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ON JORDAN REPRESENTATIONS OF UNBOUNDED OPERATOR ALGEBRAS

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ABSTRACT. Every closed Jordan *-representation of an EC^*-algebra is the sum of a closed *-representation and a closed *-antirepresentation.

Atsushi Inoue [3] has recently initiated the study of a class of unbounded operator algebras, called EC^*-algebras. They seem to be quite useful in connection with unbounded Hilbert algebras. The purpose of this note is to prove the following theorem concerning the structure of unbounded J^*-representations of these algebras.

THEOREM. *Every closed J^*-representation π of an EC^*-algebra A is a direct sum of a closed *-representation π_1 and a closed *-antirepresentation π_2.*

A mapping π of a *-algebra A with identity 1 into linear operators (not necessarily bounded) all defined on a common domain $D(\pi)$ dense in a Hilbert space K is called a J^*-representation of A on K if, for all $x, y \in A$, $\alpha, \beta \in \mathbf{C}$ and $\xi, \eta \in D(\pi)$, the following hold:

1. $\pi(\alpha x + \beta y)\xi = \alpha\pi(x)\xi + \beta\pi(y)\xi$,
2. $\langle \pi(x)\xi, \eta\rangle = \langle \xi, \pi(x^*)\eta\rangle$,
3. $\pi(x)D(\pi) \subset D(\pi)$ and $\pi(xy + yx)\xi = \pi(x)\pi(y)\xi + \pi(y)\pi(x)\xi$,
4. $\pi(1) = I$.

The *closure* $\bar{\pi}$ *of* π is a J^*-representation on K defined as $\bar{\pi}(x) = \overline{\pi(x)}$ ($x \in A$), with domain $D(\bar{\pi}) = \bigcap\{D(\overline{\pi(x)}) \mid x \in A\}$ where $D(\overline{\pi(x)})$ denotes the domain of the closure $\overline{\pi(x)}$ of the operator $\pi(x)$ in K. The map π is *closed* if $\pi = \bar{\pi}$.

Our theorem is an extension of a result of Størmer [6, Theorem 3.3] on C^*-algebras. Let A be an EC^*-algebra on a dense subspace D of a Hilbert space H. Then its bounded part $\bar{A}_b = \{\bar{T} \mid T$ is a bounded operator in $A\}$ is a C^*-algebra. We apply Størmer's theorem to $\bar{A}_b$. Since EC^*-algebras are concrete realizations of locally convex GB^*-algebras [2, Theorem 7.11], the proof is completed by applying next a density theorem [1] viz. If A is a GB^*-algebra with unit ball B_0, then the B^*-algebra $A(B_0)$ is sequentially dense in A. Note that $A(B_0) = \{\lambda x \mid \lambda \in \mathbf{C}, x \in B_0\}$ with the Minkowski functional $\|\cdot\|_{B_0}$ as the norm.

We shall need the following elementary lemmas.

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LEMMA A. *Let A be a locally convex GB^*-algebra with unit ball B_0. Let π be a J^*-representation of A on K. Then $x \to \overline{\pi(x)}$ defines a J^*-homomorphism of the B^*-algebra $A(B_0)$ into $B(K)$, the algebra of all bounded linear operators on K.*

We omit the easy proof.

LEMMA B. *Let A be a locally convex GB^*-algebra. Let τ be the largest locally convex GB^*-topology on A. Let π be a J^*-representation of A on K. Let σ_π be the weak topology on $\pi(A)$ defined by the seminorms $\pi(x) \to |\langle \pi(x)\xi, \eta\rangle|$ for ξ, η in $D(\pi)$. Then π: $(A, \tau) \to (\pi(A), \sigma_\pi)$ is continuous.*

PROOF. By the polarization identity, σ_π is also determined by the seminorms $p(\pi(x)) = |\langle \pi(x)\xi, \xi\rangle|$ for $\xi \in D(\pi)$. For any such ξ, let f on A be defined by $f(x) = \langle \pi(x)\xi, \xi\rangle$. Let $x \in A$. By [**2**, Proposition 5.1], $x^*x = h^2$ for some $h \geqslant 0$. Then $f(x^*x) = \langle \pi(h^2)\xi, \xi\rangle = \langle \pi(h)^2\xi, \xi\rangle = \|\pi(h)\xi\|^2 \geqslant 0$. Thus f is a positive linear functional on A which by [**2**, §8] is τ-continuous. Now the result follows immediately.

