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AN IRREDUCIBLE REPRESENTATION OF A SYMMETRIC STAR ALGEBRA IS BOUNDED

BY

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ABSTRACT. A $*$-algebra A is called symmetric if $(1 + x^*x)$ is invertible in A for each x in A. An irreducible hermitian representation of a symmetric $*$-algebra A maps A onto an algebra of bounded operators.

1. THEOREM. *Let A be a symmetric $*$-algebra with identity* 1. *Let $(\pi, D(\pi), H)$ be a closed $*$-representation of A on a Hilbert space H. If the only π-invariant selfadjoint subspaces of $D(\pi)$ are* (0) *and $D(\pi)$, then π is a bounded representation.*

COROLLARY. *Every closed (algebraically) irreducible $*$-representation of a symmetric $*$-algebra is bounded.*

The purpose of this paper is to prove the above theorem. A *$*$-algebra A* is a linear associative algebra with identity 1 over the complex field **C** such that A admits an involution $a \in A \to a^* \in A$ satisfying the usual axioms. If $(1 + a^*a)^{-1}$ exists in A, for every $a \in A$, then A is called *symmetric*.

A representation $(\pi, D(\pi), H)$ of a $*$-algebra A on a Hilbert space H is a mapping π of A into the linear operators (not necessarily bounded), all defined on a common domain $D(\pi)$, a dense linear subspace in H, such that for all a, b in A, α, β in **C** and ξ in $D(\pi)$,

(i) $\pi(\alpha a + \beta b)\xi = \alpha\pi(a)\xi + \beta\pi(b)\xi$,

(ii) $\pi(a)D(\pi) \subset D(\pi)$ and $\pi(a)\pi(b)\xi = \pi(ab)\xi$,

(iii) $\pi(1) = I$.

It is called a *$*$-representation* if for each $a \in A$,

(iv) $D(\pi) \subset D(\pi(a)^*)$, the domain of the operator adjoint $\pi(a)^*$ of $\pi(a)$, and $\pi(a^*) \subset \pi(a)^*$. π is called a *bounded representation* if $\pi(a)$ is a bounded operator for each $a \in A$. Throughout by a representation we always mean a $*$-representation.

The analysis of the representations of abstract $*$-algebras has been motivated in Quantum Field Theory to avoid starting with (and staying within) a specific Hilbert space (the Fock space) scheme and rather to stress that the basic objects of the theory are observables considered as purely algebraic quantities forming a $*$-algebra. Realizations of these algebraic objects as Hilbert space operators naturally lead to unbounded representations defined above. In **[15]**, R. T. Powers developed a basic representation theory for $*$-algebras admitting unbounded observables. Representations of symmetric $*$-algebras have been investigated in **[11]**. On the other hand,

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certain symmetric algebras of unbounded operators (symmetric $*$-algebras, EC^*-algebras, EW^*-algebras) have been studied by Dixon [9] and Inoue [12]. EC^*-algebras occur naturally in the unbounded generalizations of left Hilbert algebras and standard von Neumann algebras [13]. The above theorem for EC^*-algebras was established in [6].

Given a representation $(\pi, D(\pi), H)$ of a $*$-algebra A, the *induced topology* t_A on $D(\pi)$ is the locally convex topology defined by the seminorms $\xi \to \|\pi(a)\xi\|$ $(a \in A)$. The completion of $(D(\pi), t_A)$ is $\overline{D(\pi)} = \bigcap\{D(\overline{\pi(a)})|a \in A\}$, $\overline{\pi(a)}$ denoting the closure of $\pi(a)$. Then $\bar{\pi}(a) = \overline{\pi(a)}|_{D(\bar{\pi})}$ defines a representation $(\bar{\pi}, D(\bar{\pi}), H)$, called the *closure of* π; π is *closed* if $D(\pi) = D(\bar{\pi})$. A representation π is called *selfadjoint* if $D(\pi) = D(\pi^*)$, where $D(\pi^*) = \bigcap\{D(\pi(a^*)^*)|a \in A\}$. A π-invariant subspace M of $D(\pi)$ is *selfadjoint* if the restriction of π to M is selfadjoint. For further details, we refer to [11].

