Complete positivity, tensor products and C*-nuclearity for inverse limits of C*-algebras

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Abstract. The paper aims at developing a theory of nuclear (in the topological algebraic sense) pro-C*-algebras (which are inverse limits of C*-algebras) by investigating completely positive maps and tensor products. By using the structure of matrix algebras over a pro-C*-algebra, it is shown that a unital continuous linear map between pro-C*-algebras A and B is completely positive if and only if, it defines a completely positive map between the C*-algebras b(A) and b(B) consisting of all bounded elements of A and B. In the metrizable case, A and B are homeomorphically isomorphic if they are matricially order isomorphic. The injective pro-C*-topology α and the projective pro-C*-topology ν on $A \otimes B$ are shown to be minimal and maximal pro-C*-topologies; and α coincides with the topology of biequicontinuous convergence if either A or B is abelian. A nuclear pro-C*-algebra A is one that satisfies, for any pro-C*-algebra (or a C*-algebra) B, any of the equivalent requirements; (i) $\alpha = \nu$ on $A \otimes B$ (ii) A is inverse limit of nuclear C*-algebras (iii) there is only one admissible pro-C*-topology on $A \otimes B$ (iv) the bounded part $b(A)$ of A is a nuclear C*-algebra (v) any continuous complete state map $A \to B^*$ can be approximated in simple weak* convergence by certain finite rank complete state maps. This is used to investigate permanence properties of nuclear pro-C*-algebras pertaining to subalgebras, quotients and projective and inductive limits. A nuclearity criterion for multiplier algebras (in particular, the multiplier algebra of Pedersen ideal of a C*-algebra) is developed and the connection of this C*-algebraic nuclearity with Grothendieck's linear topological nuclearity is examined. A $\sigma$-C*-algebra A is a nuclear space if it is an inverse limit of finite dimensional C*-algebras; and if abelian, then A is isomorphic to the algebra (pointwise operations) of all scalar sequences.

Keywords. Inverse limits of C*-algebras; completely positive maps; tensor products; nuclear C*- and nuclear pro-C*-algebras; multiplier algebras; nuclear space.

1. Introduction and Preliminaries

A topological *algebra A is an involutive linear associative algebra (with identity 1) over complex scalars admitting a Hausdorff topology such that A is a topological vector space in which the multiplication and the involution are continuous. A pro-C*-algebra is a complete topological *algebra A the topology on which is determined by the collection S(A) of all continuous C*-seminorms on it; equivalently, A is homeomorphically *isomorphic to an inverse limit of C*-algebras. A $\sigma$-C*-algebra is a metrizable pro-C*-algebra. Besides an intrinsic interest in pro-C*-algebras as topological algebras ([1], [11], [13], [14], [19], [20] and references therein), it has been shown recently that they provide an important tool in investigation of certain aspects of C*-algebras (like multipliers of the Pedersen ideal, tangent algebra of a C*-algebra,
cross-product and \(K\)-theory, as well as non-commutative algebraic topology [23], [24], pseudodifferential operators [8] and quantum field theory [9]. In the literature, inverse limits of \(C^\ast\)-algebras have been given different names such as \(b^*\)-algebras, \(m\)-convex-\(C^\ast\)-algebras or LMC*-algebras; the more appropriate pro(jective limits of) \(C^\ast\)-algebras is a recent suggestion [23], [24] following [28].

The present paper aims at developing a theory of nuclear pro-\(C^\ast\)-algebras; and this requires investigating tensor products of such algebras and complete positivity of linear maps. The significance of this has been noted in [23, p. 175]. Unlike \(C^\ast\)-algebras and locally convex spaces, there are at least two concepts of nuclearity for pro-\(C^\ast\)-algebras, viz. nuclearity in topological algebraic sense (an extension of \(C^\ast\)-nuclearity [5], [6], [17]), and Grothendieck’s linear topological nuclearity [26, Chapter III, §7]. Except for a final remark, we mean the former.

