ON JORDAN REPRESENTATIONS OF UNBOUNDED OPERATOR ALGEBRAS

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Abstract. Every closed Jordan *-representation of an EC*-algebra is the sum of a closed *-representation and a closed *-antirepresentation.

Atsushi Inoue [3] has recently initiated the study of a class of unbounded operator algebras, called EC*-algebras. They seem to be quite useful in connection with unbounded Hilbert algebras. The purpose of this note is to prove the following theorem concerning the structure of unbounded J*-representations of these algebras.

Theorem. Every closed J*-representation π of an EC*-algebra A is a direct sum of a closed *-representation π₁ and a closed *-antirepresentation π₂.

A mapping π of a *-algebra A with identity 1 into linear operators (not necessarily bounded) all defined on a common domain D(π) dense in a Hilbert space K is called a J*-representation of A on K if, for all x, y ∈ A, α, β ∈ C and ξ, η ∈ D(π), the following hold:
1. π(αx + βy)ξ = απ(x)ξ + βπ(y)ξ,
2. ⟨π(x)ξ, η⟩ = ⟨ξ, π(x*)η⟩,
3. π(x)D(π) ⊂ D(π) and π(xy + yx)ξ = π(x)π(y)ξ + π(y)π(x)ξ,
4. π(1) = I.

The closure ̄π of π is a J*-representation on K defined as ̄π(x) = π(x) (x ∈ A), with domain D(̄π) = ⋃{D(π(x)) | x ∈ A} where D(π(x)) denotes the domain of the closure π(x) of the operator π(x) in K. The map π is closed if π = ̄π.

Our theorem is an extension of a result of Størmer [6, Theorem 3.3] on C*-algebras. Let A be an EC*-algebra on a dense subspace D of a Hilbert space H. Then its bounded part ̄A_b = {T | T is a bounded operator in A} is a C*-algebra. We apply Størmer’s theorem to ̄A_b. Since EC*-algebras are concrete realizations of locally convex GB*-algebras [2, Theorem 7.11], the proof is completed by applying next a density theorem [1] viz. If A is a GB*-algebra with unit ball B₀, then the B*-algebra A(B₀) is sequentially dense in A. Note that A(B₀) = {λx | λ ∈ C, x ∈ B₀} with the Minkowski functional || · || B₀ as the norm.

We shall need the following elementary lemmas.
Lemma A. Let $A$ be a locally convex GB*-algebra with unit ball $B_0$. Let $\pi$ be a $J^*$-representation of $A$ on $K$. Then $x \to \pi(x)$ defines a $J^*$-homomorphism of the $B^*$-algebra $A(B_0)$ into $B(K)$, the algebra of all bounded linear operators on $K$.

We omit the easy proof.

Lemma B. Let $A$ be a locally convex GB*-algebra. Let $\tau$ be the largest locally convex GB*-topology on $A$. Let $\pi$ be a $J^*$-representation of $A$ on $K$. Let $\sigma_\pi$ be the weak topology on $\pi(A)$ defined by the seminorms $\pi(x) \to \langle \pi(x)\xi, \eta \rangle$ for $\xi, \eta \in D(\pi)$. Then $\pi: (A, \tau) \to (\pi(A), \sigma_\pi)$ is continuous.

Proof. By the polarization identity, $\sigma_\pi$ is also determined by the seminorms $\rho(\pi(x)) = \|\langle \pi(x)\xi, \eta \rangle\|$ for $\xi \in D(\pi)$. For any such $\xi$, let $f$ on $A$ be defined by $f(x) = \langle \pi(x)\xi, \eta \rangle$. Let $x \in A$. By [2, Proposition 5.1], $x^*x = h^2$ for some $h \geq 0$. Then $f(x^*x) = \langle \pi(h^2)\xi, \eta \rangle = \langle \pi(h)\xi, \eta \rangle = \|\pi(h)\xi\|^2 \geq 0$. Thus $f$ is a positive linear functional on $A$ which by [2, §8] is $\tau$-continuous. Now the result follows immediately.