The idea of the proof is borrowed from the enveloping C^*-algebra of a Banach $*$-algebra. Form a suitable reducing ideal I, represent the quotient algebra $X = A/I$ faithfully as an unbounded operator algebra (not an EC^*-algebra, though symmetric). Modify the EC^*-techniques of [6] avoiding the completeness of the underlying algebra X_b of bounded operators. π extends to an irreducible representation σ of the completion $\tilde{X}_b$ of X_b. By standard C^*-theory, σ turns out to be algebraically irreducible which quickly leads to boundedness of π. The technicalities of some of our steps are modifications of those that are scattered in [1, 4, 6 and 8]. However, for the sake of completeness we have briefly included all details.

Finally, connections with the work of Mathot [14] on the disintegration of representations is discussed.

2. Peliminary constructions.

(2.1) *The $*$-algebra $X = A/I$.* Let A be a $*$-algebra. Let $P(A)$ and $R(A)$ denote respectively the sets of all positive (linear) forms on A and of all closed strongly cyclic representations of A. By the GNS construction, each $f \in P(A)$ yields a $\pi_f \in R(A)$ as follows: Let $N_f = \{x \in A | f(x^*x) = 0\} = \{x \in A | f(y^*x) = 0$ for all $y \in A\}$; $X_f = A/N_f$, a pre-Hilbert space with inner product $\langle a + N_f, b + N_f\rangle = f(b^*a)$; $H_f =$ the Hilbert space obtained by completing X_f. Define π'_f on A as $\pi'_f(a)(b + N_f) = ab + N_f$, $D(\pi'_f) = X_f$. Then π'_f is an ultracyclic representation of A. Let π_f be the closure of π'_f with $D(\pi_f) = D(\bar{\pi}'_f)$. Then $\pi_f \in R(A)$, $\xi_f = 1 + N_f$ being a strongly cyclic vector. Also, modulo unitary equivalence, every strongly cyclic representation of A is of this form [15, §VI].

Now let $I = \bigcap\{N_f | f \in P(A)\}$. Then $I = \bigcap\{\ker \pi | \pi \in R(A)\}$. For, if $\pi_f(x) = 0$ for some $f \in P(A)$, then $xy + N_f = 0$ for all $y \in A$. Taking $y = 1$, $f(x^*x) = 0$, $x \in N_f$. On the other hand, given $f \in P(A)$, $y \in A$, define $f_y \in P(A)$ by $f_y(x) = f(y^*xy)$. Then for a given $x \in A$, $f(x^*x) = 0$ for all $f \in P(A)$ implies that for an arbitrary $f \in P(A)$, $f_y(x^*x) = 0$ for all $y \in A$. Hence $\pi_f(x) = 0$ giving $x \in \ker \pi_f$.

It follows from the above that I is a $*$-ideal of A. Let $X = A/I$ be the quotient algebra. Define

$$B_0 = \{x \in X | f(x^*x) \leqslant f(1) \text{ for all } f \in P(X)\},$$

$$B_0' = \left\{x \in X | \text{for each } f \in R(X),\ \pi(x) \text{ is bounded and } \|\pi(x)\| \leqslant 1\right\}.$$

Then $B_0 = B_0'$. Indeed, let $x \in B_0$. Then for every $f \in P(X)$, $y \in X$,

$$\left\|\pi_f(x)(y + N_f)\right\|^2 = f(y^*x^*xy) = f_y(x^*x) \leqslant f_y(1)$$
$$= f(y^*y) = \left\|y + N_f\right\|^2.$$

Hence $\|\pi_f(x)\| \leqslant 1$. On the other hand, if $x \in B_0'$, then $f(y^*xy) \leqslant f(y^*y)$ for all $y \in X$, $f \in P(X)$. Again taking $y = 1$, $x \in B_0$. Thus $B_0 = B_0'$. We verify the following properties of B_0.

(i) $B_0 = B_0^*$.

This is immediate in view of the fact that $\|\pi_f(x)\| \leqslant 1$ iff $\|\pi_f(x^*)\| \leqslant 1$.

(ii) B_0 is absolutely convex.

That it is balanced is obvious. For x, y in B_0, $0 \leqslant t \leqslant 1$, taking $z = tx + (1 - t)y$, the Cauchy–Schwarz inequality gives

$$f(z^*z) \leqslant \left\{tf(x^*x)^{1/2} + (1 - t)f(y^*y)^{1/2}\right\}^2 \leqslant f(1)$$

showing that B_0 is convex.

(iii) $B_0^2 \subset B_0$, $1 \in B_0$.