Given a pro-\(C^\ast\)-algebra \(A\), each \(p \in \mathcal{S}(A)\) determines a \(C^\ast\)-algebra \(A_p = A / N_p\) \((N_p = \{x \in A | x_p = 0\}\), a \(*\)-ideal in \(A\), with \(C^\ast\)-norm \(\|x_p\|_p = p(x)\) where \(x_p = x + N_p\). The point is that \((A_p, \|\cdot\|_p)\) is complete [1], [27]; and \(A\) admits an inverse limit decomposition \(A = \lim_{\leftarrow} A_p\), \(\Delta\) being a confinal subset of \(\mathcal{S}(A)\). The \textit{bounded part} of \(A\) is the \(*\)-subalgebra \(b(A) = \{x \in A \mid \sup_{p \in \Delta} p(x) < \infty\}\), a \(C^\ast\)-algebra with norm \(\|x\| = \sup_{p \in \Delta} p(x)\) continuously embedded in \(A\) [23]. A crucial fact about a pro-\(C^\ast\)-algebra is that \(b(A)\) is dense in \(A\) [1], [23]; and this, in fact, characterizes pro-\(C^\ast\)-algebras [2]. Let \(M_n(C)\) be the \(C^\ast\)-algebra of all \(n \times n\) matrices. Given pro-\(C^\ast\)-algebras \(A\) and \(B\), a linear map \(\phi : A \to B\) is \textit{completely positive} if for each \(n\), \(\phi_\ast = \phi \otimes id : A \otimes M_n(C) \to B \otimes M_n(C) = M_n(B)\) is positive. In §2, by obtaining a pro-\(C^\ast\)-analogue of Stinespring’s Theorem, it is shown that a continuous \(\phi\) is completely positive iff \(\phi(b(A)) \subset b(B)\) and \(\phi : b(A) \to b(B)\) is a completely positive map between \(C^\ast\)-algebras. This is used to show that \(\sigma\)-\(C^\ast\)-algebras \(A\) and \(B\) are homeomorphically \(\ast\)-isomorphic iff they are matricially order isomorphic. In §3, the tensor product of pro-\(C^\ast\)-algebras \(A\) and \(B\) is investigated. Two standard pro-\(C^\ast\)-topologies on \(A \otimes B\) are introduced in [13], viz. the \textit{injective tensorial} pro-\(C^\ast\)-\textit{topology} \(\alpha\) and the \textit{projective tensorial} pro-\(C^\ast\)-\textit{topology} \(\tau\), which correspond respectively to the least \(C^\ast\)-norm \(\|\cdot\|_{\text{min}}\) and the greatest \(C^\ast\)-norm \(\|\cdot\|_{\text{max}}\) in case of \(C^\ast\)-algebras [27, Chapter IV]. It is shown in Theorem 3.2 that any admissible pro-\(C^\ast\)-topology \(\tau\) (in the sense of [13, Definition 2.1]) on \(A \otimes B\) satisfies \(\alpha \leq \tau \leq \gamma\); and if either \(A\) or \(B\) is abelian, then \(\alpha = \tau = \gamma\), \(\gamma\) being the topology of biequicontinuous convergence. (In fact, a slightly stronger result is proved, see Remark 3.3). This improves [13, Proposition 3.1], wherein the conclusion \(\tau \leq \gamma\) has been obtained under the additional assumptions that both \(A\) and \(B\) are metrizable and the completion \(A \hat{\otimes} B\) is a \(Q\)-algebra (i.e., the invertible elements form an open set). The other half of our Theorem also improves [13, Theorem 3.1] wherein the conclusion \(\alpha = \gamma\) is obtained under the assumptions that \(A\) and \(B\) are \(Q\)-algebras one being barreled and the other commutative. In fact, a pro-\(C^\ast\)-algebra that is a \(Q\)-algebra is a \(C^\ast\)-algebra [23, Proposition 1.14]; and hence the relevant results in [13] dealing with \(Q\)-pro-\(C^\ast\)-algebras are just the usual \(C^\ast\)-algebra results. Thus our result provides a complete analogue of [27, Chapter IV, Theorem 4.19] modulo the problem whether any pro-\(C^\ast\)-topology on \(A \otimes B\) is necessarily admissible. Further, it is shown in Theorem 3.4 that the above conclusion \(\alpha = \tau = \gamma\) characterizes commutativity of either \(A\) or \(B\). This extends [27, Chapter IV, Theorem 4.14]. Continuous states on \(A \hat{\otimes} B\) are shown to correspond to continuous completely positive maps \(A \to B^*\)(= the dual). The machinery developed in §2 and §3 is used to develop a theory of nuclear pro-\(C^\ast\)-algebras.
in §4. Following [23, p. 175], a pro-$C^*$-algebra $A$ is called nuclear if for each $p \in S(A)$, the $C^*$-algebra $A_p$ is nuclear in the sense [17] that for any $C^*$-algebra $B$, $\| \|_{\min} = \| \|_{\max}$ on $A_p \otimes B$. It is shown that $A$ is nuclear iff $A$ is an inverse limit of nuclear $C^*$-algebras (with the maps of the inverse system assumed surjective) iff for any pro-$C^*$-algebra (respectively, $C^*$-algebra) $B$, there is only one admissible pro-$C^*$-topology on $A \otimes B$. A basic result in $C^*$-theory is that a $C^*$-algebra $A$ is nuclear iff $A^{**}$ (the second dual) is semidiscrete. Using this via the universal representation, it is shown in the main Theorem 4.5 that a pro-$C^*$-algebra $A$ is nuclear iff $b(A)$ is a nuclear $C^*$-algebra. This is used to investigate permanence properties of nuclear pro-$C^*$-algebras pertaining to hereditary subalgebras, quotients and products, as well as projective and inductive limits. By using the structure of continuous state space of an $m$-convex *algebra [4], it is shown in Theorem 4.11 that if $A$ is nuclear, then for any pro-$C^*$-algebra $B$, any continuous complete state map $A \rightarrow B^*$ can be approximated in simple weak* convergence by continuous complete state maps of finite ranks; and a weaker converse holds. This is a pro-$C^*$-analogue of a basic result of Lance [16]. To illustrate an application, in §5 a nuclearity criterion for the multiplier algebra of the Pedersen ideal of a $C^*$-algebra [18] is obtained. For this, it is shown that for a pro-$C^*$-algebra $A$, its multiplier algebra $M(A)$ [23] is nuclear pro-$C^*$-algebra iff $b(A)$ and the generalized Calkin algebra $M(b(A))/b(A)$ are nuclear $C^*$-algebras. Finally, we examine the interrelation between Grothendieck's linear topological nuclearity and nuclearity in the sense of the present paper. In fact, a $\sigma$-$C^*$-algebra $A$ is linear topologically nuclear iff it is an inverse limit of finite dimensional $C^*$-algebras; and further, if abelian, it is isomorphic to the algebra of all scalar sequences with pointwise operations and pointwise convergence.

