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Complete positivity, tensor products and C*-nuclearity for inverse limits of C*-algebras

SUBHASH J BHATT and DINESH J KARIA
Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar 388 120, India

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Abstract. The paper aims at developing a theory of nuclear (in the topological algebraic sense) pro-C^*-algebras (which are inverse limits of C^*-algebras) by investigating completely positive maps and tensor products. By using the structure of matrix algebras over a pro-C^*-algebra, it is shown that a unital continuous linear map between pro-C^*-algebras A and B is completely positive iff by restriction, it defines a completely positive map between the C^*-algebras $b(A)$ and $b(B)$ consisting of all bounded elements of A and B. In the metrizable case, A and B are homeomorphically isomorphic iff they are matricially order isomorphic. The injective pro-C^*-topology α and the projective pro-C^*-topology ν on $A \otimes B$ are shown to be minimal and maximal pro-C^*-topologies; and α coincides with the topology of biequicontinous convergence iff either A or B is abelian. A nuclear pro-C^*-algebra A is one that satisfies, for any pro-C^*-algebra (or a C^*-algebra) B, any of the equivalent requirements; (i) $\alpha = \nu$ on $A \otimes B$ (ii) A is inverse limit of nuclear C^*-algebras (iii) there is only one admissible pro-C^*-topology on $A \otimes B$ (iv) the bounded part $b(A)$ of A is a nuclear C^*-algebra (v) any continuous complete state map $A \to B^*$ can be approximated in simple weak* convergence by certain finite rank complete state maps. This is used to investigate permanence properties of nuclear pro-C^*-algebras pertaining to subalgebras, quotients and projective and inductive limits. A nuclearity criterion for multiplier algebras (in particular, the multiplier algebra of Pedersen ideal of a C^*-algebra) is developed and the connection of this C^*-algebraic nuclearity with Grothendieck's linear topological nuclearity is examined. A σ-C^*-algebra A is a nuclear space iff it is an inverse limit of finite dimensional C^*-algebras; and if abelian, then A is isomorphic to the algebra (pointwise operations) of all scalar sequences.

Keywords. Inverse limits of C^*-algebras; completely positive maps; tensor products; nuclear C^*- and nuclear pro-C^*-algebras; multiplier algebras; nuclear space.

1. Introduction and Preliminaries

A *topological *algebra A* is an involutive linear associative algebra (with identity 1) over complex scalars admitting a Hausdorff topology such that A is a topological vector space in which the multiplication and the involution are continuous. A *pro-C^*-algebra* is a complete topological *algebra A the topology on which is determined by the collection $S(A)$ of all continuous C^*-seminorms on it; equivalently, A is homeomorphically *isomorphic to an inverse limit of C^*-algebras. A *σ-C^*-algebra* is a metrizable pro-C^*-algebra. Besides an intrinsic interest in pro-C^*-algebras as topological algebras ([1], [11], [13], [14], [19], [20] and references therein), it has been shown recently that they provide an important tool in investigation of certain aspects of C^*-algebras (like multipliers of the Pedersen ideal, tangent algebra of a C^*-algebra, cross-product and K-theory, as well as non-commutative algebraic topology) [23], [24], pseudodifferential operators [8] and quantum field theory [9]. In the literature, inverse limits of C^*-algebras have been given different names such as b^*-algebras, m-convex-C^*-algebras or LMC*-algebras; the more appropriate pro(jective limits of) C^*-algebras is a recent suggestion [23], [24] following [28].

The present paper aims at developing a theory of nuclear pro-C^*-algebras; and this requires investigating tensor products of such algebras and complete positivity of linear maps. The significance of this has been noted in [23, p. 175]. Unlike C^*-algebras and locally convex spaces, there are at least two concepts of nuclearity for pro-C^*-algebras, viz. nuclearity in topological algebraic sense (an extension of C^*-nuclearity [5], [6], [17]), and Grothendieck's linear topological nuclearity [26, Chapter III, § 7]. Except for a final remark, we mean the former.

Given a pro-C^*-algebra A, each $p \in S(A)$ determines a C^*-algebra $A_p = A/N_p$ ($N_p = \{x \in A \mid x_p = 0\}$, a *ideal in A), with C^*-norm $\|x_p\|_p = p(x)$ where $x_p = x + N_p$. The point is that $(A_p, \|\cdot\|_p)$ is complete [1], [27]; and A admits an inverse limit decomposition $A = \varprojlim_{p \in \Delta} A_p$, Δ being a confinal subset of $S(A)$. The *bounded part* of A is the *subalgebra $b(A) = \{x \in A \mid \sup_{p \in S(A)} p(x) < \infty\}$, a C^*-algebra with norm $\|x\|_\infty = \sup_{p \in S(A)} p(x)$ continuously embedded in A [23]. A crucial fact about a pro-C^*-algebra is that $b(A)$ is dense in A [1], [23]; and this, in fact, characterizes pro-C^*-algebras [2]. Let $M_n(\mathbb{C})$ be the C^*-algebra of all $n \times n$ matrices. Given pro-C^*-algebras A and B, a linear map $\phi: A \to B$ is *completely positive* if for each n, $\phi_n = \phi \otimes id: A \otimes M_n(\mathbb{C}) = M_n(A) \to B \otimes M_n(\mathbb{C}) = M_n(B)$ is positive. In § 2, by obtaining a pro-C^*-analogue of Stinespring's Theorem, it is shown that a continuous ϕ is completely positive iff $\phi(b(A)) \subset b(B)$ and $\phi: b(A) \to b(B)$ is a completely positive map between C^*-algebras. This is used to show that σ-C^*-algebras A and B are homeomorphically *isomorphic iff they are matricially order isomorphic. In § 3, the tensor product of pro-C^*-algebras A and B is investigated. Two standard pro-C^*-topologies on $A \otimes B$ are introduced in [13], viz. the *injective tensorial pro-C^*-topology* α and the *projective tensorial pro-C^*-topology* v, which correspond respectively to the least C^*-norm $\|\cdot\|_{\min}$ and the greatest C^*-norm $\|\cdot\|_{\max}$ in case of C^*-algebras [27, Chapter IV]. It is shown in Theorem 3.2 that any admissible pro-C^*-topology τ (in the sense of [13, Definition 2.1]) on $A \otimes B$ satisfies $\alpha \leqslant \tau \leqslant v$; and if either A or B is abelian, then $\alpha = \tau = v = \varepsilon$, ε being the topology of biequicontinuous convergence. (In fact, a slightly stronger result is proved, see Remark 3.3). This improves [13, Proposition 3.1], wherein the conclusion $\tau \leqslant v$ has been obtained under the additional assumptions that both A and B are metrizable and the completion $A \hat{\otimes}_\tau B$ is a Q-algebra (i.e., the invertible elements form an open set). The other half of our Theorem also improves [13, Theorem 3.1] wherein the conclusion $\alpha = \varepsilon$ is obtained under the assumptions that A and B are Q-algebras one being barreled and the other commutative. In fact, a pro-C^*-algebra that is a Q-algebra is a C^*-algebra [23, Proposition 1.14]; and hence the relevant results in [13] dealing with Q-pro-C^*-algebras are just the usual C^*-algebra results. Thus our result provides a complete analogue of [27, Chapter IV, Theorem 4.19] modulo the problem whether any pro-C^*-topology on $A \otimes B$ is necessarily admissible. Further, it is shown in Theorem 3.4 that the above conclusion $\varepsilon = \alpha = \tau = v$ characterizes commutativity of either A or B. This extends [27, Chapter IV, Theorem 4.14]. Continuous states on $A \hat{\otimes}_v B$ are shown to correspond to continuous completely positive maps $A \to B^*$($=$ the dual). The machinery developed in § 2 and § 3 is used to develop a theory of nuclear pro-C^*-algebras in § 4. Following [23, p. 175], a pro-C^*-algebra A is called *nuclear* if for each $p \in S(A)$, the C^*-algebra A_p is nuclear in the sense [17] that for any C^*-algebra B, $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A_p \otimes B$. It is shown that A is nuclear iff A is an inverse limit of nuclear C^*-algebras (with the maps of the inverse system assumed surjective) iff for any pro-C^*-algebra (respectively, C^*-algebra) B, there is only one admissible pro-C^*-topology on $A \otimes B$. A basic result in C^*-theory is that a C^*-algebra A is nuclear iff A^{**} (the second dual) is semidiscrete. Using this via the universal representation, it is shown in the main Theorem 4.5 that a pro-C^*-algebra A is nuclear iff $b(A)$ is a nuclear C^*-algebra. This is used to investigate permanence properties of nuclear pro-C^*-algebras pertaining to hereditary subalgebras, quotients and products, as well as projective and inductive limits. By using the structure of continuous state space of an m-convex *algebra [4], it is shown in Theorem 4.11 that if A is nuclear, then for any pro-C^*-algebra B, any continuous complete state map $A \to B^*$ can be approximated in simple weak* convergence by continuous complete state maps of finite ranks; and a weaker converse holds. This is a pro-C^*-analogue of a basic result of Lance [16]. To illustrate an application, in § 5 a nuclearity criterion for the multiplier algebra of the Pedersen ideal of a C^*-algebra [18] is obtained. For this, it is shown that for a pro-C^*-algebra A, its multiplier algebra $M(A)$ [23] is nuclear pro-C^*-algebra iff $b(A)$ and the generalized Calkin algebra $M(b(A))/b(A)$ are nuclear C^*-algebras. Finally, we examine the interrelation between Grothendieck's linear topological nuclearity and nuclearity in the sense of the present paper. In fact, a σ-C^*-algebra A is linear topologically nuclear iff it is an inverse limit of finite dimensional C^*-algebras; and further, if abelian, it is isomorphic to the algebra of all scalar sequences with pointwise operations and pointwise convergence.