Let $z = xy$ with x, y in B_0. If $f_y = 0$, then $f(z^*z) = f_y(x^*x) = 0$; otherwise, for some $u \in X$, $f_y(u) \neq 0$, and by the Cauchy–Schwarz inequality, $f_y(1) \neq 0$. Then again by the same inequality, $f(z^*z) \leqslant f_y(1) = f(y^*y) \leqslant f(1)$. Thus $z \in B_0$.

(2.2) *Topologies on X*. (a) Let X^P be the complex linear span of all positive forms on X. Let $\sigma_P = \sigma(X, X^P)$ be the weak topology on X determined by the duality $\langle X, X^P\rangle$. By the construction of X, given $x \neq 0$ in X, there exists an $f \in P(X)$ such that $f(x^*x) \neq 0$. Hence the direct sum [**15**, Remark following Theorem 7.5]

$$\pi_u = \sum_{f \in P(X)} \oplus\, \pi_f$$

(note that $\pi_f \in R(X)$) defines a faithful representation of X on the Hilbert space

$$H_u = \sum_{f \in P(X)} \oplus\, H_f$$

with domain

$$D(\pi_u) = \left\{\xi = (\xi_f) | \xi_f \in H_f \text{ for all } f \in P(X) \text{ and } \sum_{f \in P(X)}^{\oplus} \left\|\pi_f(x)\xi_f\right\|^2 < \infty \text{ for all } x \in X\right\}.$$

This is the *universal representation* of X. Let σ_{π_u} be the topology on X defined by the seminorms $x \in X \to p_{\xi,\eta}(x) = |\langle\pi_u(x)\xi, \eta\rangle|$ for ξ, η in $D(\pi_u)$; or equivalently, by the seminorms $x \in X \to p_\xi(x) = |\langle\pi_u(x)\xi, \xi\rangle|$ ($\xi \in D(\pi_u)$) by using the polarization identity. Since all positive forms on X are taken into account to construct π_u, it is easily seen that $\sigma_{\pi_u} = \sigma_P$.

Also, it is easy to check that X with σ_P (or with the Mackey topology $\tau(X, X^P)$) is a locally convex $*$-algebra (with separately continuous multiplication and continuous involution). Also, every positive form on X is σ_P-continuous and

$$B_0 = \left\{x \in X | \left\|\pi_u(x)\right\| \leqslant 1\right\}.$$

Now we verify the following additional property of B_0.

(iv) Let $\mathscr{B}^*(\sigma_P)$ be the collection of all σ_P-closed, σ_P-bounded, absolutely convex subsets B of X satisfying $B^2 \subset B$, $B^* = B$, $1 \in B$. Then B_0 is the greatest member of $\mathscr{B}^*(\sigma_P)$. That B_0 is bounded in σ_P follows from the definition of B_0. Let $B \in \mathscr{B}^*(\sigma_P)$. Let $x \in B$. If $\|\pi_u(x)\| > 1$, then for some $\xi \in D(\pi_u)$, $\|\xi\| = 1$, we have $\|\pi_u(x)\xi\| > 1$. For all $n = 1, 2, 3, \dots$

$$\left|\left\langle \pi_u(x^*x)^{2^n}\xi, \xi\right\rangle\right| \geqslant \|\pi_u(x)\xi\|^{2^{n+1}} \to \infty \quad \text{as } n \to \infty.$$

On the other hand, $x^* \in B$ as $x \in B$; and so $(x^*x)^{2^n} \in B$. This contradiction shows that $\|\pi_u(x)\xi\| \leqslant 1$ for all $\xi \in D(\pi_u)$, $\|\xi\| = 1$. Hence $x \in B_0$. Thus $B \subset B_0$. The above argument applied to B_0 also shows that B_0 is σ_P-closed. This gives (iv).

(b) We shall also need two other topologies on X induced from those on X via π_u; viz. the *quasiweak topology* defined by the seminorms $x \in X \to |\langle \pi_u(x)\xi, \eta\rangle| = p_{\xi,\eta}(x)$, where $\xi \in D(\pi_u)$, $\eta \in H_u$; and the *strong topology* defined by the seminorms $x \in X \to \|\pi_u(x)\xi\|$ for $\xi \in D(\pi_u)$.

