closed *representation π as an Op*-algebra. Further, if A is pseudo-complete and hermitian, then π represents A as a selfadjoint Op*-blgebra.*

Proof. Let $x \neq 0$ be in A. Let $y=x^*x$. Then $-y \in A \sim A^+$. Hahn-Banach Theorem gives a continuous linear functional f on A such that $f(A^+) \geqq 0$ and $f(y)<0$. Thus f is positive which, by above result, is representable. The conclusion follows from Corollary 2.4 except for the final part which is a consequence of the result that follows.

Theorem 3.5. *Every representation $(\pi, D(\pi), H)$ of a pseudo-complete hermitian locally convex *algebra is essentially selfadjoint.*

Proof. We can assume π to be closed. Let β^* denote the set of all $B \subset A$ such that $B^2=B$, $B^*=B$, B is absolutely convex and is closed and bounded. For each such B, we consider the *normed algebra $A(B)=\{\lambda x \mid \lambda \in \boldsymbol{C}, x \in B\}$ with the norm $\|x\|_B=\{\lambda>0 \mid x \in \lambda B\}$. Pseudo-completeness of A implies that it is Banach. First we show:

(i) $x \to \overline{\pi(x)}$ defines a continuous *homomorphism of $A(B)$ into $\beta(H)$.

For $\xi \in D(\pi)$, consider $F_\xi(x)=(\pi(x)\xi, \xi)$ on A. By Corollary 2.3, it is representable on A and so on $A(B)$; hence is norm continuous by [5, § 37]. Then $\|\pi(x)\xi\|^2=F_\xi(x^*x) \leqq \|F_\xi\| \, \|x^*x\| \leqq \|F_\xi\| \, \|x\|^2$; and if $\tilde{F}_\xi$ is the positive extension of F_ξ to $(A(B))_e$, then $\|F_\xi\| \leqq \tilde{F}_\xi(1) \leqq \|\xi\|^2$. It follows that $\|\pi(x)\xi\| \leqq \|x\| \, \|\xi\|$ which gives

(i) with $\|\overline{\pi(x)}\| \leqq \|x\|$ $(x \in A(B))$.

Next we show:

(ii) For each $h=h^*$ in A, $(\overline{\pi(h)})^* = \overline{\pi(h)} = \pi(h)^*$.

Indeed, let h be as above. Let $\alpha \in \mathbb{C} \sim R$. Since A is hermitian, the quasi inverse [15, Ch. I, §5] $k=(\alpha^{-1}h)_{-1}$ is bounded. Then for some $\beta \neq 0$, $S=\{(\beta^{-1}k)^n \mid n=1, 2, \cdots\}$ is a bounded set. Obviously, its closed absolutely convex hull B is in β^*. By (i) above, $\pi(k)$ is a bounded operator. Also, for each $\xi \in D(\pi)$, $\pi((\alpha^{-1}h)_{-1})\xi = (\alpha^{-1}\pi(h))_{-1}\xi = (\alpha^{-1}\overline{\pi(h)})_{-1}\xi$. Hence $\overline{\pi((\alpha^{-1}h)_{-1})} \subset (\alpha^{-1}\overline{\pi(h)})_{-1}$ each a bounded closed operator. Therefore $(I-\alpha^{-1}\overline{\pi(h)})^{-1} = I-(\alpha^{-1}\overline{\pi(h)})_{-1}$ is a bounded operator defined on the whole of H. Thus the (operator theoretic) spectrum of $\overline{\pi(h)}$ is real. But $\pi(h)=\pi(h^*) \subset \pi(h)^* = (\overline{\pi(h)})^*$, and so $\overline{\pi(h)} \subset (\overline{\pi(h)})^*$. Since a closed symmetric operator with real spectrum is selfadjoint, $\overline{\pi(h)} = (\overline{\pi(h)})^*$.

Finally, (ii) is used to prove:

(iii) For each $x \in A$, $\overline{\pi(x)} = \pi(x^*)^*$.

The argument for this is standard, e.g. as in [8, Lemma 7.10]. This, with the definition of π^*, gives $D(\pi)=D(\bar{\pi})=D(\pi^*)$ which firnishes the proof of the theorem.

An important case to which Proposition 3.3 applies is to unbounded Hilbert algebras [10]; more generally to the GB^*-algebras. We can suitably modify [2, Definition in §3] (or [8, Definition 2.5]) to define a GB^*-algebra without identity. It is a routine matter to verify that if A is a locally convex GB^*-algebra with unit ball B_0, then $A(B_0)=\{\lambda x \mid \lambda \in \mathbb{C},\ x \in B_0\}$ is a B^*-algebra (with the Minkowski functional $\|\cdot\|_{B_0}$ of B_0 as norm), and A_e is also a GB^*-algebra with underlying B^*-subalgebra $(A(B_0))_e$. A recent result due to the author [4] (proved for GB^*-algebra with identity and holds in the non unital can also) is: if A is a GB^*-algebra with unit ball B_0, then the B^*-algebra $A(B_0)$ is sequentially dense in A. This gives the following important result.

Theorem 3.6. *A locally convex GB^*-algebra possesses a bounded approximate identity consisting of positive elements.*

Proof. Given a locally convex GB^*-algebra A with unit ball B_0, let $(u_\lambda \mid \lambda \in \Lambda)$ be an approximate identity for $A(B_0)$ [3, Theorem 1.8.2] contained in B_0 and consisting of positive elements. Continuity of $(A(B_0), \|\cdot\|_{B_0}) \to (A, t)$ (t denotes the topology of A) implies that for each $x \in A(B_0)$, $xu_\lambda \xrightarrow[t]{} x$ and $u_\lambda x \xrightarrow[t]{} x$. Let t' be the associated barrel topology on A [2, §5]. Then as in [2, §5], (A, t') is easily seen to be a locally convex GB^*-blgebra with the same unit ball B_0 [8, Corollary

7.8]. Further t' is finer than t and, though t' need not be barrelled, an adaptation of [8, Lemma 6.3] shows that (A, t') is hypocontinuous. We show that $(u_\lambda \mid \lambda \in \Lambda)$ is the desired approximate identity, and for this it suffices to prove that for each $x \in A$, $xu_\lambda \underset{t'}{\to} x$ and $u_\lambda x \underset{t'}{\to} x$.

Given x in A and a o-neighbourhood W in (A, t'), continuity of addition and hypocontinuity of multiplication gives o-neighbourhoods V and U in (A, t') such that $U+U\subset W$, $V+V\subset U$, $B_0V\subset U$ and $VB_0\subset U$. By the sequential denseness of $A(B_0)$ in (A, t'), there exists a sequence (x_n) in $A(B_0)$ such that $x_n - x \in V$ for all $n \geqq n_0$, some n_0. Then

$$u_\lambda x - x = (u_\lambda x - u_\lambda x_{n_0}) + (u_\lambda x_{n_0} - x_{n_0}) + (x_{n_0} - x) \in B_0V + V + V \subset W \quad \text{eventually.}$$

Thus $u_\lambda x \underset{t'}{\to} x$ and similarly $xu_\lambda \underset{t'}{\to} x$. This proves the assertion.

It now follows from Proposition 3.3 and the automatic continuity of a positive functional in the largest locally convex GB^*-topology [8, §§ 6, 8] that every positive functional on a GB^*-algebra A is representable, which by Theorem 3.5 is represented by a selfadjoint representation. This can be used to construct a faithful selfadjoint representation of A as an extended C^*-algebra without identity as in [8, § 7]. In particular, this applies to unbounded Hilbert algebras and to b^*-algebras [2, Example 3.3].

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Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar-388120
India