2. Completely positive maps

Let A be a pro-C^*-algebra. Let $M_n(A)$ denote the *algebra of all $n \times n$ matrices over A, with the usual algebraic operations and the topology obtained by regarding it as a direct sum of n^2 copies of A. Since $A = \varprojlim A_p$, as p runs through $S(A)$, we have $M_n(A) = \varprojlim_{p \in S(A)} M_n(A_p)$. Thus $M_n(A)$ is a pro-C^*-algebra. Algebraically, $M_n(A) = A \otimes M_n(\mathbb{C})$; and since $M_n(\mathbb{C})$ is finite dimensional, all tensor topologies on $M_n(A)$ agree.

Lemma 2.1. *$b(M_n(A)) = M_n(b(A)) = b(A) \otimes M_n(\mathbb{C})$ as C^*-algebras.*

Proof. We only have to prove the first equality. If A is a C^*-algebra, and $a = [a_{ij}] \in M_n(A)$, it is easily seen that $\max_{i,j} \|a_{ij}\| \leqslant \|a\| \leqslant \Sigma_{i,j} \|a_{ij}\|$. Therefore, this inequality holds for any continuous C^*-seminorm p on a pro-C^*-algebra A. Taking supremum over all p yields $\max_{i,j} \sup_p p(a_{ij}) \leqslant \sup_p p(a) \leqslant \Sigma_{i,j} \sup_p p(a_{ij})$. Thus a is bounded iff each a_{ij} is bounded. The norms on $M_n(b(A))$ and $b(M_n(A))$ must now agree by the uniqueness of C^*-norms. This completes the proof.

It is easily seen that for $a = [a_{ij}] \in M_n(A)$, $a \geqslant 0$ iff a is a sum of matrices of the form $[a_i^* a_j]$ for a_i, a_j in A iff for all $x_1, \ldots, x_n$ in A, $\Sigma_{i,j} x_i^* a_{ij} x_j \geqslant 0$ in A. A linear map $\phi: A \to B$ between pro-C^*-algebras is *completely positive* if for all $n = 1, 2, 3, \ldots$, the linear maps $\phi_n: M_n(A) \to M_n(B)$, $\phi_n[a_{ij}] = [\phi(a_{ij})]$ are positive. Thus ϕ is completely positive iff for each $q \in S(B)$, $\phi_n: A \to B_q$; $\phi_q(x) = \phi(x) + N_q$ is completely positive. If ϕ is a continuous positive linear map, then it is easily seen that ϕ is completely positive if either A or B is abelian. Following is an analogue of Stinespring's Theorem [27, Chapter IV, Theorem 3.6].

Theorem 2.2. *Let A be a pro-C^*-algebra. Let $B(H)$ denote the C^*-algebra of all bounded linear operators on H.*

(i) *If $\pi:A\to B(K)$ is a continuous representation of A, $V:H\to K$ (a Hilbert space) is a bounded linear operator, then $\phi:A\to B(H)$, $\phi(x)=V^*\pi(x)V$ is a continuous completely positive map.*
(ii) *If $\phi:A\to B(H)$ is a continuous completely positive map, then there exists a Hilbert space K, a continuous representation $\pi:A\to B(K)$, a normal representation $\rho:\phi(A)'\to B(K)$ ($\phi(A)'=$ commutant of $\phi(A)$) and a bounded linear operator $V:H\to K$ such that $\phi(a)=V^*\pi(a)V$, $\rho(x)V=Vx(x\in\phi(A)')$, $\rho(\phi(A)')\subset\pi(A)'$ and $K=[\pi(A)VH]$, closed linear span of $\pi(A)VH$.*

(i) is a straightforward verification. For (ii), continuity of ϕ implies that there exists $p\in S(A)$ such that $\|\phi(a)\|\leqslant Mp(a)$ for all $a\in A$, for some $M>0$. Thus $\phi_p:A_p\to B(H)$, $\phi_p(x_p)=\phi(x)$, $x_p=x+N_p$ is a well defined completely positive map between C^*-algebras to which the C^*-algebras Stinespring's Theorem applies.

COROLLARY 2.3

Let A and B be pro-C^*-algebras.

(i) *A unital continuous linear map $\phi:A\to B$ is completely positive (respectively positive) iff $\phi(b(A))\subset b(B)$ and $\phi(b(A))\to b(B)$ is a completely positive (respectively positive) map between C^*-algebras. If ϕ is completely positive, then $\phi(a)^*\phi(a)\leqslant\phi(a^*a)$ for all $a\in A$.*
(ii) *A is homeomorphically *isomorphic to B iff there exits a unital continuous bijective completely positive map $\phi:A\to B$ such that ϕ^{-1} is continuous and completely positive. In particular, if A and B are σ-C^*-algebras, then A is homeomorphically *isomorphic to B iff A and B are matricially order isomorphic.*

Proof. (i) Let $\phi:A\to B$ be positive (in particular, completely positive). We show that $\phi(b(A))\subset b(B)$. Let $a\in b(A)$ be positive, and so $0\leqslant a\leqslant\|a\|_\infty 1$. Hence $0\leqslant\phi(a)\leqslant\|a\|_\infty\phi(1)=\|a\|_\infty$. Thus $\phi(a)\in b(B)$. Now $\phi(b(A))\subset b(B)$ follows by writing an arbitrary element as a linear combination of positive ones. Conversely, let given $\phi:A\to B$ be such that $\phi(b(A))\subset b(B)$ and $\phi:b(A)\to b(B)$ is completely positive. In view of Lemma 3.4, for all n, $\phi_n(b(M_n(A)))=\phi_n(M_n(b(A)))\subset M_n(b(B))=b(M_n(B))$ and ϕ_n: $b(M_n(A))\to b(M_n(B))$ are positive linear maps, continuous in the relative topologies from $M_n(A)$ and $M_n(B)$. Now $b(A)$ is dense in A, in fact any $h\geqslant 0$ in A can be approximated by the sequence $h_k=h(1+(1/k)h^2)^{-1}\geqslant 0$ in $b(A)$. Applying this to each $M_n(A)$ and using $b(M_n(A))=M_n(b(A))$, each $\phi_n:M_n(A)\to M_n(B)$ is positive.

Let $\phi:A\to B$ be completely positive. For each $q\in S(B)$, $\phi_q=\pi_q\circ\phi:A\to B_q(\pi_q:B\to B_q$ being $\pi_q(y)=(y+N_q))$ is a continuous completely positive map. Identifying B_q with C^*-subalgebra of $B(H)$ for some Hilbert space H, Theorem 2.2 implies that $\phi_q=V_q^*\sigma_q(a)V_q$ $(a\in A)$ for some continuous *homomorphism $\sigma_q:A\to B(K)$ (for some Hilbert space K) and some bounded linear map $V_q:H\to K$. Since $\phi_q(1)=1$, it is seen as in [27, Chapter IV, Remark 3.7] that V_q is an isometry. Thus $\pi_q(\phi(a)^*\phi(a))=(\pi_q(\phi(a)))^*(\pi_q(\phi(a)))=\phi_q(a)^*\phi_q(a)=V_q^*\sigma_q(a)V_qV_q^*\sigma_q(a)V_q\leqslant\|V_q\|^2V_q^*\sigma_q(a^*a)V_q\leqslant$

$\phi_q(a^*a) \leqslant \phi(a^*a)$ for all q in $S(B)$. It follows that $\phi(a)^*\phi(a) \leqslant \phi(a^*a)$ for all $a \in A$. This completes the proof of (i).

For the proof of (ii), we shall need the following. A self adjoint unital linear map $\phi: A \to B$ is a *C^*-homomorphism* if $\phi(h^2) = \phi(h)^2$ for all $h = h^*$ in A.

Lemma 2.4. *Let $\phi: A \to B$ be a continuous bijective C^*-homomorphism between pro-C^*-algebras such that for each $n \geqslant 2$, $\phi_n: M_n(A) \to M_n(B)$ is also a C^*-homomorphism. Then ϕ is a *isomorphism (not necessarily a homeomorphism).*

Proof of lemma. A C^*-homomorphism $\phi: A \to B$ being a positive map, Corollary 2.3 (i) implies that $\phi(b(A)) \subset b(B)$. Applying this to ϕ and ϕ^{-1}, it follows that $\phi(b(A)) = b(B)$ and $\phi: b(A) \to b(B)$ is bijective C^*-isomorphism between C^*-algebras. The same arguments show, in view of Lemma 2.1, that for each n, $\phi_n: b(M_n(A)) = M_n(b(A)) \to M_n(b(B)) = b(M_n(B))$ is also C^*-isomorphism. Thus by [27, Example 1, p. 202], $\phi: b(A) \to b(B)$ is a *isomorphism. Now density of $b(A)$ in A, joint continuity of multiplication in a pro-C^*-algebra and continuity of ϕ implies that $\phi: A \to B$ is a *isomorphism.

Proof of part (*ii*) *of Corollary* 2.3. The inequality in part (i) applied to the completely positive maps ϕ and ϕ^{-1} shows that $\phi(a)^*\phi(a) = \phi(a^*a)$ for all $a \in A$. In particular, ϕ is a C^*-isomorphism. By the same arguments, each ϕ_n is a C^*-isomorphism; and hence ϕ is a *isomorphism by above Lemma. The remaining assertions are trivial.

We note the following consequence of the fact that a positive linear functional on a complete locally m-convex *algebra with 1 maps a bounded set to a bounded set [7].

PROPOSITION 2.5

A positive linear map $\phi: A \to B$ from pro-C^*-algebra A to pro-C^*-algebra B maps a bounded set to a bounded set. If A is a σ-C^*-algebra, then ϕ is continuous.

3. Tensor products

Given pro-C^*-algebras A and B, we are concerned with the following four topologies on the tensor product $A \otimes B$. For a locally convex topology τ on $A \otimes B$, $A \hat{\otimes}_\tau B$ denotes the completion of $(A \otimes B, \tau)$.

(i) [26] projective tensorial topology π: For $p \in S(A)$, $q \in S(B)$, $(p \otimes q)(z) = \inf\{\Sigma_i p(x_i) q(y_i) | z = \Sigma_i x_i \otimes y_i$ in $A \otimes B$ with $x_i \in A$, $y_i \in B\}$; and π is the locally convex topology defined by the seminorms $\{p \otimes q | p \in S(A), q \in S(B)\}$. Each $p \otimes q$ is a submultiplicative seminorm satisfying $(p \otimes q)(z^*) = (p \otimes q)(z)$. Thus $A \hat{\otimes}_\pi B$ is a locally m-convex *algebra, though not a pro-C^*-algebra in general.

