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AUTOMATIC CONTINUITY OF HOMOMORPHISMS IN TOPOLOGICAL ALGEBRAS

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ABSTRACT. A homomorphism from a locally convex Q-algebra to a uniform topological algebra is continuous. A one-to-one homomorphism from a regular complete spectrally bounded uniform topological algebra onto a dense subalgebra of a semisimple locally m-convex Q-algebra is open. Examples are discussed to show that none of the assumptions in these results can be omitted.

1. PRELIMINARIES AND NOTATION

A *uniform seminorm* on a linear associative algebra A (over complex scalars) is a seminorm p satisfying (i) $p(xy) \leq p(x)p(y)$ for all x, y, and (ii) $p(x^2) = p(x)^2$ for all x. A (*locally convex*) *topological algebra* [8] is an algebra A with a Hausdorff topology t on it so that (A, t) is a (locally convex) topological vector space in which the multiplication is separately continuous. It is a *Q-algebra* [9, Appendix E] if the set of all quasi-regular elements is an open set. A *locally m-convex algebra* (*lmc algebra*) [9] is a locally convex topological algebra whose topology is determined by a separating family $P = (p_\alpha)$ of seminorms each satisfying (i). For each α, let $N_\alpha = \{x \in A | p_\alpha(x) = 0\}$ and A_α be the Banach algebra obtained by completing A/N_α in the norm $\|x_\alpha\|_\alpha = p_\alpha(x)$, $x_\alpha = x + N_\alpha$. If A is complete, then A is an inverse limit of Banach algebras $A = \varprojlim A_\alpha$ [9, Theorem 5.1]. A *uniform topological algebra* (*uT-algebra*) [3] is an lmc algebra A in which each p_α additionally satisfies (ii) so each A_α is a uniform Banach algebra. A *uniform Banach algebra* (*uB-algebra*) is a Banach algebra $(A, \|\cdot\|)$ such that $\|x^2\| = \|x\|^2$ for all x. By [4, Theorem 3.10, p. 32], a uB-algebra is commutative; via Gelfand theory, it is a closed point separating subalgebra of the supnorm Banach algebra $C(X)$ of all continuous complex-valued functions on a compact Hausdorff space X. Thus, a uT-algebra A is commutative, and if complete, A is an inverse limit of uB-algebras. An algebra A is *spectrally bounded* (*sb*) if for each $x \in A$ its spectrum $\mathrm{sp}_A(x)$ in A is a bounded subset of the complex plane. Throughout, $r(x)$ $(= r_A(x))$ denotes the spectral radius of x in A. The *bounded part* of a uT-algebra A is the

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subalgebra $b(A) = \{x \in A | \sup_\alpha p_\alpha(x) < \infty\}$. If A is complete, then $b(A)$ is a uB-algebra with norm $\|x\|_\infty = \sup_\alpha p_\alpha(x)$ continuously embedded in A [3]. For a commutative lmc algebra A, let $\sigma(A)$ denote the Gelfand space of A consisting of all nonzero continuous multiplicative linear functionals with the relative weak* topology. A is *semisimple* if, for any $x \in A$, $f(x) = 0$ for all $f \in \sigma(A)$ implies that $x = 0$. As in Banach algebras [8, Chapter 7], A is *regular* if given a closed subset $F \subset \sigma(A)$ and $f \in \sigma(A)$, $f \notin F$, there exists an $x \in A$ such that $\hat{x}|_F = 0$ and $\hat{x}(f) \neq 0$, $\hat{x}: \sigma(A) \to \mathbb{C}$, $\hat{x}(g) = g(x)$, is the Gelfand transform of x.

2. Main results

Theorem 2.1. *Let A be an sb algebra, B a uT-algebra, and $\phi: A \to B$ a homomorphism. Then $\phi(A) \subset b(B)$, and, for each continuous uniform seminorm q on B, $q(\phi(x)) \leq r(x)$ for all x in A. In particular, if A is a locally convex Q-algebra, then ϕ is continuous.*

Theorem 2.2. *Let A be an sb, regular, complete, uT-algebra and B be an lmc algebra. Let $\phi: A \to B$ be a one-to-one homomorphism such that $\overline{\mathrm{Im}(\phi)}$ is a semisimple Q-algebra. Then $\phi^{-1}|_{\mathrm{Im}(\phi)}$ is continuous.*

Note that, in the theorems, in the absence of metrizability and completeness, automatic continuity is guaranteed by a ring theoretic condition of topological nature. In §3 we discuss several examples exhibiting that various assumptions in the theorems cannot be omitted.