2. Completely positive maps

Let $A$ be a pro-$C^*$-algebra. Let $M_n(A)$ denote the *algebra of all $n \times n$ matrices over $A$, with the usual algebraic operations and the topology obtained by regarding it as a direct sum of $n^2$ copies of $A$. Since $A = \lim A_p$, as $p$ runs through $S(A)$, we have $M_n(A) = \lim_{p \in S(A)} M_n(A_p)$. Thus $M_n(A)$ is a pro-$C^*$-algebra. Algebraically, $M_n(A) = A \otimes M_n(\mathbb{C})$; and since $M_n(\mathbb{C})$ is finite dimensional, all tensor topologies on $M_n(A)$ agree.

Lemma 2.1. $b(M_n(A)) = M_n(b(A)) = b(A) \otimes M_n(\mathbb{C})$ as $C^*$-algebras.

Proof. We only have to prove the first equality. If $A$ is a $C^*$-algebra, and $a = [a_{ij}] \in M_n(A)$, it is easily seen that $\max_{i,j} \| a_{ij} \| \leq \| a \| \leq \sum_{i,j} \| a_{ij} \|$. Therefore, this inequality holds for any continuous $C^*$-seminorm $p$ on a pro-$C^*$-algebra $A$. Taking supremum over all $p$ yields $\max_{i,j} \sup_p p(a_{ij}) \leq \sup_p p(a) \leq \sum_{i,j} \sup_p p(a_{ij})$. Thus $a$ is bounded iff each $a_{ij}$ is bounded. The norms on $M_n(b(A))$ and $b(M_n(A))$ must now agree by the uniqueness of $C^*$-norms. This completes the proof.

It is easily seen that for $a = [a_{ij}] \in M_n(A)$, $a \geq 0$ if $a$ is a sum of matrices of the form $[a^* a]$ for $a_i, a_j$ in $A$ iff for all $x_1, \ldots, x_n$ in $A$, $\sum_{i,j} a_{ij} x_i x_j \geq 0$ in $A$. A linear map $\phi: A \rightarrow B$ between pro-$C^*$-algebras is completely positive if for all $n = 1, 2, 3, \ldots$, the linear maps $\phi_n: M_n(A) \rightarrow M_n(B), \phi_n(a_{ij}) = [\phi(a_{ij})]$ are positive. Thus $\phi$ is completely positive iff for each $q \in S(B), \phi_n: A \rightarrow B_q, \phi_n(x) = \phi(x) + N_q$ is completely positive. If $\phi$ is a continuous positive linear map, then it is easily seen that $\phi$ is completely positive if
either $A$ or $B$ is abelian. Following is an analogue of Stinespring’s Theorem [27, Chapter IV, Theorem 3.6].