(2.3) *The pre-C^* algebra $X(B_0)$.* From the properties (i)–(iv) of B_0, it follows that

$$X(B_0) = \{\lambda x | \lambda \in \mathbf{C}, x \in B_0\}$$

is a $*$-subalgebra of X containing the identity and, for $x \in X(B_0)$.

$$\begin{aligned} \|x\|_{B_0} &= \inf\{\lambda > 0 | x \in \lambda B_0\} \\ &= \sup\left\{ f(x^*x)^{1/2} | f \in P(X)\right\} = \|\pi_u(x)\| \end{aligned}$$

defines a norm on $X(B_0)$ satisfying $\|x^*x\|_{B_0} = \|x\|^2_{B_0}$. However, $(X(B_0), \|\cdot\|_{B_0})$ need not be complete. Also, $x \in X(B_0)$ iff $\pi_u(x)$ is a bounded operator.

(2.4) We note in passing that X provides a solution of the universal problem for selfadjoint representations. If A is a selfadjoint representation of a $*$-algebra A on a Hilbert space H with domain $D(\pi)$, then there exists a unique selfadjoint representation $\tilde{\pi}$ of X on H such that $D(\tilde{\pi}) = D(\pi)$ and $\pi = \tilde{\pi} \circ \psi$, where ψ: $A \to X$ is the natural map. This follows from the fact **[15]** that π being selfadjoint is a direct sum of closed strongly cyclic representations; and by the construction of X, every positive form on A, and hence every closed strongly cyclic representation of A, factors through X.

(2.5) LEMMA. *Let A be a symmetric $*$-algebra.*

(a) *For each $x \in X$, $(1 + x^*x)^{-1} \in X(B_0)$.*

(b) *For each $h = h^*$ in X and for each $n = 1, 2, \dots,$ $h(1 + (1/n)h^2)^{-1} \in X(B_0)$.*

(c) *If τ is any topology on X making (X, τ) a locally convex $*$-algebra such that B_0 is τ-bounded, then*

$$h = \lim_{n \to \infty} h\left(1 + \frac{1}{n}h^2\right)^{-1} \quad \text{in } \tau.$$

PROOF. (a) Obviously X is also symmetric. By [**11**, Lemma 3.2], applied to X and π_u, $(I + \pi(x)^\sharp\pi(x))^{-1}$ is bounded for each $x \in X$, and $\|(I + \pi(x)^\sharp\pi(x))^{-1}\| \leqslant 1$ where $\pi(x) = \pi_u(x)|_{D(\pi_u)}$. Thus $\|(1 + x^*x)^{-1}\|_{B_0} \leqslant 1$.

(b) By (a), for each $h = h^*$ in X, $(1 + h^2)^{-1} \in X(B_0)$, $\|(1 + h^2)^{-1}\|_{B_0} \leqslant 1$. Hence

$$(1 + h^2)^{-1} - (1 + h^2)^{-2} = h^2(1 + h^2)^{-2} \in X(B_0).$$

If necessary, by passing to the completion of $X(B_0)$ and taking a Gelfand representation, $\|h^2(1 + h^2)^{-2}\|_{B_0} \leqslant 1$. Let $h_n = h(1 + h^2/n)^{-1}$. Then for all $f \in P(X)$,

$$\begin{aligned} f(h_n^2) &= f\left(n(h/\sqrt{n})^2(1 + (h/\sqrt{n})^2)^{-2}\right) \\ &\leqslant n\left\|(h^2/n)(1 + h^2/n)^{-2}\right\|_{B_0} f(1) \leqslant nf(1). \end{aligned}$$

Hence $f((h_n/\sqrt{n})^2) \leqslant f(1)$. Thus $h_n/\sqrt{n} \in B_0$, $h_n \in X(B_0)$ for all n.

(c) follows by an argument exactly as in [**5**, Lemma 3.3].

3. Proof of the Theorem. Since π is closed and A is symmetric, [**11**, Lemma 3.5] implies that π is selfadjoint, and the von Neumann algebra $\pi(A)' = \mathbf{C}1$. Let $\tilde{\pi}$ be the representation of X on H induced by π; viz., $\tilde{\pi}(a + I) = \pi(a)$ $(a \in A)$ with $D(\tilde{\pi}) = D(\pi)$. The only $\tilde{\pi}$-invariant selfadjoint subspaces of $D(\tilde{\pi})$ are (0) and $D(\tilde{\pi})$. (Note that on $D(\pi)$, the induced topology defined by A coincides with the induced topology defined by X.) Further, every nonzero vector $\xi \in D(\pi) = D(\tilde{\pi})$ is strongly cyclic for both π and $\tilde{\pi}$; hence $D(\pi) = \mathrm{Cl}_{t_A}[\pi(A)\xi] = \mathrm{Cl}_{t_X}[\tilde{\pi}(X)\xi]$.