Proof of Theorem 2.1. We can assume B to be complete since it is easy to verify that the completion of a uT-algebra is a uT-algebra. Also, the topology of B is determined by the collection $S(B)$ of all continuous uniform seminorms on B; thus, $B = \varprojlim_{q \in S(B)} B_q$, where B_q is the uB-algebra obtained by completing B/N_q $(N_q = \{x \in B | q(x) = 0\})$ in the norm $\|y_q\|_q = q(y)$, $y_q = y + N_q$. Then $b(B) = \{y \in B | \mathrm{sp}_B(y)$ is bounded$\}$. Indeed, B being complete and lmc, [9, Corollary 5.3] implies that, for each $y \in B$, $\mathrm{sp}_B(y) = \bigcup\{\mathrm{sp}_{B_q}(y_q) | q \in S(B)\}$ and $r_B(y) = \sup_{q \in S(B)} \limsup_{n\to\infty} q(y^n)^{1/n} = \sup_{q \in S(B)} q(y)$ in view of $q(y^2) = q(y)^2$. Now let $x \in A$, $q \in S(B)$. Since $\mathrm{sp}_A(x) \supset \mathrm{sp}_B(\phi(x))$, it follows that $q(\phi(x)) = \|(\phi(x))_q\|_q = r_{B_q}(\phi(x)_q) \leq r_B(\phi(x)) \leq r_A(x) < \infty$; moreover, $\phi(A) \subset b(B)$. Further, assume A to be a Q-algebra (so that it is sb by [9, Lemma E3]) which is also locally convex. By [9, Proposition 13.5] $s(A) = \{x \in A | r(x) < 1\}$ is a neighbourhood of 0; hence there exists a convex, balanced, open set $U \subset A$ such that $0 \in U \subset s(A)$. Let $p = p_U$ be the Minkowski functional of U in A; it is a continuous seminorm. As in the proof of [10, Theorem 1.36], $U = \{x \in A | p(x) < 1\}$. For $x \in A$, $\delta > 0$, $p(y) < 1$, where $y = x/(p(x) + \delta)$. Thus $y \in s(A)$; hence, $r_A(x) < p(x) + \delta$. This gives $q(\phi(x)) \leq r_A(x) = \sup_{q \in S(B)} q(\phi(x)) \leq p(x)$, showing that ϕ is continuous.

Remark. Since $q \in S(B)$ is arbitrary, it follows that

$$\|\phi(x)\|_\infty = \sup_{q \in S(B)} q(\phi(x)) \leq p(x) \qquad (x \in A),$$

giving the stronger assertion that $\phi: A \to (b(B), \|\cdot\|_\infty)$ is continuous.

Proof of Theorem 2.2. It follows by the description of the bounded part of a uT-algebra in the first proof that $A = b(A)$ as sets. We can assume, without loss of generality, that A possesses an identity 1; thus, $\sigma(A)$ is a compact Hausdorff space. As $\overline{\mathrm{Im}(\sigma)}$ is a (commutative) lmc Q-algebra, the inversion is continuous; hence, [7, Proposition 1.6, p. 168] implies that $\sigma(\overline{\mathrm{Im}(\phi)}) = \sigma(\mathrm{Im}(\sigma))$ is also a compact Hausdorff space. Let $C = \overline{\mathrm{Im}(\phi)}$. Now the adjoint map $\phi^*\colon \sigma(C) \to \sigma(A)$, $\phi^*(f) = f \circ \phi$ is continuous; hence, $F = \phi^*(\sigma(C))$ is closed in $\sigma(A)$. In fact, ϕ^* is surjective because if not, regularity of A implies that there exists an $x \in A$, $x \neq 0$, such that $\hat{x}|_F = 0$; thus, $\phi^*(f)(x) = f(\phi(x)) = 0$ for all $f \in \sigma(C)$. Since C is semisimple, $\phi(x) = 0$, contradicting that ϕ is one-to-one. Thus $F = \sigma(A)$. Then by [9, Corollary 5.6], for any $x \in A$, $\mathrm{sp}_A(x) = \{f(x) | f \in \sigma(A)\} = \{f(\phi(x)) | f \in \sigma(C)\} = \mathrm{sp}_C(\phi(x))$. Now C being an lmc Q-algebra, there exists a continuous seminorm q on C such that $r_C(\phi(x)) \leq q(\phi(x))$ $(x \in A)$ [9, Proposition 13.5]. Then for any $p \in S(A)$, $x \in A$, $p(x) = \|x_p\|_p = r_{A_p}(x) \leq r_A(x) = r_C(\phi(x)) \leq q(\phi(x))$, showing that $\phi^{-1}|_{(\mathrm{Im}(\phi))}$ is continuous.

