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BANACH ALGEBRAS IN WHICH EVERY ELEMENT IS A TOPOLOGICAL ZERO DIVISOR

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ABSTRACT. Every element of a complex Banach algebra $(A, \|\cdot\|)$ is a topological divisor of zero, if at least one of the following holds: (i) A is infinite dimensional and admits an orthogonal basis, (ii) A is a nonunital uniform Banach algebra in which the Silov boundary ∂A coincides with the Gelfand space $\Delta(A)$; and (iii) A is a nonunital hermitian Banach *-algebra with continuous involution. Several algebras of analysis have this property. Examples are discussed to show that (a) neither hermiticity nor $\partial A = \Delta(A)$ can be omitted, and that (b) in case (ii), $\partial A = \Delta(A)$ is not a necessary condition.

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Theorem. *Every element of a complex Banach algebra* $(A, \|\cdot\|)$ *is a topological divisor of zero* (*TDZ*), *if at least one of the following holds*:

(i) A *is infinite dimensional and admits an orthogonal basis.*

(ii) A *is a nonunital uniform Banach algebra* (*uB-algebra*) *in which the Silov boundary* ∂A *coincides with the carrier space* (*the Gelfand space*) $\Delta(A)$ (*in particular,* A *is a nonunital regular uB-algebra*).

(iii) A *is a nonunital hermitian Banach*-algebra with continuous involution* (*in particular,* A *is a nonunital* C^**-algebra*).

An element x in a Banach algebra A is a TDZ if there exists a sequence (x_n), $\|x_n\| = 1$, for $n = 1, 2, \ldots$, in A such that either $x_n x \to 0$ or $x x_n \to 0$. An orthogonal basis [1, 3] in A is a sequence (e_n) in A such that: (i) each $x \in A$ can be expressed as $x = \sum \alpha_n e_n$, α_n's are scalars; and (ii) $e_m e_n = \delta_{mn} e_n$, δ_{mn} being the Kronecker delta. If (e_n) is an orthogonal basis in A, then (e_n) is a Schauder basis [1] and A is semisimple, commutative, and nonunital [3]. A is a uB-algebra if $\|x^2\| = \|x\|^2$ $(x \in A)$. Such a Banach algebra A is commutative and semisimple [2]. A hermitian Banach *-algebra [2, 5] is a Banach *-algebra in which each $h = h^*$ has real spectrum. The above theorem supplements the well-known result [5, Theorem 2.3.5, p. 57] that every element in a radical Banach algebra is a (two-sided) TDZ.

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Proof of the Theorem. (i) Let (e_n) be an orthogonal basis in A. Let $x \in A$, $x = \sum \alpha_n e_n$. Since $e_m e_n = \delta_{mn} e_n$ for all m, n, it follows that for any k, $\|e_k\| = \|e_k^2\| \leq \|e_k\|^2$, and hence $\|e_k\| \geq 1$; and $xe_k = (\sum \alpha_n e_n)e_k = \sum \alpha_n e_n e_k = \alpha_k e_k \to 0$ as $k \to \infty$. Letting $f_k = e_k/\|e_k\|$, $\|f_k\| = 1$, and $\|xf_k\| \leq \|xe_k\| \to 0$.

(ii) Let $x \in A$, $\varepsilon > 0$. Since A is nonunital, $\Delta(A)$ is a noncompact locally compact Hausdorff space and the Gelfand transform $\hat{x}$ of x vanishes at infinity. Hence, there exists a complex homomorphism $f \in \Delta(A)$ such that $|f(x)| < \varepsilon/2$. Let $U = \{g \in \Delta(A)\colon |g(x) - f(x)| < \varepsilon/2\}$, a neighborhood of f in the Gelfand topology on $\Delta(A)$. For $g \in U$, $|g(x)| \leq |g(x) - f(x)| + |f(x)| < \varepsilon$. Also, since A is a uB-algebra, for any $y \in A$, $\|y\| = \|\hat{y}\|_\infty = \sup\{|g(y)|\colon g \in \Delta(A)\}$. Since $f \in \Delta(A) = \partial A$, [4, Corollary 9.2.2, p. 225] implies that there exists $y \in A$ such that $\|y\| = \|\hat{y}\|_\infty = 1$ and, for $g \in \Delta(A)\backslash U$, $|g(y)| < \varepsilon/\|x\|$. It follows that $\|xy\| = \|\hat{x}\hat{y}\|_\infty \leq \varepsilon$. Hence there exists a sequence (y_n), $\|y_n\| = 1$, in A, such that $xy_n \to 0$, and the proof of (ii) is complete. In a regular commutative Banach algebra A, $\partial A = \Delta(A)$ [4, Theorem 9.2.3, p. 227].