Convention. For typographical convenience, from now on, we denote $\tilde{\pi}$ by π itself; and the context makes it clear whether it is a representation of A or of X.

Let $\rho(x) = \overline{\pi(x)}$ $(x \in X(B_0))$. We note that ρ: $(X(B_0), \|\cdot\|_{B_0}) \to B(H)$ is a continuous $*$-homomorphism. (Here $B(H)$ is the C^*-algebra of all bounded linear operators on H with the operator norm.) Indeed, let $x \in X(B_0)$, $\xi \neq 0$ in $D(\pi)$. Then for each $y \in X$,

$$\begin{aligned} \|\pi(x)\pi(y)\xi\|^2 &= \langle \pi(y^*x^*xy)\xi, \xi\rangle w_\xi(y^*x^*xy) \\ &\leqslant \|x\|_{B_0}^2 w_\xi(y^*y) = \|x\|_{B_0}^2\|\pi(y)\xi\|^2. \end{aligned}$$

Since ξ is strongly cyclic, it is cyclic, i.e. $[\pi(X)\xi]$ is norm dense in H. It follows that $D(\rho(x)) = H$ and $\|\rho(x)\| \leqslant \|x\|_{B_0}$.

Now let $X(B_0)^{\tilde{}}$ be the C^*-algebra obtained by completing $(X(B_0), \|\cdot\|_{B_0})$. Let σ: $X(B_0)^{\tilde{}} \to B(H)$ be the $*$-homomorphism that is the unique extension of ρ satisfying $\|\sigma(x)\| \leqslant \|x\|_{B_0}$ $(x \in X(B_0))$. Then the following hold.

Statement (I). ρ, and hence σ, is topologically irreducible (in the sense of usual C^*-representation theory).

Indeed, let H_1 be a norm closed subspace of H such that $H_1 \neq (0)$, $H_1 \neq H$, $\rho(X(B_0))H_1 \subset H_1$. Let

$$D_1 = \{\xi \in D(\pi) | \rho(x)\xi \in H_1 \text{ for all } x \in X(B_0)\}.$$

Then $\rho(X(B_0))D_1 \subset D_1$. We show that $\pi(X)D_1 \subset D_1$, and for this, it is sufficient to show that $\pi(\operatorname{sym} X)D_1 \subset D_1$ where $\operatorname{sym} X = \{h \in X | h = h^*\}$.

Let $\xi \in D_1$, $h \in \operatorname{sym} X$. Then $\pi(h)\xi \in D_1$ if for all $y \in X(B_0)$ (or equivalently, for all $y \in \operatorname{sym} X(B_0)$), $\rho(y)\pi(h)\xi = \pi(yh)\xi \in H_1$. For $n = 1, 2, 3, \ldots$, taking $h_n = h(1 + h^2/n)^{-1}$, Lemma (2.5) implies that $h_n \in X(B_0)$ and $h_n \to h$ in τ where

$\tau = \sigma_P$ or $\tau(X, X^P)$. Now

$$h - h_n = \frac{1}{n} h^3 \left(1 + \frac{1}{n} h^2\right)^{-1}$$

gives

$$(h - h_n)^* y^* y (h - h_n) = \frac{1}{n^2} h^3 \left(1 + \frac{1}{n} h^2\right)^{-1} y^2 h^3 \left(1 + \frac{1}{n} h^2\right)^{-1}.$$

Hence

$$\begin{aligned} \|\pi(yh)\xi - \pi(yh_n)\xi\|^2 &= \left\langle \pi\left((h - h_n) y^2 (h - h_n)\right)\xi, \xi \right\rangle \\ &= w_\xi\left((h - h_n) y^2 (h - h_n)\right) \\ &= \frac{1}{n^2} w_\xi \left(h^3 \left(1 + \frac{1}{n} h^2\right)^{-1} y^2 \left(1 + \frac{1}{n} h^2\right)^{-1} h^3 \right) \\ &\leqslant \frac{1}{n^2} \|y^2\|_{B_0} w_\xi \left(h^6 \left(1 + \frac{1}{n} h^2\right)^{-2} \right) \\ &\leqslant \frac{1}{n^2} \|y\|_{B_0}^2 \left\| \left(1 + \frac{1}{n} h^2\right)^{-2} \right\|_{B_0} w_\xi(h^6) \\ &\leqslant \frac{1}{n^2} \|y\|_{B_0}^2 w_\xi(h^6) \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

But $\pi(yh_n)\xi \in H_1$ and H_1 is norm closed. Therefore $\pi(yh)\xi \in H_1$. Thus $\pi(X)D_1 \subset D_1$.

