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Enveloping σ-C^*-algebra of a smooth Frechet algebra crossed product by $\mathbb{R}$, K-theory and differential structure in C^*-algebras

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Abstract. Given an m-tempered strongly continuous action α of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra A, it is shown that the enveloping σ-C^*-algebra $E(S(\mathbb{R}, A^\infty, \alpha))$ of the smooth Schwartz crossed product $S(\mathbb{R}, A^\infty, \alpha)$ of the Frechet algebra A^∞ of C^∞-elements of A is isomorphic to the σ-C^*-crossed product $C^*(\mathbb{R}, E(A), \alpha)$ of the enveloping σ-C^*-algebra $E(A)$ of A by the induced action. When A is a hermitian Q-algebra, one gets K-theory isomorphism $RK_*(S(\mathbb{R}, A^\infty, \alpha)) = K_*(C^*(\mathbb{R}, E(A), \alpha)$ for the representable K-theory of Frechet algebras. An application to the differential structure of a C^*-algebra defined by densely defined differential seminorms is given.

Keywords. Frechet *-algebra; enveloping σ-C^*-algebra; smooth crossed product; m-tempered action; K-theory; differential structure in C^*-algebras.

1. Introduction

Given a strongly continuous action α of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra A, several crossed product Frechet algebras can be constructed [11,14]. They include the smooth Schwartz crossed product $S(\mathbb{R}, A, \alpha)$, the L^1-crossed products $L^1(\mathbb{R}, A, \alpha)$ and $L^1_{|\cdot|}(\mathbb{R}, A, \alpha)$, and the σ-C^*-crossed product $C^*(\mathbb{R}, A, \alpha)$. Let $E(A)$ denote the enveloping σ-C^*-algebra of A [1,6]; and (A^∞, τ) denote the Frechet *-algebra consisting of all C^∞-elements of A with the C^∞-topology τ ([14], Appendix I). The following theorem shows that for a smooth action, the eveloping algebra of smooth crossed product is the continuous crossed product of the enveloping algebra.

Theorem 1. *Let α be an m-tempered strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra A. Let A admit a bounded approximate identity which is contained in A^∞ and which is a bounded approximate identity for the Frechet algebra A^∞. Then $E(S(\mathbb{R}, A^\infty, \alpha)) \cong E(L^1_{|\cdot|}(\mathbb{R}, A^\infty, \alpha)) \cong C^*(\mathbb{R}, E(A), \alpha)$. Further, if α is isometric, then $E(L^1(\mathbb{R}, A, \alpha)) \cong C^*(\mathbb{R}, E(A), \alpha)$.*

Notice that neither $L^1(\mathbb{R}, A, \alpha)$ nor $S(\mathbb{R}, A^\infty, \alpha)$ need be a subalgebra of $C^*(\mathbb{R}, E(A), \alpha)$. A particular case of Theorem 1 when A is a dense subalgebra of C^*-algebra has been treated in [2]. Let RK_* (respectively K_*) denote the representable K-theory functor (respectively K-theory functor) on Frechet algebras [10]. We have the following isomorphism of K-theory, obtained without direct appeal to spectral invariance.

Theorem 2. *Let A be as in the statement of Theorem* 1. *Assume that A is hermitian and a Q-algebra. Then $RK_*(S(\mathbb{R}, A^\infty, \alpha) \cong K_*(C^*(\mathbb{R}, E(A), \alpha))$. Further if the action α is isometric on A, then $RK_*(L^1(\mathbb{R}, A, \alpha)) \cong K_*(C^*(\mathbb{R}, E(A), \alpha))$.*

We apply this to the differential structure of a C^*-algebra. Let α be an action of $\mathbb{R}$ on a C^*-algebra A leaving a dense *-subalgebra $\mathcal{U}$ invariant. Let $T \sim (T_k)_0^\infty$ be a differential *-seminorm on $\mathcal{U}$ in the sense of Blackadar and Cuntz [5] with $T_0(x) = \|\cdot\|$ the C^*-norm from A. Let T be α-invariant. Let $\mathcal{U}_{(k)}$ be the completion of $\mathcal{U}$ in the submultiplicative *-norm $p_k(x) = \sum_{i=0}^k T_i(x)$. The differential Frechet *-algebra defined by T is $\mathcal{U}_\tau = \varprojlim \mathcal{U}_{(k)}$, the inverse limit of Banach *-algebras $\mathcal{U}_{(k)}$.

Now consider $\tilde{\mathcal{U}}$ to be the α-invariant smooth envelope of $\mathcal{U}$ defined to be the completion of $\mathcal{U}$ in the collection of all α-invariant differential *-seminorms. Notice that neither $\mathcal{U}_\tau$ nor $\tilde{\mathcal{U}}$ is a subalgebra of A, though each admits a continuous surjective *-homomorphism onto A induced by the inclusion $\mathcal{U} \to A$. There exists actions of $\mathbb{R}$ on each of $\mathcal{U}_\tau$ and $\tilde{\mathcal{U}}$ induced by α. The following is a smooth Frechet analogue of Connes' analogue of Thom isomorphism [7]. It supplements an analogues result in [11].

Theorem 3.

(a) $RK_*(S(\mathbb{R}, \mathcal{U}_\tau^\infty, \alpha)) = K_{*+1}(A)$.
(b) *Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $RK_*(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = K_{*+1}(A)$.*

2. Preliminaries and notations

A *Frechet *-algebra* (A, t) is a complete topological involutive algebra A whose topology t is defined by a separating sequence $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of seminorms satisfying $\|xy\|_n \leq \|x\|_n\|y\|_n$, $\|x^*\|_n = \|x\|_n$, $\|x\|_n \leq \|x\|_{n+1}$ for all x, y in A and all n in $\mathbb{N}$. If each $\|\cdot\|_n$ satisfies $\|x^*x\|_n = \|x\|_n^2$ for all x in A, then A is a *σ-C^*-algebra* [9]. A is called a *Q-algebra* if the set of all quasi-regular elements of A is an open set. For each n in $\mathbb{N}$, let A_n be the Hausdorff completion of $(A, \|\cdot\|_n)$. There exists norm decreasing surjective *-homomorphisms $\pi_n : A_{n+1} \to A_n$, $\pi_n(x + \ker\|\cdot\|_{n+1}) = x + \ker\|\cdot\|_n$ for all $x \in A$. Then the sequence

$$A_1 \xleftarrow{\pi_1} A_2 \xleftarrow{\pi_2} A_3 \xleftarrow{\pi_3} \cdots \xleftarrow{\pi_{n-1}} A_n \xleftarrow{\pi_n} A_{n+1} \longleftarrow \cdots$$

is an inverse limit sequence of Banach *-algebras and $A = \varprojlim A_n$, the inverse limit of Banach *-algebras. Let $\mathrm{Rep}(A)$ be the set of all *-homomorphisms $\pi : A \to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. Let

$$\mathrm{Rep}_n(A) := \{\pi \in \mathrm{Rep}(A) : \text{ there exists } k > 0 \text{ such that } \|\pi(x)\| \leq k\|x\|_n \text{ for all } x\}.$$

Then $|x|_n := \sup\{\|\pi(x)\| : \pi \in \mathrm{Rep}_n(A)\}$ defines a C^*-seminorm on A. The *star radical* of A is

$$\mathrm{srad}(A) = \{x \in A : |x|_n = 0 \text{ for all } n \text{ in } \mathbb{N}\}.$$

The enveloping σ-C^*-algebra $(E(A), \tau)$ of A is the completion of $A/\mathrm{srad}(A)$ in the topology τ defined by the C^*-seminorms $\{|\cdot|_n : n \in \mathbb{N}\}$, $|x + \mathrm{srad}(A)|_n = |x|_n$ for x in A.