**Theorem 2.2.** Let $A$ be a pro-$C^*$-algebra. Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on $H$.

(i) If $\pi: A \to B(K)$ is a continuous representation of $A$, $V: H \to K$ (a Hilbert space) is a bounded linear operator, then $\phi: A \to B(H)$, $\phi(x) = V^* \pi(x) V$ is a continuous completely positive map.

(ii) If $\phi: A \to B(H)$ is a continuous completely positive map, then there exists a Hilbert space $K$, a continuous representation $\pi: A \to B(K)$, a normal representation $\rho: \phi(A)' \to B(K)$ ($\phi(a)'$ is commutant of $\phi(A)$) and a bounded linear operator $V: H \to K$ such that $\phi(a) = V^* \pi(x) V$, $\rho(x) V = V x(x \in \phi(A)' \cap \pi(A)' \cap K = [\pi(A) VH]$, closed linear span of $\pi(A) VH$.

(i) is a straightforward verification. For (ii), continuity of $\phi$ implies that there exists $p \in S(A)$ such that $\|\phi(a)\| \leq M \rho(a)$ for all $a \in A$, for some $M > 0$. Thus $\phi_p: A \to B(H)$, $\phi_p(a) = \phi(x)$, $x_p = x + N_p$ is a well defined completely positive map between $C^*$-algebras to which the $C^*$-algebras Stinespring’s Theorem applies.

**COROLLARY 2.3**

Let $A$ and $B$ be pro-$C^*$-algebras.

(i) A unital continuous linear map $\phi: A \to B$ is completely positive (respectively positive) iff $\phi(b(A)) \subset b(B)$ and $\phi(b(A)) \to b(B)$ is a completely positive (respectively positive) map between $C^*$-algebras. If $\phi$ is completely positive, then $\phi(a)^* \phi(a) \leq \phi(a^* a)$ for all $a \in A$.

(ii) $A$ is homeomorphically *isomorphic to $B$ iff there exits a unital continuous bijective completely positive map $\phi: A \to B$ such that $\phi^{-1}$ is continuous and completely positive. In particular, if $A$ and $B$ are $\sigma$-$C^*$-algebras, then $A$ is homeomorphically *isomorphic to $B$ iff $A$ and $B$ are matricially order isomorphic.

**Proof.** (i) Let $\phi: A \to B$ be positive (in particular, completely positive). We show that $\phi(b(A)) \subset b(B)$. Let $a \in b(A)$ be positive, and so $0 \leq \|a\|_\infty \leq 1$. Hence $0 \leq \phi(a) \leq \|a\|_\infty$. $\phi(1) = \|a\|_\infty$. Thus $\phi(a) \in b(B)$. Now $\phi(b(A)) \subset b(B)$ follows by writing an arbitrary element as a linear combination of positive ones. Conversely, let given $\phi: A \to B$ be such that $\phi(b(A)) \subset b(B)$ and $\phi(b(A)) \to b(B)$ is completely positive. In view of Lemma 3.4, for all $n$, $\phi_n(b(M_n(A))) = \phi_n(M_n(b(A))) \subset M_n(b(B)) = b(M_n(B))$ and $\phi_n: b(M_n(A)) \to b(M_n(B))$ are positive linear maps, continuous in the relative topologies from $M_n(A)$ and $M_n(B)$. Now $b(A)$ is dense in $A$, in fact any $h \geq 0$ in $A$ can be approximated by the sequence $h_k = h(1 + (1/k)h)^{-1} \geq 0$ in $b(A)$. Applying this to each $M_n(A)$ and using $b(M_n(A)) = M_n(b(A))$, each $\phi_n: M_n(A) \to M_n(B)$ is positive.