(iii) We can take $\|x^*\| = \|x\|$ for all $x \in A$. Let $h \in A$, $h = h^*$. Then, for any $n \in \mathbb{N}$, inh is quasiregular. Hence [5, Corollary 1.5.10, p. 25] implies that h is a TDZ in A. It follows from this that for any $x \in A$, x^*x is a TDZ. We may assume that x^*x is a left TDZ. Hence by [5, Lemma 1.5.1 (ii), p. 20], either x^* or x is a left TDZ. By continuity of the involution, it follows that x is a TDZ.

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Examples. (3.1) For the unit circle T, the Lebesgue space $L^p(T)$, $1 < p < \infty$, is a convolution Banach algebra with orthogonal basis $e_n(z) = z^n$ $(n \in \mathbb{N})$ [3]. This is because, for each $f \in L^p(T)$, the Fourier series of f converges to f in the norm of $L^p(T)$. The Banach sequence algebras c_0, l^p $(1 \leq p < \infty)$, with pointwise multiplication, have $e_n = (\delta_{nm})_{m=1}^\infty$ as orthogonal basis. The Hardy space $H^p(U)$ $(1 < p < \infty)$ on an open unit disc U is a Banach algebra with Hadamard product

$$f * g(z) = \frac{1}{2\pi i} \int_{|u|=r} f(u)g(zu^{-1})u^{-1}\,du, \qquad |z| < r < 1,\ z \in U.$$

It has orthogonal basis $e_n(z) = z^n$ $(n \in \mathbb{N})$ [3]. In all these algebras, every element is TDZ.

(3.2) For a locally compact nondiscrete abelian group G, the convolution Banach algebra $L^1(G)$ is a nonunital hermitian Banach *-algebra with involution $f^*(t) = \overline{f(-t)}$. Thus every element of $L^1(G)$ is TDZ. For $G = T$ (the unit circle), the subspaces $C(T)$ (continuous functions) and $C^m(T)$ (C^m-functions), $1 \leq m < \infty$, of $L^1(T)$ are convolution Banach algebras with respective norms

$$\|f\|_\infty = \sup\{|f(z)|\colon z \in T\} \quad \text{and} \quad \|f\|_m = \sup_{z \in T} \sum_{k=0}^{m} \frac{|f^k(z)|}{k!}.$$

In fact, $C(T)$ and each $C^m(T)$ are ideals in $L^1(T)$; hence, they are spectrally invariant in $L^1(T)$. Thus $C(T)$ and $C^m(T)$ are hermitian algebras and each of their elements is TDZ.

(3.3) Let A be a normed algebra with completion $\overline{A}$. If every element of A is TDZ, then so is in $\overline{A}$, because TDZs in A form a closed set. Let $\mathscr{F}$ be the *-algebra of all finite rank operators on an infinite-dimensional Hilbert space H. By [5, p. 279], every element of $\mathscr{F}$ is a zero divisor in $\mathscr{F}$. Hence it follows that in the Banach algebras $(C^p(H), \|\cdot\|_p)$, $1 \leq p < \infty$, of operators of Schatten class C^p, every element is TDZ.

(3.4) In the above Theorem, neither the condition $\partial A = \Delta(A)$ in (ii) nor the hermiticity condition in (iii) can be omitted. Let $A = A(D)$, the supnorm disc algebra of the closed unit disc D, viz., the algebra of all continuous functions on D that are analytic in the interior of D, with the involution $f^*(z) = \overline{f(\overline{z})}$ $(f \in A)$. Let $I = \{f \in A: f(0) = 0\}$, a closed ideal in A. Note that I (as well as $A(D)$) is not hermitian and $\partial I \neq \Delta(I)$. The function $f(z) = z$, $z \in D$, in I is not TDZ in I.

(3.5) There exists a uB-algebra A in which every element is TDZ but $\partial A \neq \Delta(A)$. For $1 < r < R$, define $U_R = \{z \in C: |z| < R\}$, $\overline{U}_R =$ the closure of U_R, $A = \{f \in C(\overline{U}_R): f$ is analytic in open unit disc$\}$, and $B = \{f \in A: f(z) = 0,$ $r \leq |z| \leq R\}$. Then B is a nonunital uB-algebra in which [4, Corollary 9.5.1, p. 246] implies that every element of B is TDZ. But $\partial B = \{z: 1 \leq |z| < r\}$ and $\Delta(B) = \{z: |z| < r\}$.

(3.6) Let T be a bijective bounded linear operator on an infinite-dimensional Hilbert space H. Let $C^*(T) =$ the operator norm closure of polynomials (without constant terms) in T and T^*. It follows from (iii) of the above Theorem that the identity operator belongs to $C^*(T)$.

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