Let α be a strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of A. The C^{∞}*-elements of A for the action* α are

$$A^{\infty} := \{x \in A : t \to \alpha_t(x) \text{ is a } C^{\infty}\text{-function}\}.$$

It is a dense *-subalgebra of A which is a Frechet algebra with the topology defined by the submultiplicative *-seminorms

$$\|x\|_{k,n} = \|x\|_n + \sum_{j=0}^{k}(1/j!)\|\delta^j x\|_n, \quad n \in \mathbb{N},\ k \in \mathbb{Z}^+ = \mathbb{N} \cup (0)$$

where δ is the derivation $\delta(x) = (\mathrm{d}/\mathrm{d}t)\alpha_t(x)|_{t=0}$. By Theorem A.2 of [14], α leaves A^{∞} invariant and each α_t restricted to A^{∞} gives a continuous *-automorphism of the Frechet algebra A^{∞}. The action α is *smooth* if $A^{\infty} = A$.

2.1 *Smooth Schwartz crossed product* [14]

Assume that α is *m-tempered* in the sense that for each $n \in \mathbb{N}$, there exists a polynomial P_n such that $\|\alpha_r(x)\|_n \le P_n(r)\|x\|_n$ for all $r \in \mathbb{R}$ and all $x \in A$. Let $S(\mathbb{R})$ be the Schwartz space. The completed (projective) tensor product $S(\mathbb{R}) \otimes A = S(\mathbb{R}, A)$ consisting of A-valued Schwartz functions on $\mathbb{R}$ is a Frechet algebra with the twisted convolution

$$(f * g)(r) = \int_R f(s)\alpha_s(g(r-s))\mathrm{d}s$$

called the *smooth Schwartz crossed product by* $\mathbb{R}$ denoted by $S(\mathbb{R}, A, \alpha)$. The algebra $S(\mathbb{R}, A^{\infty}, \alpha)$ is a Frechet *-algebra with the involution $f^*(r) = \alpha_r(f(-r)^*)$ (Corollary 4.9 of [14]) whose topology τ_s is defined by the seminorms

$$\|f\|_{n,l,m} = \sum_{i+j=n} \int_R (1+|r|)^i \|f^{(j)}(r)\|_{l,m}\mathrm{d}r, \quad n \in \mathbb{Z}^+, l \in \mathbb{Z}^+, m \in \mathbb{N}$$

where

$$\|f^{(j)}(r)\|_{l,m} = \sum_{k=0}^{l}(1/k!)\|\delta^k(\alpha_s((\mathrm{d}^j/\mathrm{d}r^j)f(r))|_{s=0}\|_m$$

(Theorem 3.1.7 of [14], [11]). These seminorms are submultiplicative if α is isometric on A in the sense that $\|\alpha_r(x)\|_n = \|x\|_n$ for all $n \in \mathbb{N}$ and all $x \in A$.

2.2 L^1*-crossed products* [11,14]

Let F_d be the set of all functions $f : \mathbb{R} \to A$ for which

$$\|f\|_{d,m} := \int_R (1+|r|)^d \|f(r)\|_m \mathrm{d}r < \infty$$

for all m in $\mathbb{N}$. Here $\int$ denotes the upper integral. Let $\mathbb{L}_d$ be the closure in F_d of the set of all measurable simple functions $f : \mathbb{R} \to A$ in the topology on F_d given by the seminorms $\{\|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Let $N_d = \cap\{\ker \|\cdot\|_{d,m} : m \in \mathbb{N}\}$. Then $N_d = N_{d+1}$; $L_d := \mathbb{L}_d/N_d$

is complete in $\{\|\cdot\|_{d,m}: m \in \mathbb{N}\}$ and $L_{d+1} \to L_d$ continuously. The *space of* $|\cdot|$*-rapidly vanishing* L^1*-functions from* $\mathbb{R}$ *to* A is $L^1_{|\cdot|}(\mathbb{R}, A, \alpha) := \cap\{L_d: d \in \mathbb{Z}^+\}$, a Frechet algebra with the topology given by the seminorms $\{\|\cdot\|_{d,m}: m \in \mathbb{N},\ d \in \mathbb{Z}^+\}$ and with twisted convolution. Assume that α is isometric on $(A, \{\|\cdot\|_n\})$. Then the completed projective tensor product $L^1(\mathbb{R}) \otimes A = L^1(\mathbb{R}, A)$ is a Frechet *-algebra with twisted convolution and the involution $f \to f^*$. This L^1*-crossed product* is denoted by $L^1(\mathbb{R}, A, \alpha)$. Notice that α is isometric on $(A^\infty, \{\|\cdot\|_{n,m}\})$ also, so that the Frechet *- algebra $L^1(\mathbb{R}, A^\infty, \alpha)$ is defined; and then the induced actions $(\alpha_r f)(s) = \alpha_r(f(s))$ on $L^1(\mathbb{R}, A^\infty, \alpha)$ and on $L^1(\mathbb{R}, A, \alpha)$ are also isometric.

2.3 *σ-C^*-crossed product*

Assume that α is isometric. We define the *σ-C^*-crossed product* $C^*(\mathbb{R}, A, \alpha)$ of A by $\mathbb{R}$ to be the enveloping σ-C^*-algebra $E(L^1(\mathbb{R}, A, \alpha))$ of $L^1(\mathbb{R}, A, \alpha)$.

3. Technical lemmas

Lemma 3.1. *Let α be m-tempered on A. Then α extends as a strongly continuous isometric action of $\mathbb{R}$ by continuous *-automorphisms of the σ-C^*-algebra $E(A)$.*

Proof. By the m-temperedness of α, for each $n \in \mathbb{N}$, there exists a polynomial P_n such that for all $x \in A$ and all $r \in \mathbb{R}$, $\|\alpha_r(x)\|_n \le P_n(r)\|x\|_n$. Let $r \in \mathbb{R}$. Let $x \in \mathrm{srad}(A)$. Then for all $\pi \in \mathrm{Rep}(A)$, $\pi(x) = 0$, so that $\sigma(\alpha_r(x)) = 0$ for all $\sigma \in \mathrm{Rep}(A)$, hence $\alpha_r(x) \in \mathrm{srad}(A)$. Thus $\alpha_r(\mathrm{srad}(A)) \subseteq \mathrm{srad}(A)$, and the map

$$\tilde{\alpha}_r: A/\mathrm{srad}(A) \to A/\mathrm{srad}(A), \quad \tilde{\alpha}_r([x]) = [\alpha_r(x)],$$

where $[x] = x + \mathrm{srad}(A)$, is a well-defined *-homomorphism. Further, let $\tilde{\alpha}_r[x] = 0$. Then $\alpha_r(x) \in \mathrm{srad}(A)$. Hence $x = \alpha_{-r}(\alpha_r(x)) \in \mathrm{srad}(A)$, $[x] = 0$. Thus $\tilde{\alpha}_r$ is one-to-one, which is clearly surjective. Now, for each $n \in \mathbb{N}$, and for all $x \in A$,

$$|\tilde{\alpha}_r[x]|_n = |[\alpha_r(x)]|_n \le \|\alpha_r(x)\|_n \le P_n(r)\|x\|_n.$$