PROOF OF THE THEOREM. Let A be an EC^*-algebra over a domain D dense in H, a Hilbert space. Let $\overline{A} = \{\overline{T} \mid T \in A\}$. Let σ be the weak topology on A defined by the seminorms $T \to p_{\xi,\eta}(T) = |\langle T\xi, \eta\rangle|$ for $\xi, \eta \in D$. Then [**2**, Theorem 7.12] shows that $(\overline{A}, \sigma)$ is a locally convex GB^*-algebra with unit ball $B_0 = \{\overline{T} \in \overline{A} \mid \|T\| \leqslant 1\}$ so that the underlying B^*-algebra is $\overline{A}_b = \overline{A} \cap \beta(H)$. Let τ be the largest locally convex GB^*-topology on A [**2**, §6]. From now on, for the sake of simplicity, we omit writing the bar over the elements of A.

By Lemma A, $T \to \phi(T) = \overline{\pi(T)}$ defines a J^*-representation of $\overline{A}_b$ into $\beta(K)$. Let B be the C^*-algebra in $B(K)$ generated by $\phi(\overline{A}_b)$. Hence [**6**, Theorem 3.3] there exist orthogonal central projections E and F in the von Neumann algebra generated by B such that $T \to \phi_1(T) = \phi(T)E$ defines a $*$-representation of $\overline{A}_b$ on $K_1 = EK$ and $T \to \phi_2(T)F$ defines a $*$-antirepresentation of $\overline{A}_b$ on $K_2 = FK$. Also $E + F = I$. Let $D(\pi_1) = ED(\pi)$, $D(\pi_2) = FD(\pi)$.

We show that, for each $T \in A$, $\xi, \eta \in D(\pi)$, $\langle E\pi(T)\xi, \eta\rangle = \langle E\xi, \pi(T)^*\eta\rangle$. Indeed, by the density theorem, there is a sequence T_n in A_b such that $T_n \to T$ in τ. By Lemma B,

$$\begin{aligned}\langle E\pi(T)\xi, \eta\rangle &= \langle \pi(T)\xi, E\eta\rangle = \lim_n \langle \pi(T_n)\xi, E\eta\rangle = \lim_n \langle E\pi(T_n)\xi, \eta\rangle \\ &= \lim_n \langle \pi(T_n)E\xi, \eta\rangle \quad \text{as } E \text{ is central} \\ &= \lim_n \langle E\xi, \pi(T_n)^*\eta\rangle \\ &= \lim_n \langle E\xi, \pi(T_n^*)\eta\rangle \quad \text{as } \pi(T_n^*) \subset \pi(T_n)^* \\ &= \langle E\xi, \pi(T^*)\eta\rangle \\ &\qquad \text{again by Lemma B and the continuity of the involution} \\ &= \langle E\xi, \pi(T)^*\eta\rangle.\end{aligned}$$

This shows that $\eta \to \langle E\xi, \pi(T)^*\eta\rangle$ defines, for each $\xi \in D(\pi)$, a norm bounded linear functional on $D(\pi)$, and so $E\xi \in D(\pi(T)^{**}) = D(\overline{\pi(T)})$. Thus

$E\xi \in \bigcap\{D(\overline{\pi(T)}) \mid T \in A\} = D(\bar{\pi}) = D(\pi)$. Thus $ED(\pi) \subset D(\pi)$. Similarly $FD(\pi) \subset D(\pi)$ and so $D(\pi_1) + D(\pi_2) \subset D(\pi)$, $D(\pi_1) + D(\pi_2) = D(\pi)$.

Clearly $D(\pi_1)$ is dense in K_1 and $D(\pi_2)$ is dense in K_2. Let, for each $T \in A$, $\pi_1(T) = \pi(T)E$, $\pi_2(T) = \pi(T)F$ with domains $D(\pi_1)$ and $D(\pi_2)$ respectively. We show that π_1 and π_2 are the required maps.