Let $\phi: A \to B$ be completely positive. For each $q \in S(B)$, $\phi_q = \pi_q \circ \phi: A \to B_q (\pi_q: B \to B_q$ being $\pi_q(v) = (v + N_q))$ is a continuous completely positive map. Identifying $B_q$ with $C^*$-subalgebra of $B(H)$ for some Hilbert space $H$, Theorem 2.2 implies that $\phi_q = V_q^* \sigma_q(a) V_q (a \in A)$ for some continuous *homomorphism $\sigma_q: A \to B(K)$ (for some Hilbert space $K$) and some bounded linear map $V_q: H \to K$. Since $\phi_q(1) = 1$, it is seen as in [27, Chapter IV, Remark 3.7] that $V_q$ is an isometry. Thus $\pi_q(\phi(a)^* \phi(a)) = (\pi_q(\phi(a)))^* (\pi_q(\phi(a))) = \phi_q(a)^* \phi_q(a) = V_q^* \sigma_q(a) V_q V_q^* \sigma_q(a) V_q \leq \|V_q\|^2 V_q^* \sigma_q(a^* a) V_q \leq \|V_q\|^2 \|\sigma_q(a^* a)\| \leq \|\sigma_q(a^* a)\|$. Hence $\phi_q(a)^* \phi_q(a) \leq \phi_q(a^* a)$ for all $a \in A$. For the other direction, $\phi_q(a)$ is a positive element of $B_q$, and so $\|\phi_q(a)\|_\infty = \|\phi(a)\|_\infty$. Thus $\phi(a)^* \phi(a) \leq \phi(a^* a)$ for all $a \in A$. Therefore, $\phi: A \to B$ is completely positive.
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\[ \phi_q(a^*a) \leq \phi(a^*a) \text{ for all } q \in S(B). \] It follows that \( \phi(a)^*\phi(a) \leq \phi(a^*a) \) for all \( a \in A \). This completes the proof of (i).

For the proof of (ii), we shall need the following. A self-adjoint unital linear map \( \phi : A \to B \) is a \( C^* \)-homomorphism if \( \phi(h^2) = \phi(h)^2 \) for all \( h = h^* \) in \( A \).

**Lemma 2.4.** Let \( \phi : A \to B \) be a continuous bijective \( C^* \)-homomorphism between pro-C*-algebras such that for each \( n \geq 2 \), \( \phi_n : M_n(A) \to M_n(B) \) is also a \( C^* \)-homomorphism. Then \( \phi \) is a *isomorphism (not necessarily a homeomorphism).

**Proof of lemma.** A \( C^* \)-homomorphism \( \phi : A \to B \) being a positive map, Corollary 2.3 (i) implies that \( \phi(b(b(A)) \subseteq b(B) \). Applying this to \( \phi \) and \( \phi^{-1} \), it follows that \( \phi(b(A)) = b(B) \) and \( \phi : b(A) \to b(B) \) is bijective \( C^* \)-isomorphism between \( C^* \)-algebras. The same arguments show, in view of Lemma 2.1, that for each \( n \), \( \phi_n : b(M_n(A)) = M_n(b(A)) \to M_n(b(B)) = b(M_n(B)) \) is also \( C^* \)-isomorphism. Thus by [27, Example 1, p. 202], \( \phi : b(A) \to b(B) \) is a *isomorphism. Now density of \( b(A) \) in \( A \), joint continuity of multiplication in a pro-\( C^* \)-algebra and continuity of \( \phi \) implies that \( \phi : A \to B \) is a *isomorphism.

**Proof of part (ii) of Corollary 2.3.** The inequality in part (i) applied to the completely positive maps \( \phi \) and \( \phi^{-1} \) shows that \( \phi(a)^*\phi(a) = \phi(a^*a) \) for all \( a \in A \). In particular, \( \phi \) is a \( C^* \)-isomorphism. By the same arguments, each \( \phi_n \) is a \( C^* \)-isomorphism; and hence \( \phi \) is a *isomorphism by above Lemma. The remaining assertions are trivial.

We note the following consequence of the fact that a positive linear functional on a complete locally \( m \)-convex *algebra with 1 maps a bounded set to a bounded set [7].

**PROPOSITION 2.5.**

A positive linear map \( \phi : A \to B \) from pro-\( C^* \)-algebra \( A \) to pro-\( C^* \)-algebra \( B \) maps a bounded set to a bounded set. If \( A \) is a \( \sigma \)-\( C^* \)-algebra, then \( \phi \) is continuous.