(ii) [26] topology ε **of biequicontinuous convergence**: For $p \in S(A)$, $q \in S(B)$, let $U_p^0(1) = \{f \in A^* \,||f(x)| \leqslant 1$ for all x such that $p(x) \leqslant 1\}$, $U_q^0(1) = \{g \in B^* \,||g(y)| \leqslant 1$ for all y such that $q(y) \leqslant 1\}$, A^* and B^* denoting topological duals of A and B respectively. For $z = \Sigma x_i \otimes y_i$ in $A \otimes B$, define $\varepsilon_{p,q}(z) = \sup\{|\Sigma f(x_i) g(y_i)| \,|f \in U_p^0(1),\ g \in U_q^0(1)\}$. The family of seminorms $\{\varepsilon_{p,q} | p \in S(A),\ q \in S(B)\}$ (called the *natural ε-calibration*) defines the biprojective tensorial topology ε, also called the topology of biequicontinuous convergence, i.e., the topology of uniform convergence on sets of form $S \otimes T$ where

$S \subset A^*$, $T \subset B^*$ are equicontinuous. The seminorms $\varepsilon_{p,q}$ need not be submultiplicative, and $A \hat{\otimes}_\varepsilon B$ need not be an algebra.

(iii) [13] projective tensorial pro-C*-topology ν: By a bounded representation (π, H_π) of a *algebra K is meant to be a *homomorphism π of K into the C^*-algebra $B(H_\pi)$. Let $R(K)$ be the collection of all continuous bounded representations of a topological *algebra K. Then $R(K) = \cup \{R_s(K) | s \in S(K)\}$, $R_s(K) = \{\pi \in R(K) | \|\pi(x)\| \leqslant s(x)$ for all $x \in K\}$. Now for $p \in S(A)$, $q \in S(B)$, let $R_{p,q}(A \hat{\otimes}_\pi B) = \{\sigma \in R(A \hat{\otimes}_\pi B) | \|\sigma(z)\| \leqslant (p \otimes q)(z)$ for all $z \in A \otimes B\}$. Define a C^*-seminorm $\nu_{p,q}(z) = \sup\{\|\sigma(z)\| | \sigma \in R_{p,q}(A \hat{\otimes}_\pi B)\}$. The projective tensorial pro-C^*-topology ν is the topology defined by the C^*-seminorms $\{\nu_{p,q} | p \in S(A), q \in S(B)\}$.

(iv) [13] injective tensorial pro-C*-topology α: In the above notations let $t_{p,q}(z) = \sup\{\|\pi \otimes \sigma(z)\| | \pi \in R_p(A), \sigma \in R_q(B)\}$, a C^*-seminorm on $A \otimes B$. The topology α is defined by the C^*-seminorms $\{t_{p,q} | p \in S(A), q \in S(B)\}$.

Note that $\varepsilon \leqslant \alpha \leqslant \nu \leqslant \pi$; and for C^*-algebras A and B, the topologies ε, α, ν and π reduce to the topologies respectively due to the injective cross norm $\|\cdot\|_\lambda$, the minimal C^*-norm $\|\cdot\|_{\min}$, the maximal C^*-norm $\|\cdot\|_{\max}$ and the projective cross norm $\|\cdot\|_\gamma$ [27, Chapter IV].

Lemma 3.1. *For pro-C^*-algebras A and B, the following hold*

(i) $A \hat{\otimes}_\nu B = \varprojlim_{p,q} A_p \hat{\otimes}_{\max} B_q$ (ii) $A \hat{\otimes}_\alpha B = \varprojlim_{p,q} A_p \hat{\otimes}_{\min} B_q$

(iii) $A \hat{\otimes}_\pi B = \varprojlim_{p,q} A_p \hat{\otimes}_\gamma B_q$ (iv) $A \hat{\otimes}_\varepsilon B = \varprojlim_{p,q} A_p \hat{\otimes}_\lambda B_q$.

Given pro-C^*-algebras A and B, a Hausdorff topology τ on $A \otimes B$ is an admissible topology [13] if: (i) $(A \otimes B, \tau)$ is a locally m-convex *algebra; i.e., there exists a family $\Gamma = \{r_\alpha | \alpha \in \Delta\}$ of submultiplicative seminorms satisfying $r_\alpha(z^*) = r_\alpha(z)$ for all z such that Γ determines τ. (ii) for each $\alpha \in \Delta$, there exist $p \in S(A)$, $q \in S(B)$ such that $r_\alpha(x \otimes y) \leqslant p(x)q(y)$ for all $x \in A$, $y \in B$; and (iii) given equicontinuous subsets $M \subset A^*$, $N \subset B^*$, $M \otimes N$ is an equicontinuous subset of $(A \hat{\otimes}_\tau B)$. A topology τ on a *algebra K is a pro-C^*-topology if the completion of (K, τ) is a pro-C^*-algebra. It is shown in [13, Proposition 3.1] that if A and B are σ-C^*-algebras and if τ is an admissible pro-C^*-topology on $A \otimes B$ such that $(A \otimes B, \tau)$ is a Q-algebra, then $\tau \leqslant \nu$. The following substantially improves this. It can also be regarded as an analogue of the C^*-algebra result [27, Chapter IV, Theorem 4.19] that if β is a C^*-norm on $A \otimes B$ for C^*-algebras A and B, then β is a cross norm satisfying $\|\cdot\|_{\min} \leqslant \beta(\cdot) \leqslant \|\cdot\|_{\max}$; and if either A or B is abelian, then $\beta(\cdot) = \|\cdot\|_\lambda = \|\cdot\|_{\min} = \|\cdot\|_{\max}$ [27, Chapter IV, Lemma 4.18]. The other half of the following also improves [13, Theorem 3.1] wherein it is shown that for Q-pro-C^*-algebras A and B one being barrelled and the other abelian, $\varepsilon = \alpha$ on $A \otimes B$. In fact, Proposition 3.1 and Theorem 3.1, both of [13] referred to above, reduce to be the usual C^*-algebra results, because, by [23, Proposition 1.14], a Q-pro-C^*-algebra is a C^*-algebra.

Theorem 3.2. *Let A and B be pro-C^*-algebras. Let τ be an admissible pro-C^*-topology on $A \otimes B$. Then $\alpha \leqslant \tau \leqslant \nu$. If either A or B is abelian, then $\varepsilon = \alpha = \tau = \nu$.*

Proof. Let $K_\tau = A \hat{\otimes}_\tau B$. Let (z_α) be a net in $A \otimes B$, $z_\alpha \to 0$ in τ. By [26, Chapter IV, p. 127], the topology of a locally convex space is the topology of uniform convergence on equicontinuous subsets of its dual. Hence $z_\alpha \to 0$ uniformly on every equicontinuous subset of $(A \hat{\otimes}_\tau B)^*$. In view of admissibility of τ, $z_\alpha \to 0$ uniformly on $M \otimes N$ for equicontinuous sets $M \subset A^*$, $N \subset B^*$. Hence $z_\alpha \to 0$ in ε by [26, Chapter III, p. 96], and

$$\varepsilon \leqslant \tau. \tag{1}$$

Now given $\gamma \in S(A \hat{\otimes}_\tau B)$, choose $p \in S(A)$, $q \in S(B)$ such that for all x, y, $\gamma(x \otimes y) \leqslant p(x)q(y)$. Hence for $z = \Sigma x_i \otimes y_i$ in $A \otimes B$, $\gamma(z) \leqslant \Sigma \gamma(x_i \otimes y_i) \leqslant \Sigma p(x_i)q(y_i)$. By definition of $p \otimes q$, $\gamma(z) \leqslant (p \otimes q)(z)$; and $\tau \leqslant \pi$. Thus

$$\varepsilon \leqslant \tau \leqslant \pi. \tag{2}$$

Again for a net (z_α) in $A \otimes B$, let $z_\alpha \to 0$ in ν, so that for each $p \in S(A)$, $q \in S(B)$, $\gamma_{p,q}(z_\alpha) \to 0$. Let $\gamma \in S(A \hat{\otimes}_\tau B)$. Then there exist p and q as in (2) above satisfying $\gamma(z) \leqslant (p \otimes q)(z)$ for all z. Since γ is a τ-continuous C^*-seminorm, a faithful representation of the C^*-algebra $(A \hat{\otimes}_\tau B)_\gamma = ((A \hat{\otimes}_\tau B)/\ker \gamma$ with C^*-norm induced by γ) as an operator algebra gives a τ-continuous bounded representation (σ, H_σ) of $A \hat{\otimes}_\tau B$ satisfying $\|\sigma(z)\| = \gamma(z) \leqslant p \otimes q(z)$ for all z. Thus $\sigma \in R_{p,q}(A \hat{\otimes}_\pi B)$. Hence $\gamma(z_\alpha) \leqslant \nu_{p,q}(z_\alpha) \to 0$ showing

$$\tau \leqslant \nu. \tag{3}$$

Finally, we show that $\alpha \leqslant \tau$. For given $p \in S(A)$, $q \in S(B)$, the sets $P_p(A) = \{f \in A^* \,|\, |f(x)| \leqslant p(x) \text{ for all } x \text{ in } A\} \subset A^*$ and $P_q(B) = \{g \in B^* \,|\, |g(y)| \leqslant q(y) \text{ for all } y \text{ in } B\} \subset B^*$ being equicontinuous, $P_p(A) \otimes P_q(B)$ is an equicontinuous subset of $(A \hat{\otimes}_\tau B)^*$. Hence there exist a $\gamma \in S(A \hat{\otimes}_\tau B)$ such that

$$P_p(A) \otimes P_q(B) \subset P_\gamma(A \hat{\otimes}_\tau B) = \{h \in (A \hat{\otimes}_\tau B)^* \,|\, |h(z)| \leqslant \gamma(z) \text{ for all } z\}. \tag{4}$$