Corollary 2.3. *Let A be a unital locally convex Q-algebra. Then every uniform seminorm p on A is continuous. Further, if the inversion in A is continuous, then $p(x) \leq r(x)$ for all x.*

Proof. The quotient map $\phi\colon A \to (A_p, \|\cdot\|_p)$, $\phi(x) = x_p$, is continuous by Theorem 2.1, and there exists a continuous seminorm q on A such that, for all $x \in A$, $p(x) = \|\phi(x)\|_p \leq q(x)$. In fact, $p(x) = \lim_n p(x^{2^n})^{1/2^n} \leq \sup \limsup_{n\to\infty} q(x^{2^n})^{1/2^n} = \beta(x)$ [1, Theorem 3.12], where $\beta(x)$ is the radius of boundedness in the sense of [1], but $\beta(x) \leq r(x)$ again by [1, Theorem 3.12]. (The assumption that the inversion map on A is continuous is required [1, Theorem 4.1] for the equality of the spectral radii for the usual spectrum considered here and the spectrum considered in [1].)

Corollary 2.4. *For a compact Hausdorff space X, let $C(X)$ denote the Banach algebra with supnorm $\|f\|_\infty = \sup\{|f(x)| \,|\, x \in X\}$ of all continuous complex-valued functions on X. Let $|\cdot|$ be any norm on $C(X)$ such that $(C(X), |\cdot|)$ is a normed linear space (not necessarily complete) satisfying $|f^2| = |f|^2$ for all f in $C(X)$. Then $|\cdot| = \|\cdot\|_\infty$.*

Proof. By [3], $|\cdot|$ satisfies $|fg| \leq |f|\,|g|$ for all f, g; hence, $(C(X), |\cdot|)$ is a normed algebra with the result $\|\cdot\|_\infty \leq |\cdot|$ [11, Theorem 1.2.4]. By Corollary 2.3, $|\cdot| \leq r(\cdot) = \|\cdot\|_\infty$.

Corollary 2.5. *Let A be an sb algebra, B be a complete barreled uT-algebra, and $\phi\colon A \to B$ be a surjective homomorphism. Then the topology of B is normable.*

Proof. By Theorem 2.1, $B = \phi(A) = b(B)$, and the conclusion follows by applying the open mapping theorem to the identity map $i\colon (b(B), \|\cdot\|_\infty) \to B$.

3. Remarks

(3.1) In Theorem 2.1 the assumption that A is a Q-algebra cannot be omitted, even if A is a complete uT-algebra and B is a uB-algebra. Consider the complete uT-algebra $(C[0, 1], \tau)$ with the topology τ of uniform convergence on all countable compact subsets of $[0, 1]$. It is not a Q-algebra, because the topology of a complete uT-algebra, which is a Q-algebra, has to be normable [3, Theorem 2], and τ fails to be normable. The identity map $i\colon (C[0,1],\tau) \to (C[0,1], \|\cdot\|_\infty)$ is not continuous.

(3.2) In Corollary 2.5 the assumption that B be barreled cannot be omitted. This is seen by considering the identity map $i\colon (C[0,1], \|\cdot\|_\infty) \to (C[0,1],\tau)$, where τ is as in (3.1).