Since, by definition, $|\cdot|_n$ is the greatest C^*-seminorm on $A/\mathrm{srad}(A)$ satisfying that for some $k_n > 0$, $|[z]|_n \le k_n\|z\|_n$ for all $z \in A$, it follows that $|\tilde{\alpha}_r[x]|_n \le |[x]|_n$ for all x in A. Hence

$$|[x]|_n \le |\tilde{\alpha}_{-r}(\tilde{\alpha}_r[x]|_n = |\tilde{\alpha}_{-r}[\alpha_r(x)]|_n \le |[\alpha_r(x)]|_n = |\tilde{\alpha}_r[x]|_n$$

showing that $|\tilde{\alpha}_r[x]|_n = |[x]|_n$ for all $x \in A$, $r \in \mathbb{R}, n \in \mathbb{N}$. It follows that $\tilde{\alpha}_r$ extends as a *-automorphism $\tilde{\alpha}_r: E(A) \to E(A)$ satisfying $|\tilde{\alpha}_r(z)|_n = |z|_n$ for all $z \in A$ and all $n \in \mathbb{N}$; and $\tilde{\alpha}: \mathbb{R} \to \mathrm{Aut}^*(E(A))$, $r \to \tilde{\alpha}_r$ defines an isometric action of $\mathbb{R}$ on $E(A)$. We verify that $\tilde{\alpha}$ is strongly continuous. Let $z \in E(A)$. It is sufficient to prove that the map $f: \mathbb{R} \to E(A)$, $f(r) = \alpha_r(z)$ is continuous at $r = 0$. Choose $z_n = [x_n]$ in $A/\mathrm{srad}(A)$ such that $z_n \to z$ in $E(A)$. Fix $k \in \mathbb{N}$, $\varepsilon > 0$. Choose n_0 in $\mathbb{N}$ such that $|z_{n_0} - z|_k < \varepsilon/3$ with $z_{n_0} = [x_{n_0}]$. Then for all $r \in \mathbb{R}$, $|\tilde{\alpha}_r(z) - \tilde{\alpha}_r(z_{n_0})|_k = |z - z_{n_0}|_k < \varepsilon/3$. Since α is strongly continuous, there exists a $\delta > 0$ such that $|r| < \delta$ implies that $\|\alpha_r(x_0) - x_0\|_k < \varepsilon/3$. Then for all such r, $|\tilde{\alpha}_r(z) - z|_k < \varepsilon$ showing the desired continuity of f. This completes the proof. □

Notation. Henceforth we denote the action $\tilde{\alpha}$ by α.

A *covariant representation* of the Frechet algebra dynamical system $(\mathbb{R}, A, \alpha)$ is a triple (π, U, H) such that

(a) $\pi\colon A \to B(H)$ is a *-homomorphism;
(b) $U\colon \mathbb{R} \to \mathcal{U}(H)$ is a strongly continuous unitary representation of $\mathbb{R}$ on H; and
(c) $\pi(\alpha_t(x)) = U_t\pi(x)U_t^*$ for all $x \in A$ and all $t \in \mathbb{R}$.

The following is an analogue of Proposition 7.6.4, p. 257 of [12] which can be proved along the same lines. Let $C_c^\infty(\mathbb{R}, A^\infty) = C_c^\infty(\mathbb{R}) \otimes A^\infty$ (completed projective tensor product) be the space of all A^∞-valued C^∞-functions on $\mathbb{R}$ with compact supports.

Lemma 3.2. *Let A have a bounded approximate identity* (e_l) *contained in* A^∞ *which is also a bounded approximate identity for the Frechet algebra* A^∞. (*In particular, let A be unital.*)

(a) *If* (π, U, H) *is a covariant representation of* $(\mathbb{R}, A^\infty, \alpha)$, *then there exists a non-degenerate *- representation* $(\pi \times U, H)$ *of* $S(\mathbb{R}, A^\infty, \alpha)$ *such that*

$$(\pi \times U)y = \int_R \pi(y(t))U_t \mathrm{d}t$$

for every y in $C_c^\infty(\mathbb{R}, A^\infty)$. *The correspondence* $(\pi, U, H) \to (\pi \times U, H)$ *is bijective onto the set of all non-degenerate *-representations of* $S(\mathbb{R}, A^\infty, \alpha)$.

(b) *Let* α *be isometric. Then the above gives a one-to-one correspondence between the covariant representations of* $(\mathbb{R}, A, \alpha)$ *and non-degenerate *-representations of each of* $L^1(\mathbb{R}, A^\infty, \alpha)$ *and* $L^1(\mathbb{R}, A, \alpha)$.

Lemma 3.3. $E(A^\infty) = E(A)$; *and for all k in* $\mathbb{Z}^+$, *n in* $\mathbb{N}$, $\|\,\|_{n,k} = \|\,\|_n$.

Proof. Consider the inverse limit $A = \varprojlim A_n$ as in the Introduction. Since α satisfies $\|\alpha_r(x)\|_n \leq P_n(r)\|x\|_n$ for all $x \in \mathbb{R}$, all $r \in A$ and all $n \in \mathbb{N}$, it follows that for each n, α 'extends' uniquely as a strongly continuous action $\alpha^{(n)}$ of $\mathbb{R}$ by continuous *-automorphisms of the Banach *-algebra A_n. Let $(A_{n,m}, \|\cdot\|_{n,m})$ be the Banach algebra consisting of all C^m-elements y of A_n with the norm $\|\cdot\|_{n,m} = \|y\|_n + \sum_{i=1}^m (1/i!)\|\delta^i(x)\|_n$. Let $(A_n^\infty, \{\|\,\|_{m,n}\colon m \in \mathbb{Z}^+\})$ be the Frechet algebra consisting of all C^∞-elements of A_n for the action $\alpha^{(n)}$. Then

$$A^\infty = \varprojlim A_n^\infty = \varprojlim \varprojlim A_{m,n} = \varprojlim A_{n,n}.$$

By Theorem 2.2 of [15], each $A_{m,n}$ is dense and spectrally invariant in A_n. Hence each $A_{n,m}$ is a Q-normed algebra in the norm $\|\cdot\|_n$ of A_n.