Now for a $\pi \in R_p(A)$, $\xi \in H_\pi$, the linear functional $f_\xi^\pi(x) = \langle \pi(x)\xi, \xi \rangle / \|\xi\|^2$ satisfies $|f_\xi^\pi(x)| \leqslant p(x)$ for all $x \in A$; and the equicontinuous set M defined as $M = \{f_\xi^\pi \,|\, \pi \in R_p(A), \xi \in H_\pi\}$ satisfies $M \subset P_p(A)$. Similarly, $N = \{g_\eta^\sigma \,|\, \sigma \in R_q(B), \eta \in K_\sigma\} \subset P_q(B)$ where $g_\eta^\sigma(y) = \langle \sigma(y)\eta, \eta \rangle / \|\eta\|^2$. It follows from (4) that for all $(\pi, H_\pi) \in R_p(A)$, $(\sigma, K_\sigma) \in R_q(B)$, $\xi \in H_\pi$, $\eta \in K_\sigma$, $|f_\xi^\pi \otimes g_\eta^\sigma(z)| \leqslant \gamma(z)$ for all $z \in A \hat{\otimes}_\tau B$. Taking $z = \Sigma x_i \otimes y_i$ in $A \otimes B$, this implies that

$$\left| \sum_i \frac{\langle (\pi \otimes \sigma)(x_i \otimes y_i)(\xi \otimes \eta), (\xi \otimes \eta) \rangle}{\|\xi \otimes \eta\|^2} \right| = \left| \sum_i \frac{\langle \pi(x_i)\xi, \xi \rangle \langle \sigma(y_i)\eta, \eta \rangle}{\|\xi\|^2 \|\eta\|^2} \right|$$
$$= f_\xi^\pi \otimes g_\eta^\sigma(z) \leqslant \gamma(z).$$

Thus, for all $z \in A \otimes B$, $|\langle (\pi \otimes \sigma)(z)(\xi \otimes \eta), \xi \otimes \eta \rangle| \leqslant \gamma(z) \|\xi \otimes \eta\|^2$. It follows by the polarization identity, that for any ξ, ξ' in H_π and η, η' in K_σ, $|\langle \pi \otimes \sigma(z)(\xi \otimes \eta), \xi' \otimes \eta' \rangle| \leqslant \gamma(z) |\langle \xi \otimes \eta, \xi' \otimes \eta' \rangle|$. Now taking $\theta = \Sigma_i \xi_i \otimes \eta_i \in H_\pi \otimes K_\sigma$, and taking without loss of generality, (η_i) to be an orthonormal set in K_σ, we obtain

$$|\langle (\pi \otimes \sigma)(z)\theta, \theta \rangle| \leqslant \sum_{i,j} |\langle (\pi \otimes \sigma)(z)(\xi_i \otimes \eta_i), \xi_j \otimes \eta_j \rangle|$$
$$\leqslant \gamma(z) \sum_{i,j} |\langle \xi_i \otimes \eta_i, \xi_j \otimes \eta_j \rangle|$$
$$= \gamma(z) \sum_i \|\xi_i\|^2 = \gamma(z) \|\theta\|^2. \tag{5}$$

Since $\pi\otimes\sigma(z)$ is a bounded operator on the completed Hilbert space tensor product $H_\pi\bar{\otimes}K_\sigma$, it follows that (5) holds for all θ in $H_\pi\bar{\otimes}K_\sigma$. Thus, for $\pi\in R_p(A)$, $\sigma\in R_q(B)$, $z\in A\otimes B$ $\theta\in H_\pi\bar{\otimes}K_\sigma$,

$$\|\pi\otimes\sigma(z)\theta\|^2=\langle(\pi\otimes\sigma)(z^*z)\theta,\theta\rangle$$
$$\leqslant\gamma(z^*z)\|\theta\|^2=\gamma(z)^2\|\theta\|^2. \tag{6}$$

Hence for all $z\in A\otimes B$ (and so for all $z\in A\hat{\otimes}_\tau B$), $\|\pi\otimes\sigma(z)\|\leqslant\gamma(z)$, $\pi\otimes\sigma\in R_\gamma(A\hat{\otimes}_\tau B)$. Summarizing, we have shown that given $p\in S(A)$, $q\in S(B)$, there exists a $\gamma\in S(A\hat{\otimes}_\tau B)$ such that $R_p(A)\otimes R_q(B)\subset R_\gamma(A\otimes B)$.

Now if $z_\alpha\to 0$ in τ, it follows from above that for all $p\in S(A)$, $q\in S(B)$, $t_{p,q}(z_\alpha)\to 0$. Hence

$$\alpha\leqslant\tau. \tag{7}$$

Finally, assume that either A or B, say A, is abelian. Then for $p\in S(A)$, $q\in S(B)$, by standard C^*-algebra result [27, p. 212], $\lambda=\|\cdot\|_{\min}=\|\cdot\|_{\max}$ on $A_p\otimes B_q$. Lemma 3.1 implies that $A\hat{\otimes}_\alpha B=A\hat{\otimes}_\varepsilon B=A\hat{\otimes}_\nu B$ with $\alpha=\varepsilon=\nu$; in fact, $\varepsilon_{p,q}=t_{p,q}$. This completes the proof.

Remarks 3.3. (a) A closer examination of the proof reveals that we have, in fact, proved the following stronger assertions.

(i) Any admissible topology τ on $A\otimes B$ satisfies $\varepsilon\leqslant\tau\leqslant\pi$.
(ii) In the derivation of inequality (6), if γ is not a C^*-seminorm but a submultiplicative *seminorm only, then too $\|\pi\otimes\sigma(z)\theta\|^2\leqslant\gamma(z^*z)\|\theta\|^2\leqslant\gamma(z^*)\gamma(z)\|\theta\|^2\leqslant\gamma(z)^2\|\theta\|^2$ holds. Thus, for any admissible (not necessarily pro-C^*) topology τ on $A\otimes B$, $\alpha\leqslant\tau$; and α (respectively π) is the smallest (respectively finest) admissible locally m-convex *algebra topology on $A\otimes B$.
(iii) In general, $\nu<\pi$, since $A\hat{\otimes}_\nu B$ is, under certain circumstances, the enveloping pro-C^*-algebra [11] of the complete locally m-convex *algebra $A\hat{\otimes}_\pi B$ [13].

(b) It is easily seen from [27] that for C^*-algebras A and B, any C^*-topology on $A\otimes B$ is automatically an admissible topology. This suggests: Let τ be any pro-C^*-topology on the tensor product $A\otimes B$ of pro-C^*-algebras A and B. Is τ necessarily an admissible topology?

Our next result shows that the conclusion $\varepsilon=\alpha$ in abelian case of above theorem, in fact, characterizes commutativity of either A or B. It gives a complete pro-C^*-analogue of [27, Chapter IV, Theorem 4.14] and a recent result of Blecher [3]; viz. for C^*-algebras A and B, if $A\hat{\otimes}_\lambda B$ is a Banach algebra with norm λ, then either A or B is abelian. We call the natural ε-calibration $\Gamma=\{\varepsilon_{p,q}|p\in S(A),\ q\in S(B)\}$ an *m^*-calibration* if each $\varepsilon_{p,q}$ satisfies $\varepsilon_{p,q}(uv)\leqslant\varepsilon_{p,q}(u)\varepsilon_{p,q}(v)$, $\varepsilon_{p,q}(u^*)=\varepsilon_{p,q}(u)$ for all u,v in $A\otimes B$. The given pro-C^*-topologies on A and B are denoted by τ_A and τ_B respectively; and $b(A)$ and $b(B)$ carry C^*-topologies unless stated otherwise.

Theorem 3.4. *Let A and B be pro-C^*-algebras. The following are equivalent.*

(1) *Either A or B is abelian.*
(2) *Any continuous pure state ω on $A\hat{\otimes}_\alpha B$ is of form $\omega=\omega_1\otimes\omega_2$, ω_1 and ω_2 being continuous pure states on A and B respectively.*

(3) $A\hat{\otimes}_\varepsilon B = A\hat{\otimes}_\alpha B$ with $\varepsilon=\alpha$ *and the natural* ε*-calibration on* $A\hat{\otimes}_\varepsilon B$ *is an* m^**-calibration.*
(3′) *The natural* ε*-calibration on* $A\hat{\otimes}_\varepsilon B$ *is an* m^**-calibration.*
(4) $b(A)\hat{\otimes}_\lambda b(B) = b(A)\hat{\otimes}_{\min} b(B)$ *and* $\lambda = \|\cdot\|_{\min}$.
(4′) $b(A)\hat{\otimes}_\lambda b(B)$ *is a Banach algebra.*
(5) $A\hat{\otimes}_\varepsilon b(B) = A\hat{\otimes}_\alpha b(B)$ *with* $\varepsilon=\alpha$ *and the natural* ε*-calibration on* $A\hat{\otimes}_\varepsilon b(B)$ *is an* m^**-calibration.*
(5′) *The natural* ε*-calibration on* $A\hat{\otimes} b(B)$ *is an* m^**-calibration.*
(6) $b(A)\hat{\otimes}_\varepsilon B = b(A)\hat{\otimes}_\alpha B$ *with* $\varepsilon=\alpha$ *and the natural* ε*-calibration on* $b(A)\hat{\otimes}_\varepsilon B$ *is an* m^**-calibration.*
(6′) *The natural* ε*-calibration on* $b(A)\hat{\otimes}_\varepsilon B$ *is an* m^**-calibration.*

Lemma 3.5. *Let* R *be a pro-*C^**-algebra. Let* E *be *subalgebra of* R *containing the identity of* R *such that* E *is a Banach *algebra with some norm* $|\cdot|$. *Then* $E\subset b(A)$ *and on* E, $\|\cdot\|_\infty \leqslant |\cdot|$.