Let $T \in A$. For each $n = 1, 2, \ldots$ let $T_n = T(1 + \frac{1}{n}T^*T)^{-1}$. (Here the sum and the product are in the strong sense.) First we show that $T_n \to T$ in τ.

$$\begin{aligned} T - T_n &= \tfrac{1}{n}TT^*T\left(1 + \tfrac{1}{n}T^*T\right)^{-1} \\ &= \frac{1}{\sqrt{n}}(TT^*)\left(\frac{T}{\sqrt{n}}\right)\left(1 + \left(\frac{T}{\sqrt{n}}\right)^*\left(\frac{T}{\sqrt{n}}\right)\right)^{-1} \in \frac{1}{\sqrt{n}}(TT^*)B_0 \end{aligned}$$

by [5, Theorem 13.13].

Now by the separate continuity of multiplication in τ, given an o-neighbourhood V, there exists an o-neighbourhood U such that $TT^*U \subset V$. Further, as B_0 is τ-bounded, $\sqrt{r}\,B_0 \subset U$ for sufficiently small $r > 0$. It follows that $T - T_n \in V$ eventually. Thus $T_n \to T$ in τ.

Next we show that $\pi(A)D(\pi_1) \subset D(\pi_1)$ and $\pi(A)D(\pi_2) \subset D(\pi_2)$. Let $\xi, \eta \in D(\pi)$. By Lemma B,

$$\begin{aligned} \langle \pi(T)E\xi, \eta \rangle &= \lim_n \langle \pi(T_n)E\xi, \eta \rangle \\ &= \lim_n \langle E\pi(T_n)E\xi, \eta \rangle \quad \text{as } \pi(T_n)E\xi \in K \\ &= \lim_n \langle \pi(T_n)E\xi, E\eta \rangle \\ &= \langle \pi(T)E\xi, E\eta \rangle \quad \text{again by Lemma B} \\ &= \langle E\pi(T)E\xi, \eta \rangle \end{aligned}$$

and so $\pi(T)E\xi = E\pi(T)E\xi \in D(\pi_1)$. Hence $\pi(A)D(\pi_1) \subset D(\pi_1)$. Similarly $\pi(A)D(\pi_2) \subset D(\pi_2)$. Thus $\pi_1(T)$ and $\pi_2(T)$ are operators in K_1 and K_2 respectively.

Now given T, S in A, again by the density theorem, there exist sequences T_n and S_n in A such that $T_n \to T$, $S_n \to S$ in τ. As (A, τ) is barrelled [2, Lemma 6.2], it is hypocontinuous by [2, Lemma 6.3]. Since the multiplication in a hypocontinuous algebra is easily see to be sequentially jointly continuous, $T_nS_n \to TS$. Then for each ξ, η in $D(\pi_1)$, repeated uses of Lemma B give

$$\begin{aligned} \langle \pi_1(T)\pi_1(S)\xi, \eta \rangle &= \lim_n \lim_k \langle \pi_1(T_n)\pi_1(S_k)\xi, \eta \rangle \\ &= \lim_n \langle \pi_1(T_n)\pi_1(S_n)\xi, \eta \rangle \\ &= \lim_n \langle \pi_1(T_nS_n)\xi, \eta \rangle \quad \text{as } \phi_1 \text{ is a representation} \\ &= \langle \pi_1(TS)\xi, \eta \rangle. \end{aligned}$$

Thus π_1 is a *-representation. Similarly π_2 is a *-antirepresentation.

It only remains to show that each of π_1 and π_2 is closed. Let $T \in A$ and $\xi \in D(\overline{\pi_1(T)})$, the domain of the closure in K. Then for some sequence $\{\xi_n, \pi_1(T)\xi_n\}$

in the graph of $\pi_1(T)$, $\xi_n \to \xi$ in K_1. Then clearly $\xi \in D(\overline{\pi(T)})$ and so $\xi \in D(\bar{\pi}) = D(\pi)$. Also π_1 is closed, and similarly so is π_2.

This completes the proof of the theorem.

Note that what is essentially required in the proof is the fact that A is a locally convex GB^*-algebra. Hence [**4**, Corollary 3.4] immediately gives the following.

COROLLARY. *Let D be a pure unbounded Hilbert algebra over a maximal unital Hilbert algebra D_0. Then each closed J^*-representation of D is the direct sum of a closed $*$-representation and a closed $*$-antirepresentation.*

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