Let $\pi\colon A^\infty \to B(H)$ be a *-representation of A on a Hilbert space H. Since the topology of A^∞ is determined by the seminorms

$$\|x\|_{n,n} = \|x\|_n + \sum_{j=1}^n (1/j!)\|\delta^j(x)\|_n, \quad n \in \mathbb{N}$$

it follows that for some $k > 0$, $\|\pi(x)\| \leq k\|x\|_{n,n}$ for all $a \in A^\infty$. Hence π defines a *-homomorphism $\pi\colon (A_{n,n}, \|\,\|_{n,n}) \to B(H)$ satisfying $\|\pi(x)\| \leq k\|x\|_{n,n}$ for all x in

$A_{n,n}$. Since $(A_{n,n}, \| \|_n)$ is a Q-normed *-algebra, this map π is continuous in the norm $\| \|_n$ on $A_{n,n}$. In fact, for all x in A^∞,

$$\|\pi(x)\|^2 = \|\pi(x^*x)\| = r_{B(H)}(\pi(x^*x)) \le r_{A_{n,n}}(\pi(x^*x + \ker \| \|_{n,n}))$$
$$\le \|x^*x + \ker \| \|_n\| = \|x^*x\|_n \le \|x\|^2.$$

Thus $\|\pi(x)\| \le \|x\|_n$ for all x in A^∞. Since A^∞ is dense in A, π can be uniquely extended as a *-representation $\pi\colon A \to B(H)$ satisfying that $\|\pi(x)\| \le \|x\|_n$ for all x in A. Then by the definition of the C^*-seminorm $| |_n$ on A, π extends as a continuous *-homomorphism $\tilde{\pi}\colon E(A) \to B(H)$ such that $\|\tilde{\pi}(x)\| \le |x|_n$ for all x in $E(A)$. This also implies that $E(A^\infty) = E(A)$ and $|\cdot|_{n,m} = |\cdot|_n$ for all n, m.

Lemma 3.4. *Let B be a σ-C^*-algebra. Let $j\colon A \to E(A)$ be* $j(x) = x + \mathrm{srad}(A)$. *Let $\pi\colon A \to B$ be a *-homomorphism. Then there exists a unique *-homomorphism $\tilde{\pi}\colon E(A) \to B$ such that $\pi = \tilde{\pi} \circ j$.*

This follows immediately by taking $B = \lim\limits_{\leftarrow} B_n$, where B_n's are C^*-algebras, and by the universal property of $E(A)$.

4. Proof of Theorem 1

Step I. $\mathrm{Rep}(S(\mathbb{R}, A^\infty, \alpha)) = \mathrm{Rep}(S(\mathbb{R}, E(A), \alpha)) = \mathrm{Rep}(L^1(\mathbb{R}, E(A), \alpha))$ up to one-to-one correspondence.

By Lemma 3.1, the Frechet algebras $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$ are *-algebras with the continuous involution $y \to y^*, y^*(t) = \alpha_t(y(-t))^*$. By Lemma 3.2, $\mathrm{Rep}(S(\mathbb{R}, E(A), \alpha)) = \mathrm{Rep}(L^1(\mathbb{R}, E(A), \alpha))$ each identified with the set of all covariant representations. Let $\rho\colon S(\mathbb{R}, A^\infty, \alpha) \to B(H)$ be in $\mathrm{Rep}(S(\mathbb{R}, A^\infty, \alpha))$. There exists $c > 0$ and appropriate n, l, m such that for all y,

$$\|\rho(y)\| \le c\|y\|_{n,l,m} = c \sum_{i+j=n} \int_R (1 + |r|)^i \|y^{(j)}(r)\|_{l,m} \mathrm{d}r. \tag{1}$$

By Lemma 3.2, there exists a covariant representation (π, U, H) of $(\mathbb{R}, A^\infty, \alpha)$ on H such that $\rho = \pi \times U$. Thus $\pi\colon A^\infty \to B(H)$ is a *-homomorphism and $U\colon \mathbb{R} \to \mathcal{U}(H)$ is a strongly continuous unitary representation such that

(i) $\rho(f) = \int_R \pi(f(t))U_t \mathrm{d}t$ for all f in $S(\mathbb{R}, A^\infty, \alpha)$, (2)

(ii) $\pi(\alpha_t(x)) = U_t \pi(x) U_t^*$ for all $x \in A^\infty, t \in \mathbb{R}$, (3)

(iii) there exists $K > 0$ such that $\|\pi(x)\| \le k\|x\|_{l,m}$ for all $x \in A^\infty$.

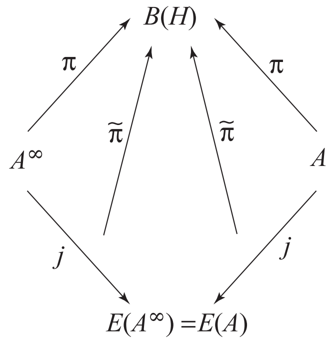
The l, m in (iii) are the same as in (1). Let $\{|\cdot|_{l,m}\colon l \text{ in } \mathbb{Z}^+, m \text{ in } \mathbb{N}\}$ be the sequence of C^*-seminorms on A^∞ (and also on $E(A^\infty)$ via srad A^∞) which are defined by the submultiplicative *-seminorms $\{\|\cdot\|_{l,m}\colon l \text{ in } \mathbb{Z}^+, m \text{ in } \mathbb{N}\}$. Then $|\cdot|_{l,m}$ is the greatest C^*-seminorm on A^∞ satisfying that there exists $M = M_{l,m} > 0$ such that $|\cdot|_{l,m} \le M\|\cdot\|_{l,m}$. Hence by (iii) above, π can be uniquely extended as a continuous *-homomorphism $\tilde{\pi}\colon E(A^\infty) \to B(H)$ such that $\tilde{\pi}(j(x)) = \pi(x)$ for all $x \in A^\infty$; and

$$\|\tilde{\pi}(x)\| \le |x|_{l,m} \text{ for all } x \in E(A^\infty). \tag{4}$$

Here j is the map $j\colon A^{\infty} \to E(A^{\infty})$, $j(x) = x + \mathrm{srad}\, A^{\infty}$. Let l denote $\max(l, m)$. Then we have

$$\|\rho(y)\| \le c\|y\|_{n,l,l} \ \text{ for all } y \in S(\mathbb{R}, A^{\infty}, \alpha);$$
$$\|\pi(x)\| \le k\|x\|_{l,l} \ \text{ for all } x \in A^{\infty};$$
$$\|\tilde{\pi}(z)\| \le |z|_{l,l} \ \text{ for all } z \in E(A^{\infty}). \tag{5}$$

By Lemma 3.3, $\tilde{\pi}\colon E(A) \to B(H)$ is a *-representation satisfying $\|\tilde{\pi}(x)\| \le |x|_l$ for all x in $E(A)$. We have the following commutative diagram.