Proof. By continuity of involution in $(R,|\cdot|)$, we assume $|x^*|=|x|$ for all $x\in E$. Let $B=\{x\in E \,|\, |x|\leqslant 1\}$. Then B is an absolutely convex *idempotent containing the identity 1 of R. By standard Banach *algebra arguments, every positive linear functional f on R, restricted to E, is $|\cdot|$-continuous satisfying $|f(x)|\leqslant f(1)|x|$. Since the dual R^* of R is a complex linear span of continuous positive functionals, B turns out to be $\sigma(R,R^*)$ bounded, hence bounded in the topology of R. Now, in R, it is easy to verify that $K=\{x\in R \,|\, \|x\|_\infty\leqslant 1\}$ is the largest (under inclusion) bounded absolutely convex *idempotent. Thus $B\subset K$, $E\subset b(R)$ and $\|\cdot\|_\infty\leqslant|\cdot|$ on E.

Proof of theorem. (1)$\Rightarrow$(2). First of all, for any A and B, not necessarily abelian, let $K(A\hat{\otimes}_\alpha B)$ be the set of all continuous states on $A\hat{\otimes}_\alpha B$. For each $j=(p,q)\in S(A)\times S(B)$, Let $U_j=\{z\in A\hat{\otimes}_\alpha B \,|\, t_{p,q}(z)\leqslant 1\}$, R_j be the C^*-algebra $A\hat{\otimes}_\alpha B/\ker t_{p,q}$ with the C^*-norm induced by $t_{p,q}$. Then R_j is *isomorphic to the C^*-algebra $A_p\hat{\otimes}_{\min} B_q$. Let $K_j(A\hat{\otimes}_\alpha B)=\{f\in K(A\hat{\otimes}_\alpha B) \,|\, f$ is bounded on $U_j\}$. Then from [4], the following hold.

(a) $K(A\hat{\otimes}_\alpha B) = U_j K_j(A\hat{\otimes}_\alpha B)$.
(b) For the sets of extreme points, $E(K(A\hat{\otimes}_\alpha B)) = U_j E(K_j(A\hat{\otimes}_\alpha B))$.
(c) $K_j(A\hat{\otimes}_\alpha B)$ is in bijective correspondence with $K(R_j)$ under the map $f\in K_j(A\hat{\otimes}_\alpha B)\to f_j$, $f_j(z_j)=f(z)$ where for $z\in A\otimes B$, $z_j\in R_j$ is $z_j=z+\ker t_{p,q}$; and this correspondence preserves the weak* topologies.

Now assume (1) say B is abelian. For all q in $S(B)$, the C^*-algebras B_q are abelian. Let ω be a continuous pure state on $A\hat{\otimes}_\alpha B$, so that, by [4, Corollary 4.3], ω is an extreme point of $K(A\hat{\otimes}_\alpha B)$. By above, there exists a $j=(p,q)\in S(A)\times S(B)$ such that ω_j is a pure state on $R_j=A_p\hat{\otimes}_{\min} B_q$. The C^*-algebra result [27, p. 211] gives pure states ω_1' of A_p and ω_2' of B_q satisfying $\omega_j=\omega_1'\otimes\omega_2'$. Again from above and [4, Theorem 4.3], $\omega_1(x)=\omega_1'(x_p)$, $\omega_2(y)=\omega_2'(y_q)$ define continuous pure states on A and B satisfying $\omega=\omega_1\otimes\omega_2$.
(2)$\Rightarrow$(1). This can be shown exactly as in [27, Chapter IV, Theorem 4.14] by using the facts [11] that a pro-C^*-algebra R admits sufficiently many bounded continuous irreducible representations. In fact, for each $s\in S(R)$, an irreducible representation π' of the C^*-algebra R_s gives an irreducible representation π of R as $\pi(z)=\pi'(z_s)$.
(1)$\Rightarrow$(3). This is already shown in Theorem 3.2 wherein $\varepsilon_{p,q}=t_{p,q}$ has been observed, i.e., the natural ε-calibration consists of C^*-seminorms. Also, (3)$\Rightarrow$(3′), (4)$\Rightarrow$(4′), (5)$\Rightarrow$(5′) and (6)$\Rightarrow$(6′) are trivial. Assume (3′). In view of Lemma 3.1, for each (p,q)

in $S(A) \times S(B)$, $A_p \hat{\otimes}_\lambda B_q$ becomes a Banach algebra with the injective cross-norm λ. Hence by [3, Corollary 4, p. 123] either A_p or B_q is abelian; and so C^*-algebra arguments [27, Theorem 4.14, p. 211] gives $A_p \hat{\otimes}_\lambda B_q = A_p \otimes_{\min} B_q$ with $\lambda = \|\cdot\|_{\min}$. This gives (3). Similarly it follows that $(4') \Rightarrow (4)$, $(5') \Rightarrow (5)$ and $(6') \Rightarrow (6)$.

$(3) \Rightarrow (1)$. Observe the following.

(i) $b(A \hat{\otimes}_\alpha B)$ is a C^*-algebra with C^*-norm $\|z\|_{\infty,\alpha} = \sup\{t_{p,q}(z) | p \in S(A), q \in S(B)\}$.

(ii) Since the ε-calibration is an m^*-calibration, $A \hat{\otimes}_\varepsilon B$ is a complete locally m-convex *algebra; and its bounded part, defined as $b(A \hat{\otimes}_\varepsilon B) = \{z \in A \hat{\otimes}_\varepsilon B | \sup_{p,q} \varepsilon_{p,q}(z) < \infty\}$ is a Banach *algebra with norm $\|z\|_{\infty,\varepsilon} = \sup\{\varepsilon_{p,q}(z) | p \in S(A), q \in S(B)\} = \|z^*\|_{\infty,\varepsilon}$. Now assume $A \hat{\otimes}_\alpha B = A \hat{\otimes}_\varepsilon B$. We assert that $b(A \hat{\otimes}_\alpha B) = b(A \hat{\otimes}_\varepsilon B)$. Indeed, for all p, q in $S(A) \times S(B)$, $\varepsilon_{p,q} \leqslant t_{p,q}$ and so $\|\cdot\|_{\infty,\varepsilon} \leqslant \|\cdot\|_{\infty,\alpha}$ with the result $b(A \hat{\otimes}_\alpha B) \subset b(A \hat{\otimes}_\varepsilon B)$. Further, above (ii) in the light of Lemma 3.5, applied to the pro-C^*-algebra $A \hat{\otimes}_\alpha B$, implies that $b(A \hat{\otimes}_\varepsilon B) \subset b(A \hat{\otimes}_\alpha B)$ and $\|\cdot\|_{\infty,\varepsilon} = \|\cdot\|_{\infty,\alpha}$.

Further, $b(A) \otimes b(B) \subset b(A \hat{\otimes}_\alpha B)$ and the norm $\|\cdot\|_{\min}$ on $b(A) \otimes b(B)$ is [27, p. 207] $\|z\|_{\min} = \sup\{\|\pi \otimes \sigma(z)\| \,|\pi \in R(b(A)), \sigma \in R(b(B))\} \geqslant \|z\|_{\infty,\alpha}$ by definition of $\|\cdot\|_{\infty,\alpha}$. But $\|\cdot\|_{\min}$ is the smallest among all C^*-norms on $b(A) \otimes b(B)$ [27, Proposition 4.19, p. 216]. Hence $\|\cdot\|_{\min} = \|\cdot\|_{\infty,\alpha} = \|\cdot\|_{\infty,\varepsilon}$ on $b(A) \otimes b(B)$ and so on $b(A) \hat{\otimes}_{\min} b(B)$. Finally, the λ-norm on $b(A) \otimes b(B)$ is [27, p. 188], with $z = \Sigma x_i \otimes y_i$, $\lambda(z) = \sup\{|\Sigma f(x_i) g(y_i)| \,| f \in b(A)^*, |f(X)| \leqslant \|x\|_\infty, g \in b(B)^*, |g(y)| \leqslant \|y\|_\infty\} \leqslant \|z\|_{\infty,\varepsilon}$ by definition of $\|\cdot\|_{\infty,\varepsilon}$.

Thus, on $b(A) \otimes b(B)$, $\|\cdot\|_{\infty,\varepsilon} \leqslant \lambda(\cdot) \leqslant \|\cdot\|_{\min} \leqslant \|\cdot\|_{\infty,\alpha} = \|\cdot\|_{\infty,\varepsilon}$ with the result that $\lambda(\cdot) = \|\cdot\|_{\min}$. It follows [27, Theorem 4.14, p. 211] that either $b(A)$ or $b(B)$ is abelian. But $b(A)$ is dense in A, $b(B)$ is dense in B and the multiplication in a pro-C^*-algebra is jointly continuous. It follows that either A or B is abelian. $(1) \Leftrightarrow (4)$ is a consequence of C^*-algebra theory together with density arguments as above. Similarly one gets $(1) \Leftrightarrow (5)$ and $(1) \Leftrightarrow (6)$ from $(1) \Leftrightarrow (3)$. This completes the proof.

Remark 3.6. In the above theorem, to conclude $(3) \Rightarrow (1)$ and similar other implications, the hypothesis that the ε-calibration is an m^*-calibration cannot be omitted, even for C^*-algebras. Take A to be a non-abelian finite dimensional C^*-algebra. Then for all C^*-algebras B, $A \hat{\otimes}_\varepsilon B = A \hat{\otimes}_\pi B$ and $\varepsilon = \pi$, i.e., λ and γ, and so λ and $\|\cdot\|_{\min}$ are equivalent. On the other hand $\lambda = \|\cdot\|_{\min}$ forces A or B to be abelian.

PROPOSITION 3.7

Let $\phi_1 : A_1 \to B_1$, $\phi_2 : A_2 \to B_2$ be continuous completely positive maps between pro-C^*-algebras. Then

$$\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_2, \quad (\phi_1 \otimes \phi_2)(x_1 \otimes x_2) = \phi_1(x_1) \otimes \phi_2(x_2)$$

is an $\alpha - \alpha$ continuous (respectively $\nu - \nu$ continuous) completely positive map that extends as a completely positive map $\phi_1 \otimes \phi_2 : A_1 \hat{\otimes}_\alpha A_2 \to B_1 \hat{\otimes}_\alpha B_2$ (respectively, $\phi_1 \otimes \phi_2 : A_1 \hat{\otimes}_\nu A_2 \to B_1 \hat{\otimes}_\nu B_2$).