Proof of the theorem. Let $A$ be an $EC^*$-algebra over a domain $D$ dense in $H$, a Hilbert space. Let $\bar{A} = \{T | T \in A\}$. Let $\sigma$ be the weak topology on $A$ defined by the seminorms $T \to p(\xi, T) = \langle T\xi, \eta \rangle$ for $\xi, \eta \in D$. Then [2, Theorem 7.12] shows that $(\bar{A}, \sigma)$ is a locally convex GB*-algebra with unit ball $B_0 = \{T \in \bar{A} | \|T\| \leq 1\}$ so that the underlying $B^*$-algebra is $\bar{A}_b = \bar{A} \cap \beta(H)$. Let $\tau$ be the largest locally convex GB*-topology on $A$ [2, §6]. From now on, for the sake of simplicity, we omit writing the bar over the elements of $A$.

By Lemma A, $T \to \phi(T) = \pi(\bar{T})$ defines a $J^*$-representation of $\bar{A}_b$ into $\beta(K)$. Let $B$ be the $C^*$-algebra in $B(K)$ generated by $\phi(\bar{A}_b)$. Hence [6, Theorem 3.3] there exist orthogonal central projections $E$ and $F$ in the von Neumann algebra generated by $B$ such that $T \to \phi_1(T) = \phi(T)E$ defines a $^*$-representation of $\bar{A}_b$ on $K_1 = EK$ and $T \to \phi_2(T)F$ defines a $^*$-antirepresentation of $\bar{A}_b$ on $K_2 = FK$. Also $E + F = I$. Let $D(\pi_1) = ED(\pi), D(\pi_2) = FD(\pi)$.

We show that, for each $T \in A$, $\xi, \eta \in D(\pi)$, $\langle E\pi(T)\xi, \eta \rangle = \langle E\xi, \pi(T)^*\eta \rangle$. Indeed, by the density theorem, there is a sequence $T_n$ in $A_b$ such that $T_n \to T$ in $\tau$. By Lemma B,

$$\langle E\pi(T)\xi, \eta \rangle = \langle \pi(T)\xi, E\eta \rangle = \lim_n \langle \pi(T_n)\xi, E\eta \rangle = \lim_n \langle E\pi(T_n)\xi, \eta \rangle$$

as $E$ is central

$$= \lim_n \langle E\xi, \pi(T_n)^*\eta \rangle$$

$$= \lim_n \langle E\xi, \pi(T_n^*)\eta \rangle$$

as $\pi(T_n^*) \subset \pi(T_n)^*$

$$= \langle E\xi, \pi(T)^*\eta \rangle$$

again by Lemma B and the continuity of the involution

$$= \langle E\xi, \pi(T)^*\eta \rangle.$$

This shows that $\eta \to \langle E\xi, \pi(T)^*\eta \rangle$ defines, for each $\xi \in D(\pi)$, a norm bounded linear functional on $D(\pi)$, and so $E\xi \in D(\pi(T)^**) = D(\pi(T^*))$. Thus
$E\xi \in \cap \{ D(\pi(T)) \mid T \in A \} = D(\pi) = D(\sigma)$. Thus $ED(\sigma) \subset D(\sigma)$. Similarly $FD(\pi) \subset D(\pi)$ and $so D(\pi_1) + D(\pi_2) \subset D(\pi_1) \cap D(\pi_2) = D(\pi).

Clearly $D(\pi_1)$ is dense in $K_1$ and $D(\pi_2)$ is dense in $K_2$. Let, for each $T \in A$, $\pi_1(T) = \pi(T)E$, $\pi_2(T) = \pi(T)F$ with domains $D(\pi_1)$ and $D(\pi_2)$ respectively. We show that $\pi_1$ and $\pi_2$ are the required maps.

Let $T \in A$. For each $n = 1, 2, \ldots$ let $T_n = T(1 + \frac{1}{n} T^*T)^{-1}$. (Here the sum and the product are in the strong sense.) First we show that $T_n \to T$ in $\tau$.

$$T - T_n = \frac{1}{\sqrt{n}} (TT^*) \left( \frac{T}{\sqrt{n}} \right) \left( 1 + \left( \frac{T}{\sqrt{n}} \right)^* \left( \frac{T}{\sqrt{n}} \right) \right)^{-1} \in \frac{1}{\sqrt{n}} (TT^*)B_0$$

by [5, Theorem 13.13].