Now, let $\alpha\colon \mathbb{R} \to \mathrm{Aut}^* E(A)$ be the action on $E(A)$ induced by α as in Lemma 3.1 satisfying

$$\alpha_t(j(x)) = j(\alpha_t(x)) \quad \text{for all } x \text{ in } A. \tag{6}$$

Then $(\tilde{\pi}, U, H)$ is a covariant representation of $(\mathbb{R}, E(A), \alpha)$. Indeed, let $x \in A^{\infty}$, $y = j(x)$. Then for all $t \in \mathbb{R}$,

$$\begin{aligned}\tilde{\pi}(\alpha_t(y)) &= \tilde{\pi}(\alpha_t(j(x))) = \tilde{\alpha}(j(\alpha_t(x))) = \pi(\alpha_t(x)) = U_t\pi(x)U_t^* \\ &= U_t\tilde{\pi}(j(x))U_t^* = U_t\tilde{\pi}(y)U_t^*.\end{aligned}$$

By the continuity of $\tilde{\pi}$ and α_t, it follows that $\tilde{\pi}(\alpha_t(y)) = U_t\tilde{\pi}(y)U_t^*$ for all $y \in E(A)$ and all $t \in \mathbb{R}$. Hence by Lemma 3.2, $\tilde{\rho} = \tilde{\pi} \times U$ is a non-degenerate *-representation of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$ satisfying, for some constants c and c', the following (using (5)):

(iv) For all f in $L^1(\mathbb{R}, E(A), \alpha)$, $\|\rho(f)\| \le c|f|_l = c\int_R |f(t)|_l \mathrm{d}t$.

(v) For all f in $S(\mathbb{R}, E(A), \alpha)$, $\|\tilde{\rho}(f)\| \le c'|f|_{n,l,m}$. (7)

Thus given a *-representation ρ of $S(\mathbb{R}, A^{\infty}, \alpha)$, there is canonically associated a *-representation $\tilde{\rho}$ of each of $S(\mathbb{R}, E(A), \alpha)$ and $L^1(\mathbb{R}, E(A), \alpha)$.

Conversely, given ρ in $\mathrm{Rep}(S(\mathbb{R}, E(A), \alpha))$, $\rho = \pi \times U$ for a covariant representation (π, U) of $(\mathbb{R}, E(A), \alpha)$, $\pi \circ j$ is a covariant representation of A, and then $(\pi \circ j) \times U$ is in $\mathrm{Rep}(S(\mathbb{R}, A^{\infty}, \alpha))$.

Step II. The σ-C^*-algebra $C^*(\mathbb{R}, E(A), \alpha)$ is universal for the *-representations of the Frechet algebra $S(\mathbb{R}, A^{\infty}, \alpha)$.

Let $\tilde{j}$: $S(\mathbb{R}, A^{\infty}, \alpha) \to L^1(\mathbb{R}, E(A), \alpha)$ be the map

$$\tilde{j}(f) = j \circ f = \tilde{f} \ \text{(say)}, \quad \text{i.e.,}$$

$$\tilde{j}(f)(r) = j(f(r)) = f(r) + \text{srad}(A)^{\infty} \quad \text{for all } r \in \mathbb{R}. \tag{8}$$

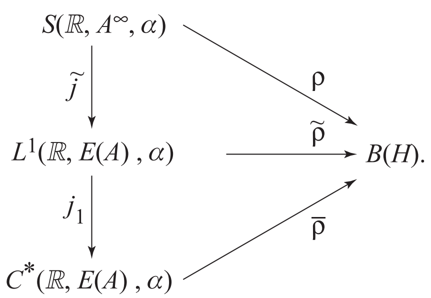
Notice that the map $\tilde{j}$ is defined and is continuous; because $(S(\mathbb{R}, A^{\infty}, \alpha)) \subset L^1(\mathbb{R}, A^{\infty}, \alpha) \subset L^1(\mathbb{R}, A, \alpha)$, and for n in $\mathbb{N}$ and m in $\mathbb{Z}^+$, all f in $S(\mathbb{R}, A^{\infty}, \alpha)$,

$$|\tilde{f}(t)|_n \leq \|f(t)\|_n \leq M\|f(t)\|_{m,n}, \ \text{ and hence}$$

$$\int_R |\tilde{f}(t)|_l \mathrm{d}t \leq \int_R \|f(t)\|_{m,n}\mathrm{d}t < \infty$$

so that $f \in L^1(\mathbb{R}, E(A), \alpha)$. Let j_1: $L^1(\mathbb{R}, E(A), \alpha) \to C^*(\mathbb{R}, E(A), \alpha)$ be the natural map $j_1(f) = f + \text{srad}(L^1(\mathbb{R}, E(A), \alpha))$. This gives the continuous *-homomorphism

$$J : j_1 \circ \tilde{j} : S(\mathbb{R}, A^{\infty}, \alpha) \to C^*(\mathbb{R}, E(A), \alpha). \tag{9}$$



Let $\rho \in \text{Rep}(S(\mathbb{R}, A^{\infty}, \alpha))$, $\rho = \pi \times U$ in usual notations with π: $A^{\infty} \to B(H)$ in $\text{Rep}(E(A))$ such that $\pi = \tilde{\pi} \circ j$. Let $\tilde{\rho}$: $L^1(\mathbb{R}, E(A), \alpha) \to B(H)$ be $\tilde{\rho} = \tilde{\pi} \times U$. Then for all f in $S(\mathbb{R}, A^{\infty}, \alpha)$,

$$\begin{aligned}\tilde{\rho}(\tilde{j}(f)) = (\tilde{\pi} \times U)(\tilde{j}(f)) &= \int_R \tilde{\pi}(\tilde{j}(f)(t))U_t\mathrm{d}t = \int_R \tilde{\pi}(j \circ f)(t)U_t\mathrm{d}t \\ &= \int_R \tilde{\pi}(j(f(t)))U_t\mathrm{d}t = \int_R \tilde{\pi}(f(t) + \text{srad}(A))U_t\mathrm{d}t \\ &= \int_R \pi(f(t))U_t\mathrm{d}t = \rho(f).\end{aligned}$$

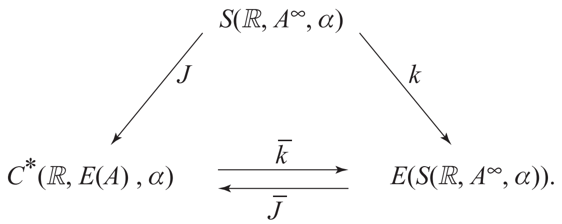
Thus $\tilde{j} \circ \tilde{\rho} = \rho$; and hence $J \circ \bar{\rho} = \rho$, where $J = j_1 \circ \tilde{j}$ and $\bar{\rho} \in \text{Rep}(C^*(\mathbb{R}, E(A), \alpha))$ is defined by $j_1 \circ \bar{\rho} = \tilde{\rho}$ in view of $C^*(\mathbb{R}, E(A), \alpha) = E(L^1(\mathbb{R}, E(A), \alpha))$.

Step III. Given a *-homomorphism ρ: $S(\mathbb{R}, A^{\infty}, \alpha) \to B$ from $S(\mathbb{R}, A^{\infty}, \alpha)$ to a σ-C^*-algebra B, there exists *-homomorphisms $\tilde{\rho}$: $L^1(\mathbb{R}, E(A), \alpha) \to B$, $\tilde{\rho}$: $C^*(\mathbb{R}, E(A), \alpha) \to B$ such that $\rho = \tilde{\rho} \circ \tilde{j} = \bar{\rho} \circ J$ and $\tilde{\rho} = \bar{\rho} \circ j_1$.