3. Tensor products

Given pro-\( C^* \)-algebras \( A \) and \( B \), we are concerned with the following four topologies on the tensor product \( A \otimes B \). For a locally convex topology \( \tau \) on \( A \otimes B \), \( A \otimes_\tau B \) denotes the completion of \( (A \otimes B, \tau) \).

(i) [26] projective tensorial topology \( \pi \): For \( p \in S(A), q \in S(B), (p \otimes q)(z) = \inf \{ \Sigma_i p(x_i)q(y_i) \} \]
\[ z = \Sigma x_i \otimes y_i \text{ in } A \otimes B \text{ with } x_i \in A, y_i \in B; \]
and \( \pi \) is the locally convex topology defined by the seminorms \( \{ p \otimes q \mid p \in S(A), q \in S(B) \} \). Each \( p \otimes q \) is a submultiplicative seminorm satisfying \( (p \otimes q)(z^*) = (p \otimes q)(z) \). Thus \( A \otimes_\pi B \) is a locally \( m \)-convex *algebra, though not a pro-\( C^* \)-algebra in general.

(ii) [26] topology \( \varepsilon \) of biequicontinuous convergence: For \( p \in S(A), q \in S(B), \) let \( U_p^q(1) = \{ f \in A^* \mid \| f(x) \| \leq 1 \text{ for all } x \text{ such that } p(x) \leq 1 \}, U_q^p(1) = \{ g \in B^* \mid \| g(y) \| \leq 1 \text{ for all } y \text{ such that } q(y) \leq 1 \} \), \( A^* \) and \( B^* \) denoting topological duals of \( A \) and \( B \) respectively. For \( z = \Sigma x_i \otimes y_i \) in \( A \otimes B \), define \( \varepsilon_{p,q}(z) = \sup \{ \| f(x_i)q(y_i) \| \| f \in U_p^q(1), \text{ and } q \in U_q^p(1) \} \). The family of seminorms \( \{ \varepsilon_{p,q} \mid p \in S(A), q \in S(B) \} \) (called the natural \( \varepsilon \)-calibration) defines the biprojective tensorial topology \( \varepsilon \), also called the topology of biequicontinuous convergence, i.e., the topology of uniform convergence on sets of form \( S \otimes T \) where
$S \subseteq A^*$, $T \subseteq B^*$ are equicontinuous. The seminorms $\varepsilon_{p,q}$ need not be submultiplicative, and $A \otimes B$ need not be an algebra.

(iii) [13] projective tensorial pro-$C^*$-topology $v$: By a bounded representation $(\pi, H_\pi)$ of a $C^*$-algebra $K$ is meant to be a $*$-homomorphism $\pi$ of $K$ into the $C^*$-algebra $B(H_\pi)$. Let $R(K)$ be the collection of all continuous bounded representations of a topological $*$-algebra $K$. Then $R(K) = \bigcup \{ R_x(K) | x \in S(K) \}$, $R_x(K) = \{ \pi \in R(K) | \| \pi(x) \| \leq s(x) \text{ for all } x \in K \}$. Now for $p \in S(A)$, $q \in S(B)$, let $R_{p,q}(A \otimes_q B) = \{ \sigma \in R(A \otimes_q B) | \| \sigma(z) \| \leq (p \otimes q)(z) \text{ for all } z \in A \otimes B \}$. Define a $C^*$-seminorm $\gamma_{p,q}(z) = \sup \{ \| \sigma(z) \| | \sigma \in R_{p,q}(A \otimes_q B) \}$. The projective tensorial pro-$C^*$-topology $v$ is the topology defined by the $C^*$-seminorms $\{ \gamma_{p,q} | p \in S(A), q \in S(B) \}$.

(iv) [13] injective tensorial pro-$C^*$-topology $\alpha$: In the above notations let $t_{p,q}(z) = \sup \{ \| p \otimes q \sigma(z) \| | \sigma \in R_p(A), \sigma \in R_q(B) \}$, a $C^*$-seminorm on $A \otimes B$. The topology $\alpha$ is defined by the $C^*$-seminorms $\{ t_{p,q} | p \in S(A), q \in S(B) \}$.

Note that $\varepsilon \leq \alpha \leq \nu \leq \pi$; and for $C^*$-algebras $A$ and $B$, the topologies $\varepsilon$, $\alpha$, $\nu$ and $\pi$ reduce to the topologies respectively due to the injective cross norm $\| \cdot \|_\alpha$, the minimal $C^*$-norm $\| \cdot \|_{\min}$, the maximal $C^*$-norm $\| \cdot \|_{\max}$ and the projective cross norm $\| \cdot \|_\gamma$.