(3.3) In Theorem 2.1 the hypothesis that B is a uT-algebra cannot be omitted, even if A is a Q-normed algebra and B is a complete metrizable lmc Q-algebra. Take $A = (C^\infty[0,1], \|\cdot\|_\infty)$, the algebra of all C^∞-functions on $[0,1]$ with the supnorm $\|\cdot\|_\infty$. For any $f \in A$, $\mathrm{sp}_A(f) = \text{range of } f$; hence, $r_A(f) = \|f\|_\infty = \limsup_{n\to\infty} \|x^{2^n}\|_\infty^{1/2^n} = \limsup_{n\to\infty} \|f^n\|_\infty^{1/n}$, so by [2, Proposition 15], the normed algebra A is a Q-algebra. Let $B = (C^\infty[0,1], t)$, a complete lmc algebra with topology t defined by submultiplicative norms

$$p_n(f) = \sup_{0\le t\le 1} \left[\sum_0^n \frac{|f^{(k)}(t)|}{k!}\right];$$

it is a Q-algebra [9, Appendix E]. The identity map $i\colon A \to B$ is not continuous.

(3.4) The topological algebra B in (3.3) is not a uT-algebra, for otherwise, B being a Q-algebra, the topology t has to be normable [3, Theorem 2], with the result that B has to be a Banach algebra. On the other hand, the algebra B fails to be a Banach algebra under any norm, as a semisimple commutative Banach algebra is known not to admit a nonzero derivation. Thus, the discontinuity of the identity map $i\colon A \to B$ in (3.3) also shows that in Corollary 2.5 the assumption that B is a uT-algebra cannot be omitted.

(3.5) In Corollary 2.4 the square property $|f^2| = |f|^2$ $(f \in C(X))$ of the norm $|\cdot|$ cannot be omitted (or weakened to square inequality). By [5], for an infinite compact Hausdorff space X, there exists a norm on $C(X)$, distinct from $\|\cdot\|_\infty$ and not equivalent to it, making $C(X)$ an incomplete normed algebra. Also,

$$|f| = \sup\left\{\frac{|f(s)+f(t)|}{2} + \frac{|f(s)-f(t)|}{2} \,\middle|\, s, t \text{ in } X\right\}$$

defines a norm on $C(X)$, equivalent to $\|\cdot\|_\infty$ but distinct from $\|\cdot\|_\infty$, making $C(X)$ a Banach algebra satisfying $\|\cdot\|_\infty \le |\cdot| \le 2\|\cdot\|_\infty$ [4, Example 7.5, p. 70] and hence satisfying the square inequality $\frac{1}{4}|f|^2 \le |f^2| \le |f|^2$ for all $f \in C(X)$.

(3.6) Corollary 2.4 does not hold for uniformly closed nonselfadjoint subalgebras of $C(X)$. On the supnorm disc algebra $A(D)$ of all those continuous functions on the closed unit disc D in the complex plane that are analytic in the interior of D, $|f|_r = \sup\{|f(z)| \,|\, 0 < |z| \le r\}$, $0 < r < 1$, define uniform norms distinct from the supnorm $\|\cdot\|_\infty$, satisfying $|\cdot|_r \le \|\cdot\|_\infty$. However, it follows [3] that if $|\cdot|$ is a uniform norm on a uB-algebra $(A, \|\cdot\|)$ such that either $(A, \|\cdot\|)$ is regular or $(A, |\cdot|)$ is a Q-algebra, then $|\cdot| = \|\cdot\|_\infty$.

(3.7) In Theorem 2.2 the hypothesis that A is regular cannot be omitted, even if A and B are uB-algebras. Take A to be the supnorm disc algebra $A(D)$ as in (3.6). Let $0 < r < 1$, $\Gamma = \{z \in D \,|\, |z| = r\}$, $B = (C(\Gamma), \|\cdot\|_\infty)$, and $\phi\colon A \to B$ be $\phi(f) = f|_\Gamma$. The algebra A is not regular [8, §7.2, p. 167], and ϕ^{-1} fails to be continuous.

(3.8) Let us note that Theorems 2.1 and 2.2 are uT-algebra analogues of a couple of automatic continuity results for *-homomorphisms between LMC*-algebras proved in [6].

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