This follows by applying Step II to each of the factor C^*-algebra B_n in the inverse limit decomposition of B.

Step IV. $C^*(\mathbb{R}, E(A), \alpha) = E(S(\mathbb{R}, A^{\infty}, \alpha))$ up to homeomorphic *-isomorphism.

Let $k: S(\mathbb{R}, E(A), \alpha) \to E(S(\mathbb{R}, A^{\infty}, \alpha))$ be $k(f) = f + \text{srad}\, S(\mathbb{R}, A^{\infty}, \alpha)$. Then there exists a *-homomorphism $\bar{k}: C^*(\mathbb{R}, E(A), \alpha) \to E(S(\mathbb{R}, A^{\infty}, \alpha))$ such that $\bar{k} \circ J = k$. We show that $\bar{k}$ is the desired homeomorphic *-isomorphism making the following diagram commutative.



(10)

By the universal property of $E(S(\mathbb{R}, A^{\infty}, \alpha))$, there exists a *-homomorphism $\bar{J}: E(S(\mathbb{R}, A^{\infty}, \alpha)) \to C^*(\mathbb{R}, E(A), \alpha)$ such that $\bar{J} \circ k = J$. We claim that $|\bar{k}|_{\text{Im}(J)}$ is injective. Indeed, let $f \in S(\mathbb{R}, A^{\infty}, \alpha)$ be such that $\bar{k}(J(f)) = 0$. Hence $k(f) = 0$, so that $f \in \text{srad}(S(\mathbb{R}, A^{\infty}, \alpha))$. Thus, for all $\rho \in \text{Rep}(S(\mathbb{R}, A^{\infty}, \alpha))$, $\rho(f) = 0$. Therefore, by Step I, $\sigma(\tilde{f}) = 0$ for all $\sigma \in \text{Rep}(L^1(\mathbb{R}, E(A), \alpha))$. (Recall that $\tilde{f} = j \circ f = \tilde{j}(f)$.) Hence $\tilde{j}(f)$ is in $\text{srad}(L^1(\mathbb{R}, E(A), \alpha))$, and so $j_1(\tilde{j}(f)) = 0$. Therefore $J(f) = 0$. It follows that $\bar{k}$ is injective on $\text{Im}(J)$.

Now by (10) and the injectivity of $\bar{k}$ on $\text{Im}(J)$, $\bar{J} \circ k = J$. Hence $J = \bar{J} \circ \bar{k} \circ J$, and so $\bar{J} \circ \bar{k} = \text{id}$ on $\text{Im}(J)$. Similarly $\bar{k} \circ \bar{J}(k(f)) = \bar{k}(J(f)) = k(f)$, hence $\bar{k} \circ \bar{J} = \text{id}$ on $\text{Im}(k)$. Thus $\bar{k} = (\bar{J})^{-1}$ on $\text{Im}(J)$. Thus $\bar{k}$ is a homeomorphic *-isomorphism from the dense *-subalgebra $J(S(\mathbb{R}, A^{\infty}, \alpha))$ of $C^*(\mathbb{R}, E(A), \alpha)$ on the dense *-subalgebra $k(S(\mathbb{R}, A^{\infty}, \alpha))$ of $E(S(\mathbb{R}, A^{\infty}, \alpha))$. It follows that $C^*(\mathbb{R}, E(A), \alpha)$ is homeomorphically *-isomorphic to $E(S(\mathbb{R}, A^{\infty}, \alpha))$.

Step V. $E(L^1_{|\cdot|}(\mathbb{R}, A^{\infty}, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$.

Let $\mathbb{R}$ act on $L^1_{|\cdot|}(\mathbb{R}, A, \alpha)$ by $xf(y) = f(x - y)$. For this action, $(L^1_{|\cdot|}((\mathbb{R}, A, \alpha))^{\infty} = S(\mathbb{R}, A, \alpha)$ by Theorem 2.1.7 of [14]. Thus $S(\mathbb{R}, A^{\infty}, \alpha) = (L^1_{|\cdot|}((\mathbb{R}, A, \alpha))^{\infty}$. Hence by Lemma 3.4, $E(L^1_{|\cdot|}(\mathbb{R}, A^{\infty}, \alpha))^{\infty} = E(S(\mathbb{R}, A^{\infty}, \alpha)) = C^*(\mathbb{R}, E(A), \alpha)$. This completes the proof of Theorem 1. □

5. Proof of Theorem 2

Let the Frechet algebra A be hermitian and a Q-algebra. Hence A is spectrally bounded, i.e., the spectral radius $r(x) = r_A(x) < \infty$ for all $x \in A$. Let $s_A(x) := r(x^*x)^{1/2}$ be the Ptak's spectral function on A. By Corollary 2.2 of [1], $E(A)$ is a C^*-algebra, the complete C^*-norm of $E(A)$ being defined by the greatest C^*-seminorm $p_{\infty}(\cdot)$ (automatically continuous) on A. Now for any $x \in A$,

$$\begin{aligned} p_{\infty}(x)^2 &= p_{\infty}(x^*x) = \|x^*x + \text{srad}(A)\| \\ &= r_{E(A)}(x^*x + \text{srad}(A)) \leq r_A(x^*x) = s_A(x)^2. \end{aligned}$$

Hence $p_{\infty}(x) \leq s_A(x)$ for all $x \in A$. By the hermiticity and Q-property, $s_A(\cdot)$ is a C^*-seminorm (Theorem 8.17 of [8]), hence $p_{\infty}(\cdot) = s(\cdot) \geq r(\cdot)$. In this case, $\text{rad}(A) = \text{srad}(A)$. Let $A_q = A/\text{rad}(A)$ which is a dense *-subalgebra of the C^*-algebra $E(A)$ and is also a Frechet Q-algebra with the quotient topology t_q. Let $[x] = x + \mathrm{rad}(A)$ for all $x \in A$. Since the spectrum

$$\mathrm{sp}_A(x) = \mathrm{sp}_{A_q}([x]), \quad r_A(x) = r_{A_q}([x]), \quad s_A(x) = s_{A_q}([x]),$$

and so $r_{A_q}([x]) \leq s_{A_q}([x]) = \|[x]\|_\infty$. Hence $\|\cdot\|_\infty$ is a spectral norm on A_q, i.e., $(A_q, \|\cdot\|_\infty)$ is a Q-algebra. Thus A_q is spectrally invariant in $E(A)$. Hence by Corollary 7.9 of [10], $K_*(A_q) = RK_*(A_q) = K_*(E(A))$.