[27, Chapter IV].

Lemma 3.1. For pro-$C^*$-algebras $A$ and $B$, the following hold

(i) $A \otimes_v B = \lim_{p,q \to v} A_p \otimes_{\max} B_q$
(ii) $A \otimes_{\alpha} B = \lim_{p,q \to \alpha} A_p \otimes_{\min} B_q$
(iii) $A \otimes_{\beta} B = \lim_{p,q \to \beta} A_p \otimes_{\beta} B_q$
(iv) $A \otimes_{\gamma} B = \lim_{p,q \to \gamma} A_p \otimes_{\gamma} B_q$

Given pro-$C^*$-algebras $A$ and $B$, a Hausdorff topology $\tau$ on $A \otimes B$ is an admissible topology [13] if:(i) $(A \otimes B, \tau)$ is a locally $m$-convex $*$-algebra; i.e., there exists a family $\Gamma = \{ r_\alpha | x \in \Delta \}$ of submultiplicative seminorms satisfying $r_\alpha(z^*) = r_\alpha(z)$ for all $z$ such that $\Gamma$ determines $\tau$. (ii) For each $x \in A$, there exist $p \in S(A), q \in S(B)$ such that $r_\alpha(x \otimes y) \leq p(x)q(y)$ for all $x \in A, y \in B$; and (iii) given equicontinuous subsets $M \subseteq A^*$, $N \subseteq B^*$, $M \otimes N$ is an equicontinuous subset of $(A \otimes B)$. A topology $\tau$ on a $*$-algebra $K$ is a pro-$C^*$-topology if the completion of $(K, \tau)$ is a pro-$C^*$-algebra. It is shown in [13, Proposition 3.1] that if $A$ and $B$ are $\sigma$-$C^*$-algebras and if $\tau$ is an admissible pro-$C^*$-topology on $A \otimes B$ such that $(A \otimes B, \tau)$ is a $Q$-algebra, then $\tau \leq v$. The following substantially improves this. It can also be regarded as an analogue of the $C^*$-algebra result [27, Chapter IV, Theorem 4.19] that if $\beta$ is a $C^*$-norm on $A \otimes B$ for $C^*$-algebras $A$ and $B$, then $\beta$ is a cross norm satisfying $\| \cdot \|_{\min} \leq \beta(\cdot) \leq \| \cdot \|_{\max}$; and if either $A$ or $B$ is abelian, then $\beta(\cdot) = \| \cdot \|_\alpha = \| \cdot \|_{\min} = \| \cdot \|_{\max}$ [27, Chapter IV, Lemma 4.18]. The other half of the following also improves [13, Theorem 3.1] wherein it is shown that for $Q$-pro-$C^*$-algebras $A$ and $B$ one being barrelled and the other abelian, $\varepsilon = \alpha$ on $A \otimes B$. In fact, Proposition 3.1 and Theorem 3.1, both of [13] referred to above, reduce to be the usual $C^*$-algebra results, because, by [23, Proposition 1.14], a $Q$-pro-$C^*$-algebra is a $C^*$-algebra.

Theorem 3.2. Let $A$ and $B$ be pro-$C^*$-algebras. Let $\tau$ be an admissible pro-$C^*$-topology on $A \otimes B$. Then $\alpha \leq \tau \leq v$. If either $A$ or $B$ is abelian, then $\varepsilon = \alpha = \tau = v$.

Proof. Let $K_\varepsilon = A \otimes \varepsilon B$. Let $(z_n)$ be a net in $A \otimes B$, $z_n \to 0$ in $\tau$. By [26, Chapter IV, p. 127], the topology of a locally convex space is the topology of uniform convergence
on equicontinuous subsets of its dual. Hence \( z_\alpha \to 0 \) uniformly on every equicontinuous subset of \((A \hat{\otimes}, B)^*\). In view of admissibility of \( \tau \), \( z_\alpha \to 0 \) uniformly on \( M \otimes N \) for equicontinuous sets \( M \subset A^*, N \subset B^* \). Hence \( z_\alpha \to 0 \) in \( \varepsilon \) by [26, Chapter III, p. 96], and

\[ \varepsilon \leq \tau. \] (1)