Proof. We shall prove for the topology α. The other assertion can be similarly established. In view of Lemma 3.1, it is sufficient to find coherent continuous completely positive maps $\psi_{p,q} : A_1 \hat{\otimes}_\alpha A_2 \to (B_1)_{q_1} \hat{\otimes}_{\min} (B_2)_{q_2}$ for $(q_1, q_2) \in S(B_1) \times S(B_2)$. By continuity of ϕ_1 and ϕ_2, there exist $p_1 \in S(A_1)$, $p_2 \in S(A_2)$ such that $\|(\pi_1 \circ \phi_1)(x_1)\|_{p_1} \leqslant M p_1(x_1)$, $\|(\pi_2 \circ \phi_2)(x_2)\| \leqslant M p_2(x_2)$ for $x_i \in A_i$ where $\pi_i : B_i \to (B_i)_{q_i}$, $i = 1, 2$ are the natural quotient maps. It follows that the maps $\pi_i \circ \phi_i$ factor through the quotient C^*-algebras $(A_i)_{p_i} \to (B_i)_{q_i}$. By the corresponding property of C^*-algebras [27, p. 218], there is a completely positive map $\theta:(A_1)_{p_1} \hat{\otimes}_{\min}(A_2)_{p_2} \to (B_1)_{q_1} \hat{\otimes}_{\min}(B_2)_{q_2}$. The desired map $\psi_{p,q}$ is the composite map $A_1 \hat{\otimes}_\alpha A_2 \to (A_1)_{p_1} \hat{\otimes}_{\min}(A_2)_{p_2} \to (B_1)_{q_1} \hat{\otimes}_{\min} (B_2)_{q_2}$. This completes the proof.

PROPOSITION 3.8

Let A and B be pro-C^*-algebras. Let B^0 (respectively B^*) denote the algebraic dual (respectively topological dual) of B. For a linear functional ω on $A \otimes B$, let $T_\omega: A \to B^0$ be defined by $\langle y, T_\omega x\rangle = \langle x \otimes y, \omega\rangle$.

(1) ω is a state on $A \otimes B$ iff T_ω is a complete state map (i.e., T_ω is a completely positive map such that $T_\omega(1)$ is a state on B).
(2) Further, ω is continuous in the topology v (so that $\omega \in (A \hat{\otimes}_v B)^*$) iff $T_\omega(A) \subset B^*$ and $T_\omega: A \to B^*$ is continuous, where B^* carries the topology τ_b of uniform convergence on all bounded subsets of B.

Proof. (1) is exactly as in [27, Chapter IV, Proposition 4·6]. Further, let ω be v-continuous. There exist $(p, q) \in S(A) \times S(B)$ and a scalar $k > 0$ such that for all $z \in A \otimes B$, $|\omega(z)| \leqslant k v_{p,q}(z)$. Hence for all $x \in A$, $y \in B$, $|\langle y, T_\omega(x)\rangle| = |\langle x \otimes y, \omega\rangle| \leqslant k v_{p,q}(x \otimes y) \leqslant k p \otimes q(x \otimes y) \leqslant k p(x) q(y)$. Hence for $K \subset B$ bounded with $q(y) \leqslant M_{q,k}(y \in K)$, $|\langle y, T_\omega x\rangle| \leqslant k M_{q,k} p(x) (x \in A)$ showing $T_\omega A \subset B^*$ and T_ω is continuous in τ_b. Conversely, let $T_\omega: A \to B^*$ be τ_b continuous. By general theory of topological tensor products [26], $\omega \in (A \hat{\otimes}_\pi B)^*$. By the GNS construction [11, Theorem 3.4] on the complete locally m-convex *algebra $A \hat{\otimes}_\pi B$, there exists a continuous bounded representation $\pi_\omega: A \hat{\otimes}_\pi B \to B(H_\pi)$ on a Hilbert space H_π having a cyclic vector ξ_0 such that $\omega(z) = \langle \pi_\omega(z)\xi_0, \xi_0\rangle$ and $\|\pi_\omega(z)\| \leqslant (p \otimes q)(z) (z \in A \otimes B)$ for some $(p, q) \in S(A) \times S(B)$. Thus $\pi_\omega \in R_{p,q}(A \hat{\otimes}_\pi B)$; and for all z, $|\langle z, \omega\rangle| = |\langle \pi_\omega(z)\xi_0, \xi_0\rangle| \leqslant \|\pi_\omega(z)\| \|\xi_0\|^2 \leqslant v_{p,q}(z) \|\xi_0\|^2$ showing that ω is a continuous in the topology v.

4. Nuclear pro-C^*-algebras

Following a suggestion in [23], a pro-C^*-algebra is called *nuclear* if for each $p \in S(A)$, the C^*-algebra $(A_p, \|\cdot\|_p)$ is a nuclear C^*-algebra [16], [17] in the sense that for any C^*-algebra B, $\|\cdot\|_{\min} = \|\cdot\|_{\max}$ on $A_p \otimes B$. Thus a nuclear C^*-algebra is a nuclear pro-C^*-algebra; and a nuclear pro-C^*-algebra is an inverse limit of nuclear C^*-algebras. Commutative pro-C^*-algebras, the matrix algebra $M_n(A)$ over a nuclear pro-C^*-algebra A and a pro-C^*-algebra of type I (in the sense that all continuous bounded representations are of type I) are all nuclear pro-C^*-algebras.

PROPOSITION 4.1

Let a pro-C^*-algebra $A = \varprojlim_{\alpha \in \Delta} B_\alpha$, an inverse limit of C^*-algebras with the maps $\pi_\alpha: A \to B_\alpha$ of the inverse system assumed surjective. Then A is a nuclear pro-C^*-algebra iff each B_α is a nuclear C^*-algebra.

Proof. One way is obvious, since $A = \varprojlim_{p \in S(A)} A_p$. Conversely, let $A = \varprojlim_{\alpha \in \Delta} B_\alpha$ where each B_α is a nuclear C^*-algebra. Then the family $\{p_\alpha | \alpha \in \Delta\}$ of continuous C^*-seminorms determines the topology of A, where $p_\alpha(x) = \|\pi_\alpha(x)\|_\alpha$, π_α being the *homomorphism from A to the C^*-algebra $(B_\alpha, \|\cdot\|_\alpha)$. The B_α is *isomorphic to the C^*-algebra $A_\alpha = A/\ker p_\alpha$ with C^*-norm $\|x + \ker p_\alpha\|_{p_\alpha} = p_\alpha(x)$ [1, Theorem 2.4]. Thus A_α is nuclear. Given $p \in S(A)$, by continuity, there exists an $\alpha \in \Delta$ such that $p \leqslant p_\alpha$. Thus $\phi: A_\alpha \to A_p$, $\phi(x + \ker p_\alpha) = x + \ker p$ is a well defined continuous surjective *homomorphism, and A_p is *isomorphic to the quotient C^*-algebra $A_\alpha/\ker \phi$, which is nuclear, since A_α is nuclear [6], [17]. This completes the proof.

For pro-C^*-algebras A and B, the identity map $A \otimes B \to A \otimes B$ extends to a continuous surjective *homomorphism $\psi: A \hat{\otimes}_v B \to A \hat{\otimes}_\alpha B$. The following shows that A is nuclear iff ψ is a homeomorphism for all B.

Theorem 4.2. *For a pro-C^*-algebra A, the following are equivalent.*

(1) *A is nuclear.*
(2) *For all pro-C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*
(3) *For all C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*
(4) *For all pro-C^*-algebras B (respectively C^*-algebras B), there is only one admissible pro-C^*-topology on $A \otimes B$.*

Further, if A is a σ-C^*-algebra, then above are equivalent to any of the following.

(5) *For all σ-C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$.*
(6) *For all C^*-algebras B, $A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$.*
(7) *For all σ-C^*-algebras B (respectively C^*-algebras B), the topology v on $A \otimes B$ is faithful.*

An admissible topology τ on $A \otimes B$ is faithful [12] *if the map $i_\tau: A \hat{\otimes}_\tau B \to A \hat{\otimes}_\varepsilon B \subset B(A^*, B^*)$, $i_\tau(z) = (x' \otimes y')(z)$, $x' \in A^*, y' \in B^*$ is one–one. The following improves* [12, *Proposition* 3.3].

Lemma 4.3. *The injective tensorial pro-C^*-topology α on $A \otimes B$ is faithful.*

Proof. The map $i_\alpha: A \hat{\otimes}_\alpha B \to A \hat{\otimes}_\varepsilon B$ is the unique continuous linear extension of the identity map $i: A \otimes B \to A \otimes B$, $i(z) = z$. Let $z \in A \hat{\otimes}_\alpha B$ with $i_\alpha(z) = 0$. Let (z_λ) be a net in $A \otimes B$, $z_\lambda = \sum_{i=1}^{k_\lambda} x_i^{(\lambda)} \otimes y_i^{(\lambda)} \to_\alpha z$. Then for all $f \in A^*$, $g \in B^*$, $\lim_\lambda i_\alpha(z_\lambda)(f, g) = \lim_\lambda f \otimes g(z_\lambda) = f \otimes g(z) = 0$. To show $z = 0$, we show that $t_{p,q}(z) = 0$ for all $(p, q) \in S(A) \times S(B)$. Let $\pi \in R_p(A)$, $\sigma \in R_q(B)$, $\xi_1 \in H_\pi$, $\eta_1 \in H_\pi$, $\xi_2 \in H_\sigma$, $\eta_2 \in H_\sigma$. Define f and g by $f(x) = \langle \pi(x)\xi_1, \eta_1 \rangle$, $g(y) = \langle \sigma(y)\xi_2, \eta_2 \rangle$. Then

$$
\begin{aligned}
&\langle \pi \otimes \sigma(z)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle \\
&= \lim_\lambda \left(\left\langle \sum_1^{k\lambda} \pi(x_i^{(\lambda)}) \otimes \sigma(y_i^{(\lambda)}) \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \right\rangle \right) \\
&= \lim_\lambda \sum f(x_i^{(\lambda)}) g(y_i^{(\lambda)}) = \lim_\lambda f \otimes g(z_\lambda) = 0.
\end{aligned}
$$

Hence $\pi \otimes \sigma(z) = 0$ on the completed Hilbert space tensor product $H_\pi \bar{\otimes} H_\sigma$. It follows that $t_{p,q}(z) = 0$.