Further D_1 is a closed subspace of $(D(\pi), t_X)$. Let $\xi \in D(\pi)$ be such that for some net $(\xi_\alpha) \subset D_1$, $\xi_\alpha \to \xi$ in t_X. Then for all $x \in X$, $\|\pi(x)(\xi_\alpha - \xi)\| \to 0$. But as above, for all such x, and in particular for $x \in X(B_0)$, $\pi(x)\xi_\alpha \in D_1 \subset H_1$ and hence, since H_1 is norm closed, $\pi(x)\xi \in H_1$ for all $x \in X(B_0)$ showing that $\xi \in D_1$.

Now since π is selfadjoint, D_1 is a selfadjoint π-invariant subspace of of $D(\pi)$ as in [**15**, Theorem 4.7]. Hence by the hypothesis, $D_1 = (0)$ or $D_1 = D(\pi)$. This gives respectively $H_1 = (0)$ or $H_1 = H$. Thus σ is topologically irreducible. This gives Statement (I).

STATEMENT (II). $\sigma(X(B_0)^{\tilde{}})D(\pi) \subset D(\pi)$.

It is enough to show that $\sigma(\operatorname{sym}(X(B_0)^{\tilde{}}))D(\pi) \subset D(\pi)$. Let $\xi \in D(\pi)$, $h \in \operatorname{sym} X(B_0)^{\tilde{}}$. Let a sequence (h_n) in sym $X(B_0)$ be such that $\|h_n - h\|_{B_0} \to 0$. Then in view of the facts

(a) $\pi(X(B_0))D(\pi) \subset D(\pi)$ and

(b) π is closed (so that $(D(\pi), t_X)$ is complete)

to conclude that $\pi(h)\xi \in D(\pi)$, it is sufficient to show that $(\pi(h_n)\xi)$ is Cauchy in $(D(\pi), t_X)$, i.e. for each $y \in X$ (or equivalently, for each $y \in \operatorname{sym} X$),

(A) $$\|\pi(y)\pi(h_n - h_m)\xi\| \to 0 \quad \text{as } n, m \to \infty.$$

Let $y \in \operatorname{sym} X$. By Lemma (2.5), $y_k = y(1 + y^2/k)^{-1} \to y$ in σ_P. Now as in the previous case,

$$\begin{aligned}
&\|\pi(y(h_n - h_m))\xi - \pi(y_k(h_n - h_m))\xi\|^2 \\
&\quad = \|\pi((y - y_k)(h_n - h_m))\xi\|^2 \\
&\quad = w_\xi\left(\frac{1}{k^2}(h_n - h_m)y^6\left(1 + \frac{1}{k}y^2\right)^{-2}(h_n - h_m)\right) \\
&\quad \leqslant \frac{1}{k^2}\left\|\left(1 + \frac{1}{k}y^2\right)^{-2}\right\|_{B_0} w_\xi((h_n - h_m)y^6(h_n - h_m)) \\
&\quad \leqslant \frac{1}{k^2} f_{h_n - h_m}(y^6),
\end{aligned}$$

where

(B) $$f_{h_n - h_m}(x) = w_\xi((h_n - h_m)x(h_n - h_m)).$$

Now $a_{nm} = h_n - h_m \to 0$ as n, $m \to \infty$ in the Mackey topology $\tau = \tau(X, X^P)$. Hence for each $u \in X$, $(L_{a_{nm}} f)(u) = f(a_{nm}u) \to 0$, $(R_{a_{nm}} f)(u) = f(ua_{nm}) \to 0$ uniformly over f in $\sigma(X^P, X)$ compact convex circled subsets of X^P. Also, $\{L_{a_{nm}} w_\xi | n, m = 1, 2, \ldots\}$ is contained in $\sigma(X^P, X)$ compact convex circled set B, and $(R_{a_{nm}} \varnothing)(u) \to 0$ uniformly over $\varnothing \in B$. It follows that for $\xi \in D(\pi)$, $u \in X$, there is a constant $M(u, \xi)$ independent of n, m such that

$$|f_{h_n - h_m}(u)| = |\langle \pi((h_n - h_m)u(h_n - h_m))\xi, \xi\rangle| \leqslant M(u, \xi).$$