Now by the separate continuity of multiplication in $\tau$, given an $o$-neighbourhood $V$, there exists an $o$-neighbourhood $U$ such that $TT^*U \subset V$. Further, as $B_0$ is $\tau$-bounded, $\sqrt{r} B_0 \subset U$ for sufficiently small $r > 0$. It follows that $T - T_n \in V$ eventually. Thus $T_n \to T$ in $\tau$.

Next we show that $\pi_1(A)D(\pi_1) \subset D(\pi_1)$ and $\pi_2(A)D(\pi_2) \subset D(\pi_2)$. Let $\xi, \eta \in D(\pi)$. By Lemma B,

$$\langle \pi_1(T)E\xi, \eta \rangle = \lim_{n} \langle \pi(T_n)E\xi, \eta \rangle = \lim_{n} \langle E\pi(T_n)E\xi, \eta \rangle \quad \text{as} \quad \pi(T_n)E\xi \in K$$

$$= \lim_{n} \langle \pi(T_n)E\xi, E\eta \rangle = \langle \pi(T)E\xi, E\eta \rangle \quad \text{again by Lemma B}$$

and so $\pi_1(T)E\xi = E\pi(T)E\xi \in D(\pi_1)$. Hence $\pi_1(A)D(\pi_1) \subset D(\pi_1)$. Similarly $\pi_2(A)D(\pi_2) \subset D(\pi_2)$. Thus $\pi_1(T)$ and $\pi_2(T)$ are operators in $K_1$ and $K_2$ respectively.

Now given $T, S$ in $A$, again by the density theorem, there exist sequences $T_n$ and $S_n$ in $A$ such that $T_n \to T, S_n \to S$ in $\tau$. As $(A, \tau)$ is barrelled [2, Lemma 6.2], it is hypocontinuous by [2, Lemma 6.3]. Since the multiplication in a hypocontinuous algebra is easily seen to be sequentially jointly continuous, $T_n S_n \to TS$. Then for each $\xi, \eta$ in $D(\pi_1)$, repeated uses of Lemma B give

$$\langle \pi_1(T)\pi_1(S)\xi, \eta \rangle = \lim_{n} \lim_{k} \langle \pi_1(T_n)\pi_1(S_k)\xi, \eta \rangle = \lim_{n} \langle \pi_1(T_n)\pi_1(S_n)\xi, \eta \rangle = \lim_{n} \langle \pi_1(T_nS_n)\xi, \eta \rangle \quad \text{as} \quad \phi_1 \text{ is a representation} = \langle \pi_1(TS)\xi, \eta \rangle.$$

Thus $\pi_1$ is a *-representation. Similarly $\pi_2$ is a *-antirepresentation.

It only remains to show that each of $\pi_1$ and $\pi_2$ is closed. Let $T \in A$ and $\xi \in D(\pi_1(T))$, the domain of the closure in $K$. Then for some sequence $\{\xi_n, \pi_1(T)\xi_n\}$
in the graph of \( \pi_1(T) \), \( \xi_n \to \xi \) in \( K_1 \). Then clearly \( \xi \in D(\pi(T)) \) and so \( \xi \in D(\pi) = D(\pi_1) \). Also \( \pi_1 \) is closed, and similarly so is \( \pi_2 \).

This completes the proof of the theorem.

Note that what is essentially required in the proof is the fact that \( A \) is a locally convex \( GB^* \)-algebra. Hence [4, Corollary 3.4] immediately gives the following.

**Corollary.** Let \( D \) be a pure unbounded Hilbert algebra over a maximal unital Hilbert algebra \( D_0 \). Then each closed \( J^* \)-representation of \( D \) is the direct sum of a closed \( * \)-representation and a closed \( * \)-antirepresentation.

**References**