Lemma 4.4. *The following are equivalent for pro-C^*-algebras A and B.*

(a) *$A \hat{\otimes}_\alpha B = A \hat{\otimes}_v B$ with $\alpha = v$.*

(b) *For all* $(p,q)\in S(A)\times S(B)$, $A_p\hat{\otimes}_{\min}B_q=A_p\hat{\otimes}_{\max}B_q$.
(c) *For all* $(p,q)\in S(A)\times S(B)$, $t_{p,q}(\cdot)=v_{p,q}(\cdot)$ on $A\otimes B$.

Proof. (c)⇔(b)⇒(a) is a consequence of uniqueness of C^*-norm and Lemma 3.1. We show (a)⇒(b). Let $p\in S(A)$, $q\in S(B)$ and $\pi_p:A\to A_p$, $\pi_q:B\to B_q$ be the quotient maps. Then $\pi_p\hat{\otimes}\pi_q:A\hat{\otimes}_\alpha B\to A_p\hat{\otimes}_{\min}B_q$ defines a continuous *homomorphism. By [27, Proposition 4.7, p. 207], the continuous *homomorphisms $\phi:A\to A_p\hat{\otimes}_{\min}B_q=C$ (say), $\psi:B\to C$, $\phi(x)=x_p\otimes 1_q$, $\psi(y)=1_p\otimes y_q$ give continuous *homomorphism $\eta:A\hat{\otimes}_v B\to C$ satisfying $\eta(z)=(\pi_p\otimes\pi_q)(z)$ on $A\otimes B$. By the assumption (a), $\eta=\pi_p\hat{\otimes}\pi_q$; and π_p and π_q being surjective, $A_p\hat{\otimes}_{\min}B_q=(\pi_p\hat{\otimes}\pi_q)(A\hat{\otimes}_\alpha B)=\eta(A\hat{\otimes}_v B)\subset A_p\hat{\otimes}_{\max}B_q$. The continuous *homomorphism $k:A_p\otimes_{\min}B_q\to A_p\hat{\otimes}_{\max}B_q$ so defined is the extension of the identity map. It follows that for all $z\in A_p\otimes B_q$, $\|z\|_{\max}\leqslant\|z\|_{\min}\leqslant\|z\|_{\max}$, and (b) follows.

Proof of theorem 4.2. Lemma 3.1 gives (1)⇒(2)⇒(3). (2)⇔(4) is a consequence of Theorem 3.2. (3)⇒(1) follows from (2)⇒(1), which is a consequence of Lemma 4.4. The remaining assertions on σ-C^*-algebras follow from the open mapping theorem and Lemma 4.3.

It is shown in [23] that the functor $b(\cdot):A\to b(A)$ from pro-C^*-algebras to C^*-algebras is not well behaved with respect to tensor products. Still, the following main result of this section holds giving an aesthetically pleasing nuclearity criterion.

Theorem 4.5. *A pro-C^*-algebras A is nuclear iff $b(A)$ is a nuclear C^*-algebra.*

Lemma 4.6. *Let A be a pro-C^*-algebra. Let $p\in S(A)$, $I_p=b(A)\cap\ker p$. The quotient C^*-algebra $b(A)/I_p$ with the quotient C^*-norm $\|\cdot\|_{\infty,p}$ induced by the C^*-norm on $b(A)$ is isometrically *isomorphic to the C^*-algebra A_p.*

Proof. Let $\pi_p:A\to A_p$ be $\pi_p(x)=x+\ker p$ so that $\pi_p(A)=A_p$. Since $b(A)$ is dense in A, [23, Proposition 1.11] implies that $\pi_p(b(A))=A_p$. The *ideal I_p in $b(A)$ is closed in $\|\cdot\|_\infty$ due to continuity of embedding $b(A)\to A$. Since $I_p=\ker\psi_p$ where $\psi_p=\pi_{p|b(A)}$, $b(A)/I_p$ is *isomorphic to A_p.

Proof of theorem 4.5. Nuclearity of the C^*-algebra $b(A)$ implies [6, Corollary 4] that $(I_p,\|\cdot\|_\infty)$ and $(b(A)/I_p,\|\cdot\|_{\infty,p})$ are nuclear C^*-algebras for each $p\in S(A)$. By Lemma 4.6, A_p is nuclear for all p, and hence A is nuclear. Conversely, let A be a nuclear pro-C^*-algebra. For each $p\in S(A)$, the C^*-algebra A_p can be regarded as a C^*-algebra of operators on some Hilbert space H_p by taking a faithful representation σ_p. Then $\theta_p=\sigma_p\circ\psi_p$ gives a representation of $b(A)$ on H_p such that $\theta_p(b(A))=\sigma_p(A_p)$ is a nuclear C^*-algebra. By [6], [17], the von Neumann algebra $[\theta_p(b(A))]^{cc}$ (second commutant, identified with the bidual $[\theta_p(b(A)]^{**}$) is semidiscrete. By [27, Chapter IV, Lemma 2.2], θ_p extends as a surjective normal homomorphism $\tilde{\theta}_p:b(A)^{**}\to[\theta_p(b(A))]^{cc}$. Thus $\{\tilde{\theta}_p\,|\,p\in S(A)\}$ is a faithful family of normal representations of the von Neumann algebra $b(A)^{**}$ such that $\tilde{\theta}_p(b(A)^{**})=\theta_p(b(A))^{cc}$ is semidiscrete. Hence by [10, Corollary 3.3], $b(A)^{**}$ is semidiscrete, and so $b(A)$ is a nuclear C^*-algebra [10, Theorem 6.4].

Remark 4.7. Lemma 4.6 depends only on the fact that $b(A)$ is dense in A and is continuously embedded in A. Thus, in above theorem, we have a slightly stronger

Lemma 4.9 *Let A be a pro-C^*-algebra, Let $x \in A$. For each $n = 1, 2, \ldots$, Let $x_n = x(1 + (1/n)x^*x)^{-1}$. Then each $x_k \in b(A)$ and $x_n \to x$ in A.*

We claim that $\cup_n b(B_n)$ is dense in A. Take $x \in A$. Choose a sequence (x_k) in $\cup_n B_n$, say $x_k \in B_{n_k}$ so that $x_k \to x$. We can assume (n_k) to be non-decreasing. By above lemma, for each k and each $n = 1, 2, \ldots$, $x_{k,n} = x_k(1 + (1/n)x_k^* x_k)^{-1} \in b(B_{n_k})$, $x_k = \lim_{n \to \infty} x_{k,n}$ in B_{n_k}. Then $x = \lim_k x_k = \lim_k \lim_n x_k(1 + (1/n)x_k^* x_k)^{-1}$ in A and $x_k(1 + (1/n)x_k^* x_k)^{-1} \in \cup_n b(B_n)$ for all k, n. Hence $\cup_n b(B_n)$ is dense in A. Now for each n, $b(B_n) \subset b(A)$, $\cup_n b(B_n) \subset b(A)$. Let K be the closure of $\cup_n b(B_n)$ in the C^*-algebra $b(A)$. Then the C^*-algebra K is continuously embedded in A with dense range. By Theorem 4.5, each $b(B_n)$ is a nuclear C^*-algebra, with the result, [15, Proposition 11.3.12, p. 859] implies that K is nuclear C^*-algebra. The conclusion follows from Remark 4.7.

Remark 4.10. We could not establish a more general result involving arbitrary pro-C^*-algebras with arbitrary inductive limit. However, the following particular case can be similarly established.

PROPOSITION

Let A be a pro-C^*-algebra. Suppose there exists a family $\{B_\alpha | \alpha \in \Delta\}$ of C^*-algebras such that

(i) *each B_α is a closed *subalgebra of A containing the identity of A,*
(ii) *each B_α is a nuclear C^*-algebra,*
(iii) *given α, β in Δ, there exists a $\gamma \in \Delta$ such that $B_\alpha \cup B_\beta \subset B_\gamma$,*
(iv) *$\cup_\alpha B_\alpha$ is dense in A.*

Then A is a nuclear pro-C^*-algebra.

Finally, we aim to discuss the analog of a basic result of Lance [16, Theorem 3.4] using Proposition 3.8.

Theorem 4.11. *Let A be a pro-C^*-algebra. If A is nuclear, then for any pro-C^*-algebra B, every continuous complete state map from A to the strong dual B^* can be approximated in simple weak* convergence by continuous complete state maps from A to B^* of finite rank.*

Lemma 4.12. *Let A and B be pro-C^*-algebras. Let f be a continuous state on $A \hat{\otimes}_\alpha B$. Then the complete state map $T_f: A \to B^*$, $\langle y, T_f x \rangle = \langle x \otimes y, f \rangle$ can be approximated in simple weak* convergence by complete state maps of finite rank.*

Proof. Let $(p, q) \in S(A) \times S(B)$. Letting the C^*-algebras A_p and B_q act faithfully on Hilbert spaces H_p and K_q respectively, $A_p \otimes_{\min} B_q$ acts faithfully on $H_p \bar{\otimes} K_q$; and by C^*-theory, the state space $K(A_p \hat{\otimes}_{\min} B_q)$ is the weak* closed convex hull of $D_j(0)$, $j = (p, q)$, where $D_j(0) = \{\omega_\xi | \xi = \Sigma \xi_i \otimes \eta_i$ unit vectors in $H_p \otimes K_q\}$ and $\omega_\xi(z) = \langle z_\xi, \xi \rangle$. Let $\pi_j: A \hat{\otimes}_\alpha B \to A \hat{\otimes} B / \ker t_{p,q} = A_p \hat{\otimes}_{\min} B_q$ be the quotient map. Let $D_j = \{\omega_\xi \circ \pi_j | \omega_\xi \in D_j(0)\}$, $D = \cup_j \{D_j | j \in S(A) \times S(B)\}$. Then in the light of statement (*a*), (*b*), (*c*) based on [4] in the proof of (1)$\Rightarrow$(2) of Theorem 3.4, $K_j(A \hat{\otimes}_\alpha B)$ is the weak* closed convex hull of D_j; and $\overline{co} D = \overline{co} \cup_j \{D_j | j \in S(A) \times S(B)\} = \overline{co} \cup \{\overline{co} D_j | j \in S(A) \times S(B)\} = \overline{co} \cup_j \{K_j(A \hat{\otimes}_\alpha B)\} = \overline{co} K(A \hat{\otimes}_\alpha B) = K(A \hat{\otimes}_\alpha B)$, being weak* closed and convex [6].