This, in particular in (B), gives

$$\|\pi(y)\pi(h_n - h_m)\xi - \pi(y_k)\pi(h_n - h_m)\xi\|^2 \leqslant \frac{1}{k^2}M(y, \xi) \to 0$$

uniformly over n and m as $k \to \infty$. This permits, in view of [**2**, Theroem 13.2], the interchange of limits in the following arguments.

$$\begin{aligned}
&\lim_{(n,m) \to \infty} \|\pi(y)\pi(h_n - h_m)\xi\| \\
&\quad = \lim_{(n,m) \to \infty} \lim_{k \to \infty} \|\pi(y_k)\pi(h_n - h_m)\xi\| \\
&\quad = \lim_{k \to \infty} \lim_{(n,m) \to \infty} \|\pi(y_k)\pi(h_n - h_m)\xi\| \\
&\quad = 0 \quad \text{since } y_k \in X(B_0),\ \|y_k(h_n - h_m)\| \to 0 \text{ as } n, m \to \infty.
\end{aligned}$$

This gives (A), thereby completing the proof of Statement (II).

Returning to the proof of the Theorem, a well-known result of Kadison [**7**, Corollary 1.12.17] implies that a topologically irreducible representation of a C^*-algebra is algebraically irreducible. Thus in view of Statements (I) and (II), for each nonzero $\xi \in D(\pi)$, $H = \sigma(X(B_0)^\sim)\xi \subset D(\pi)$, $D(\pi) = H$. The closed graph theorem implies that $\pi(x)$ is a bounded operator for each $x \in X$; and hence so is $\pi(a)$ for each $a \in A$.

This completes the proof of the Theorem.

4. A concluding remark. Let $t \to H(t)$ be a measurable field of Hilbert spaces over a compact space Z with a positive measure μ. Let $H = \int_Z^{\oplus} H(t)\,d\mu(t)$. Given a measurable field of operators $t \to T(t)$, not necessarily bounded, over Z, let

$$D(t) = \text{set of all square integrable vector fields } t \to x(t) \in H(t) \text{ such that } x(t) \in D(T(t)) \text{ for all } t \text{ and } t \to T(t)x(t) \text{ is square integrable.}$$

Then the operator T, defined on $D(T)$ by the field $t \to T(t)$, is called *decomposable*, written $T = \int_Z^{\oplus} T(t)\,d\mu(t)$. It is called *boundedly decomposable* if each $T(t)$ is bounded. It is easily seen that the set bA of all boundedly decomposable operators forms a $*$-subalgebra of the $*$-algebra (with strong operations) A of all decomposable operators containing the $*$-algebra A_b of all bounded decomposable operators.

Mathot [**14**, Theorem 3.2 and §3.3] has proved that if A is a separable locally convex $*$-algebra dominated in a given selfadjoint strongly continuous representation $(\pi, D(\pi), H)$ by a countable subset B (in the sense that given an $a \in A$, there are $b \in B$ and $k < \infty$ such that $\|\pi(a)x\| \leqslant k\|\pi(b)x\|$ for all x), then over a compact space Z with a positive measure μ, $(\pi, D(\pi), H)$ can be disintegrated as

$$H = \int_Z^{\oplus} H(t)\,d\mu(t), \quad D(\pi) = \int_Z^{\oplus} D(t)\,d\mu(t), \quad \pi = \int_Z^{\oplus} \pi_t\,d\mu(t)$$

strongly, where each π_t is irreducible with $D(\pi_t) = D(t)$. If A is symmetric, then every closed π is selfadjoint and, by our theorem, each π_t is a bounded representation. In particular, if A is a countably dominated symmetric $*$-algebra [**12**] of operators in a separable Hilbert space H admitting a separable locally convex $*$-algebra topology finer than the strong topology, then A is isomorphic to a $*$-subalgebra of the algebra of boundedly decomposable operators in some $\int_Z^{\oplus} H(t)\,d\mu(t)$.

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