Now consider the maps

$$A \xrightarrow{j} A_q \xrightarrow{\mathrm{id}} E(A)$$

and, for each positive integer n, the induced maps

$$M_n(A) \xrightarrow{j_n = j \otimes \mathrm{id}_n} M_n(A_q) = [M_n(A)]_q \xrightarrow{\mathrm{id}} M_n(E(A)) = E(M_n(A)).$$

By the spectral invariance of A in A_q via the map j, $j(\mathrm{inv}(A)) = \mathrm{inv}(A_q)$, where $\mathrm{inv}(K)$ denotes the group of invertible elements of K. Let $\mathrm{inv}_0(\cdot)$ denote the principle component in $\mathrm{inv}(\cdot)$. We use the following.

Lemma 5.1. *Let B be a Frechet Q-algebra or a normed Q-algebra. Then* $\mathrm{inv}_0(B)$ *is the subgroup generated by the range* $\exp B$ *of the exponential function.*

The Frechet Q-algebra case follows by adapting the proof of the corresponding Banach algebra result in Theorem 1.4.10 of [13]. If $(B, \|\cdot\|)$ is a Q-normed algebra, then $(B, \|\cdot\|)$ is advertably complete in the sense that if a Cauchy sequence (x_n) converges to an element $x \in \mathrm{inv}(B^\sim)$ ($B^\sim$ being the completion of B), then $x \in B$. Hence the exponential function is defined on B; and then the Banach algebra proof can be adapted.

We use the above lemma to verify the following:

Claim. $j_n(\mathrm{inv}_0(M_n(A))) = \mathrm{inv}_0(M_n(A_q))$.

Take $n = 1$. It is clear that $j(\mathrm{inv}_0(A)) \subseteq \mathrm{inv}_0(A_q)$. Let $y \in \mathrm{inv}_0(A_q)$. Hence $y = \Pi \exp(z_i)$ for finitely many $z_i = [x_i] = x_i + \mathrm{rad}(A)$ for some x_i in A. Then $y = [\Pi \exp(x_i)]$. Hence $y \in j(\mathrm{inv}_0(A))$. Thus $j(\mathrm{inv}_0(A)) = \mathrm{inv}_0(A_q)$. Now take $n > 1$. As A is spectrally invariant in A_q, it follows from Theorem 2.1 of [16] that the Frechet Q-algebra $M_n(A)$ is spectrally invariant in $M_n(A_q)$ via j_n. Also, $M_n(A_q) = (M_n(A))_q$ is a Q-algebra in both the quotient topology as well as the C^*-norm induced from $M_n(E(A)) = E(M_n A)$. Applying arguments analogous to above, it follows that $j_n(\mathrm{inv}_0(M_n(A))) = \mathrm{inv}_0(M_n(A_q))$.

Now consider the surjective group homomorphisms

$$\mathrm{inv}(M_n(A)) \xrightarrow{j_n} \mathrm{inv}(M_n(A_q)) \xrightarrow{J} \mathrm{inv}(M_n(A_q))/\mathrm{inv}_0(M_n(A_q)).$$

It follows that $\ker(J \circ j_n) = \mathrm{inv}_0(M_n(A))$, with the result, the group $\mathrm{inv}(M_n(A))/\mathrm{inv}_0(M_n(A))$ is isomorphic to the group $\mathrm{inv}(M_n(A_q))/\mathrm{inv}_0(M_n(A_q))$. Hence by the definition of the K-theory group K_1,

$$\begin{aligned} K_1(A) &= \varinjlim(\mathrm{inv}(M_n(A))/\mathrm{inv}_0(M_n(A))) \\ &= \varinjlim(\mathrm{inv}(M_n(A_q))/\mathrm{inv}_0(M_n(A_q))) = K_1(A_q). \end{aligned}$$

For B to be A or A_q, let the suspension of B be

$$SB = \{f \in C([0,1], B) \colon f(0) = f(1) = 0\} \cong C_0(\mathbb{R}, B).$$

We use the Bott periodicity theorem $K_0(B) = K_1(SB)$ to show that $K_0(A) = K_0(A_q)$. It is standard that $\mathrm{rad}(SA) = \mathrm{rad}(C_0(\mathbb{R}, A)) \cong C_0(\mathbb{R}, \mathrm{rad}(A))$. Hence

$$\begin{aligned} SA_q &= C_0(\mathbb{R}, A_q) = C_0(\mathbb{R}, A/\mathrm{rad}(A)) \cong C_0(\mathbb{R}, A)/C_0(\mathbb{R}, \mathrm{rad}(A)) \\ &= C_0(\mathbb{R}, A)/\mathrm{rad}(C_0(\mathbb{R}, A)) = SA/\mathrm{rad}(A)). \end{aligned}$$

Hence

$$K_0(A_q) = K_1(SA_q) = K_1(SA/\mathrm{rad}(SA)) = K_0(A).$$

Thus we have

$$K_*(A) = K_*(A_q) = K_*(E(A)) = RK_*(A) = RK_*(A_q).$$

Now A^∞ is spectrally invariant in A (Theorem 2.2 of [15]); and the action α on A^∞ is smooth (Theorem A.2 of [14]). Then applying the Phillips–Schweitzer analogue of Thom isomorphism for smooth Frechet algebra crossed product (Theorem 1.2 of [11]) and Connes analogue of Thom isomorphism for C^*-algebra crossed product [7], it follows that

$$\begin{aligned} RK_*(S(\mathbb{R}, A^\infty, \alpha)) &= RK_{*+1}(A^\infty) = RK_{*+1}(A) = RK_{*+1}(E(A)) \\ &= RK_*(C^*(\mathbb{R}, E(A), \alpha)) = K_*(C^*(\mathbb{R}, E(A), \alpha)). \end{aligned}$$

When α is isometric, Theorem 1.3.4 of [11] implies that $RK_*(S(\mathbb{R}, A^\infty, \alpha)) = RK_*(L^1(\mathbb{R}, A, \alpha))$. This completes the proof. □

6. An application to the differential structure in C^*-algebras

Let $\mathcal{U}$ be a unital *-algebra. Let $\|\cdot\|$ be a C^*-norm on $\mathcal{U}$. Let $(A, \|\cdot\|)$ be the completion of $(\mathcal{U}, \|\cdot\|)$. Following [5], a map $T\colon \mathcal{U} \to l^1(\mathbb{N})$ is a *differential seminorm* if $T(x) = (T_k(x))_0^\infty \in l^1(\mathbb{N})$ satisfies the following:

(i) $T_k(x) \geq 0$ for all k and for all x.
(ii) For all x, y in $\mathcal{U}$ and scalars λ, $T(x+y) \leq T(x) + T(y)$, $T(\lambda x) = |\lambda| T(x)$.
(iii) For all x, y in $\mathcal{U}$, for all k,

$$T_k(xy) \leq \sum_{i+j=k} T_i(x) T_j(y).$$

(iv) There exists a constant $c > 0$ such that $T_0(x) \leq c\|x\|$ $\forall x \in \mathcal{U}$.

By (ii), each T_k is a seminorm. We say that T is a *differential *-seminorm* if additionally;

(v) $T_k(x^*) = T_k(x)$ for all x and for all k.