The lemma follows from this, since for $f \in D$, T_f is a continuous complete state map of finite rank as in Proposition 3.8.

Proof of theorem 4.11. Let $T: A \to B^*$ be a continuous complete state map. By Proposition 3.8, there is an $f \in K(A \hat{\otimes}_\nu B)$ such that $T = T_f$. By Theorem 4.2, $\alpha = \nu$ and $f \in \overline{\text{co}}\, D$ so that f is a weak* limit of convex combinations of members of D. The assertion follows from Lemma 4.12.

It would be interesting to examine the converse of Theorem 4.11. Arguments in [16, Theorem 3.4] fail essentially because $K(A \hat{\otimes}_\nu B)$ need not be equicontinuous, nor does the equality $A \hat{\otimes}_\nu B = A \hat{\otimes}_\alpha B$ seem to imply $\nu = \alpha$ automatically. However, it is possible to obtain a version of Theorem 4.11 that admits a converse, and that too can be regarded as a generalization of [16, Theorem 3.4].

Theorem 4.13. *For pro-C^*-algebra A, the following are equivalent.*

(1) *A is nuclear.*
(2) *For every C^*-algebra B, for every continuous complete state map $\phi: A \to B^*$ and for every p in $S(A, \phi) = \{p \in S(A) |$ there exists $K > 0$ such that $\| \phi(x) \| \leqslant K p(x)$ for all $x\}$, there exists a net (ϕ_j) of continuous complete state maps $\phi_j: A \to B^*$ of finite ranks such that*
(a) *$\phi = \lim_j \phi_j$ in simple weak* convergence,*
(b) *$p \in S(A, \phi_j)$ for all j.*

This can be proved by passing to the C^*-algebra quotient A_p and applying corresponding result for C^*-algebras.

5. An application: Multipliers of the Pedersen ideal of a C^*-algebra

A *multiplier* on a *algebra A without identity is a pair (l, r) consisting of linear maps $l, r: A \to A$ such that for all x, y in A; $l(xy) = l(x)y$, $r(xy) = xr(y)$ and $xl(y) = r(x)y$. The collection $\Gamma(A)$ of all multipliers on A is a *algebra with operations: $(l_1, r_1) + (l_2, r_2) = (l_1 + l_2, r_1 + r_2)$, $\lambda(l, r) = (\lambda l, \lambda r)$, $(l_1, r_1)(l_2, r_2) = (l_1 l_2, r_2 r_1)$ and $(l, r)^* = (r^*, l^*)$ where $r^*(a) = r(a^*)^*$, $l^*(a) = l(a^*)^*$. A is embedded as a two-sided *ideal in $\Gamma(A)$ via the *isomorphism $\mu: a \in A \to (l_a, r_a) \in \Gamma(A)$, $l_a x = ax$, $r_a x = xa$; and μ is onto iff A has identity. For a topological *algebra A, let $M(A) = \{(l, r) \in \Gamma(A) | l$ and r are continuous$\}$. If A is a pro-C^*-algebra, then $M(A) = \Gamma(A)$ [30]; and then $M(A)$ is a pro-C^*-algebra with *seminorm topology* τ defined by the calibration $\{ \| \cdot \|_p | p \in S(A) \}$, $\| (l, r) \|_p = \sup \{ p(l(a)) | p(a) \leqslant 1 \}$ [23]. ($M(A)$ is a C^*-algebra iff A is a C^*-algebra) Corollary 4.8 (2) implies that if $M(A)$ is nuclear, then A is nuclear; but the converse does not hold even if A is a C^*-algebra. Take $A = K(H)$, the C^*-algebra of all compact operators on a Hilbert space H. Being a C^*-algebra of type I, $K(H)$ is nuclear; but $M(A) = B(H)$, (the C^*-algebra of all bounded operators on H) is not nuclear, for the Calkin algebra $B(H)/K(H)$ is known not to be nuclear.

Theorem 5.1. *Let A be a pro-C^*-algebra. The C^*-algebra $b(M(A))$ is isometrically *isomorphic to the C^*-algebra $M(b(A))$. Thus $M(A)$ is a nuclear pro-C^*-algebra iff $b(A)$ and the generalized Calkin algebra $M(b(A))/b(A)$ are nuclear C^*-algebras.*

Proof. In view of Corollary 4.8 (2), it is sufficient to show that $b(M(A)) = M(b(A))$ up to isomorphism. Let (e_λ), $\|e_\lambda\| \leqslant 1$ be an approximate identity for the C^*-algebra $b(A)$. Then (e_λ) is also an approximate identity for A [23]. We show that $b(M(A)) \subset M(b(A))$. Let $(l,r) \in M(A)$ with $\|(l,r)\|_\infty = \sup\{\|(l,r)\|_p | p \in S(A)\} < \infty$. It is sufficient to show that $l(b(A)) \subset b(A)$, $r(b(A)) \subset b(A)$. For $x \in b(A)$, $p \in S(A)$, $p(l(x)) = p(l(\lim_{\lambda,\tau} e_\lambda x)) = p(\lim_{\lambda,\tau}(l(e_\lambda x))) = \lim_\lambda p(l(e_\lambda x)) = \lim_\lambda p(l(e_\lambda)x) \leqslant \lim_\lambda p(l(e_\lambda))p(x) \leqslant \|(l,r)\|_\infty p(x) \leqslant \|(l,r)\|_\infty \|x\|_\infty$ showing $l(x) \in b(A)$. Similarly $r(x) \in b(A)$. This defines a *homomorphism $\phi: b(M(A)) \to M(b(A))$, $\phi(l,r) = (l_{|b(A)}, r_{|b(A)})$. Since $b(A)$ is dense in A, ϕ is one–one. We show that ϕ is surjective by establishing $M(b(A)) \subset b(M(A))$ in the sense that given $(l,r) \in M(b(A))$ each of l and r extends uniquely as continuous linear maps $L, R: A \to A$ such that $(L,R) \in M(A)$, $\|(L,R)\|_\infty < \infty$. It is sufficient to show that each $l, r: (b(A), \tau) \to (b(A), \tau)$ is continuous, where τ is the relativization of the pro-C^*-topology from A. Since l, r are continuous in the C^*-topology on $b(A)$, $M_l = \sup\|l(e_\lambda)\|_\infty$, $M_r = \sup\|r(e_\lambda)\|_\infty < \infty$. Take $M = \{M_l, M_r\}$. Let $p \in S(A)$, $a \in b(A)$. Then $\|ae_\lambda - a\|_\infty \to 0$, $\|e_\lambda a - a\|_\infty \to 0$; and $p(l(a)) = \lim p(l(e_\lambda a))$ as $\|\cdot\|_\infty$-convergence implies τ-convergence. Thus $p(l(a)) = \lim p(l(e_\lambda)a) \leqslant \lim p(l(e_\lambda))p(a) \leqslant \sup p(l(e_\lambda))p(a) \leqslant Mp(a)$ giving the desired continuity of l (and of r, by a similar argument). This gives existence of continuous linear extensions $L, R: A \to A$. That $(L,R) \in M(A)$ is a consequence of density of $b(A)$ in A and joint continuity of multiplication in (A, τ), which also implies that for each $x \in A$, $p(L(x)) \leqslant Mp(x)$, $p(R(x)) \leqslant Mp(x)$ for all $p \in S(A)$. Hence $\|(L,R)\|_\infty = \sup_{p \in S(A)} \cdot \|(L,R)\|_p < \infty$ giving $(L,R) \in b(M(A))$. This completes the proof.

Now let A be a C^*-algebra. Let $\mathscr{K}_A$ denote the Pedersen ideal of A [22, p. 175], [21]. Let X be the primitive ideal space of A. Let $\mathscr{F}$ be the collection of all compact closed subsets of X. For an open subset U of X, let $I(U)$ be the closed ideal corresponding to U. It is shown in [25, Theorem 7] that $\Gamma(\mathscr{K}_A)$ is a pro-C^*-algebra realized as the multiplier algebra $M(B)$ of the pro-C^*-algebra $B = \varprojlim_{C \in \mathscr{F}} A/I(X - C)$; and $b(\Gamma(\mathscr{K})) = M(A)$ [25, Theorem 2]. Theorem 4.1 gives the following.

COROLLARY 5.2

For a C^*-algebra A, $\Gamma(\mathscr{K}_A)$ is a nuclear pro-C^*-algebra iff $M(A)$ is a nuclear C^*-algebra iff A and $M(A)/A$ are nuclear C^*-algebra.

Thus nuclearity of A is not sufficient to guarantee nuclearity of $\Gamma(\mathscr{K}_A)$. In the case of $A = K(H)$, one has: $\mathscr{K}_A$ = all finite rank operators in $B(H)$ and $\Gamma(\mathscr{K}_A) = M(A) = B(H)$ [18, p. 30].

6. Nuclear pro-C^*-algebras and linear topological nuclearity

If a pro-C^*-algebra is nuclear as a locally convex space in the sense of [26, Chapter IV], then we call *A linearly nuclear*, in which case, for every locally convex space B, $A \hat{\otimes}_\varepsilon B = A \hat{\otimes}_\pi B$ and $\varepsilon = \pi$. Thus a linearly nuclear pro-C^*-algebra is a nuclear pro-C^*-algebra. An infinite dimensional nuclear C^*-algebra fails to be linearly nuclear in view of Dvoretzky-Roger Theorem [26, Corollary 3, p. 184].

Theorem 6.1. *Let A be a σ-C^*-algebra. The following are equivalent.*

(1) *A is linearly nuclear.*

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