Further T is a *differential norm* if $T(x) = 0$ implies $x = 0$. Throughout we assume that $T_0(x) = \|x\|$, $x \in \mathcal{U}$. The *total norm* of T is $T_{\mathrm{tot}}(x) = \sum_{k=0}^\infty T_k(x)$, $x \in \mathcal{U}$. Given T, the differential Frechet *-algebra defined by T is constructed as follows. For each k, let $p_k(x) = \sum_{i=0}^{k} T_i(x)$, $x \in \mathcal{U}$. Then each p_k is a submultiplicative *-norm; and on $\mathcal{U}$, we have

$$p_0 \leq p_1 \leq p_2 \leq \cdots \leq p_k \leq p_{k+1} \leq \cdots$$

and $(p_k)_0^\infty$ is a separating family of submultiplicative *-norms on $\mathcal{U}$. Let τ be the locally convex *-algebra topology on $\mathcal{U}$ defined by $(p_k)_0^\infty$. Let $\mathcal{U}_\tau = (\mathcal{U}, \tau)^\sim$ the completion of $\mathcal{U}$ in τ and let $\mathcal{U}_{(k)} = (\mathcal{U}, p_k)^\sim$ the completion of $\mathcal{U}$ in p_k. Then $\mathcal{U}_\tau$ is a Frechet locally m-convex *-algebra, $\mathcal{U}_{(k)}$ is a Banach *-algebra. Let $\mathcal{U}_T$ be the completion of $(\mathcal{U}, T_{\mathrm{tot}})$. Then the Banach *-algebra $\mathcal{U}_T = \{x \in \mathcal{U}_\tau\colon \sup_n p_n(x) < \infty\}$, the bounded part of $\mathcal{U}_\tau$. By the definitions, there exists continuous surjective *-homomorphisms $\phi_k\colon \mathcal{U}_{(k)} \to A$, $\phi\colon \mathcal{U}_\tau \to A$. The identity map $\mathcal{U} \to \mathcal{U}$ extends uniquely as continuous surjective *-homomorphisms $\varphi_k\colon \mathcal{U}_{(k+1)} \to \mathcal{U}_{(k)}$ such that

$$\mathcal{U}_{(0)} \xleftarrow{\varphi_0} \mathcal{U}_{(1)} \xleftarrow{\varphi_1} \mathcal{U}_{(2)} \xleftarrow{\varphi_2} \mathcal{U}_{(3)} \longleftarrow \cdots$$

is a dense inverse limit sequence of Banach *-algebras and $\mathcal{U}_\tau = \varprojlim \mathcal{U}_{(k)}$.

Lemma 6.1 [4]. *Let* $(\mathcal{U}, \|\cdot\|)$ *be a* C^**-normed algebra. Let* A *be the completion of* $\mathcal{U}$. *Let* B *denote* $\mathcal{U}_{(k)}$ *or* $\mathcal{U}_\tau$ *with respective topologies. Then the following hold*:

(i) *B is a hermitian Q-algebra.*
(ii) $E(B) = A$.
(iii) $K_*(B) = K_*(A) = RK_*(B)$.

The K-theory result follows from the following.

Lemma 6.2 [4]. *Let A be a Frechet algebra in which each element is bounded. Let A be spectrally invariant in $E(A)$. Then $K_*(A) = K_*(E(A))$.*

Now let α be an action of $\mathbb{R}$ on A leaving $\mathcal{U}$ invariant. Let T be α-invariant, i.e., $T_k(\alpha(x)) = T_k(x)$ for all k and for all x. Then α induces isometric actions of $\mathbb{R}$ on each of $\mathcal{U}_{(k)}, \mathcal{U}_\tau$ and $\mathcal{U}_T$. Let B be as above. Hence the crossed product Frechet *-algebras $L^1(\mathbb{R}, B^\infty, \alpha)$, $L^1(\mathbb{R}, B, \alpha)$, $S(\mathbb{R}, B, \alpha)$ and $S(\mathbb{R}, B^\infty, \alpha)$ are defined. Theorem 2 and Lemma 6.1 give the following, which is Theorem 3(a).

COROLLARY 6.3

$RK_*(S(\mathbb{R}, B^\infty, \alpha)) = RK_*(S(\mathbb{R}, B, \alpha)) = RK_*(C^*(\mathbb{R}, A, \alpha)) = K_{*+1}(A)$.

Now let $\tilde{\mathcal{U}}$ be the completion of $\mathcal{U}$ in the family $\mathcal{F}$ of all α-invariant differential *-norms on $\mathcal{U}$. Then $\tilde{\mathcal{U}}$ is a complete locally m-convex *-algebra admitting a continuous surjective *-homomorphism $\Psi\colon \tilde{\mathcal{U}} \to A$. This *$\alpha$-invariant smooth envelope* $\tilde{\mathcal{U}}$ is different from the smooth envelope defined in [5], and it need not be a subalgebra of A.

Lemma 6.4. *Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $\tilde{\mathcal{U}}$ is a hermitian Q-algebra, $E(\tilde{\mathcal{U}}) = A$, and $K_*(\tilde{\mathcal{U}}) = K_*(A)$.*

This supplements a comment in p. 279 of [5] that $K_*(A) = \dot{K}_*(\mathcal{U}_1)$ where $\mathcal{U}_1$ is the completion of $\mathcal{U}$ in all, not necessarily α-invariant nor closable, differential seminorms.

Proof. Since $\tilde{\mathcal{U}} = \lim_{\leftarrow} \mathcal{U}_\tau$, we have $E(\tilde{\mathcal{U}}) = \lim_{\leftarrow} E(\mathcal{U}_\tau) = A$; and $\tilde{\mathcal{U}}$ admits greatest continuous C^*-seminorm, say $p_\infty(\cdot)$ [1]. It is easily seen that for any $x \in \tilde{\mathcal{U}}$, the spectral radius in $\tilde{\mathcal{U}} r(x) \leq p_\infty(x)$; and $\tilde{\mathcal{U}}$ is a hermitian Q-algebra. This implies, in view of $\tilde{E}(\tilde{\mathcal{U}}) = A$, that the spectrum in $\tilde{\mathcal{U}} \mathrm{sp}(x) = \mathrm{sp}_A(j(x))$ for all x in $\tilde{\mathcal{U}}$, where $j(x) = x + \mathrm{srad}\, \tilde{\mathcal{U}}$.

It follows from Lemma 6.2 that $K_*(A) = K_*(E(A))$. Hence Lemma 6.4 follows. □

Now the action α induces an isometric action of $\mathbb{R}$ on $\tilde{\mathcal{U}}$, with the result that the crossed product algebras $S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ and $L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)$ are defined and are complete locally m-convex *-algebras with a C^*-enveloping algebras satisfying

$$\begin{aligned} E(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) &= E(L^1(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) \\ &= C^*(\mathbb{R}, E(\tilde{\mathcal{U}}), \alpha) \\ &= C^*(\mathbb{R}, A, \alpha). \end{aligned}$$

Theorem 2 quickly gives the following which is Theorem 3(b).

COROLLARY 6.5

Assume that $\tilde{\mathcal{U}}$ is metrizable. Then $RK_*(S(\mathbb{R}, \tilde{\mathcal{U}}, \alpha)) = K_{*+1}(A)$.

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