Now given \( \gamma \in S(A \hat{\otimes}, B) \), choose \( p \in S(A) \), \( q \in S(B) \) such that for all \( x, y \), \( \gamma(x \otimes y) \leq p(x)q(y) \). Hence for \( z = \sum x_i \otimes y_i \in A \otimes B \) \( \gamma(z) \leq \sum \gamma(x_i \otimes y_i) \leq \sum p(x_i)q(y_i) \). By definition of \( p \otimes q \), \( \gamma(z) \leq (p \otimes q)(z) \); and \( \tau \leq \pi \). Thus

\[ \varepsilon \leq \tau \leq \pi. \] (2)

Again for a net \( (z_\alpha) \) in \( A \otimes B \), let \( z_\alpha \to 0 \) in \( v \), so that for each \( p \in S(A) \), \( q \in S(B) \), \( \gamma_{p,q}(z_\alpha) \to 0 \). Let \( \gamma \in S(A \hat{\otimes}, B) \). Then there exist \( p \) and \( q \) as in (2) above satisfying \( \gamma(z) \leq (p \otimes q)(z) \) for all \( z \). Since \( \gamma \) is a \( \tau \)-continuous \( C^* \)-seminorm, a faithful representation of the \( C^* \)-algebra \( (A \hat{\otimes}, B)_\gamma = ((A \hat{\otimes}, B)/\ker \gamma \) with \( C^* \)-norm induced by \( \gamma \) as an operator algebra gives a \( \tau \)-continuous bounded representation \( (\sigma, H_\sigma) \) of \( A \hat{\otimes}, B \) satisfying \( \| \sigma(z) \| = \gamma(z) \leq (p \otimes q)(z) \) for all \( z \). Thus \( \sigma \in R_{p,q}(A \hat{\otimes}, B) \). Hence \( \gamma(z_\alpha) \leq \nu_{p,q}(z_\alpha) \to 0 \) showing

\[ \tau \leq \nu. \] (3)

Finally, we show that \( \alpha \leq \tau \). For given \( p \in S(A) \), \( q \in S(B) \), the sets \( P_{p,q}(A) = \{ f \in A^* \mid f(x) \leq p(x) \} \) for all \( x \in A \} \subset A^* \) and \( P_{p,q}(B) = \{ g \in B^* \mid |g(y)| \leq q(y) \} \) for all \( y \in B \} \subset B^* \) being equicontinuous, \( P_{p,q}(A) \) \( \otimes \) \( P_{p,q}(B) \) is an equicontinuous subset of \((A \hat{\otimes}, B)^*\). Hence there exist a \( \gamma \in S(A \hat{\otimes}, B) \) such that

\[ P_{p,q}(A) \otimes P_{p,q}(B) \subset P_{p,q}(A \hat{\otimes}, B) = \{ h \in (A \hat{\otimes}, B)^* \mid \| h(z) \| \leq \gamma(z) \} \] (4)

Now for a \( \pi \in R_{p,q}(A) \), \( \xi \in H_\pi \), the linear functional \( f^\pi_\xi(x) = \langle \pi(x)\xi, \xi \rangle / \| \xi \|^2 \) satisfies \( |f^\pi_\xi(x)| \leq p(x) \) for all \( x \in A \) and the equicontinuous set \( M \) defined as \( M = \{ f^\pi_\xi | \pi \in R_{p,q}(A), \xi \in H_\pi \} \) satisfies \( M \subset P_{p,q}(A) \). Similarly, \( N = \{ g^\pi_\eta | \sigma \in R_{p,q}(B), \eta \in K_\sigma \} \subset P_{p,q}(B) \) where \( g^\pi_\eta(y) = \langle \sigma(y)\eta, \eta \rangle / \| \eta \|^2 \). It follows from (4) that for all \( (\pi, H_\pi) \in R_{p,q}(A) \), \((\sigma, K_\sigma) \in R_{p,q}(B) \), \( \xi \in H_\pi \), \( \eta \in K_\sigma \), \( f^\pi_\xi \otimes g^\sigma_\eta(z) \leq \gamma(z) \) for all \( z \in A \hat{\otimes}, B \) that \( x_i \otimes y_i \in A \otimes B \), this implies that

\[ \sum_{i=1}^{\infty} \left( \frac{\langle \pi(x_i)\xi, \xi \rangle \langle \sigma(y_i)\eta, \eta \rangle}{\| \xi \otimes \eta \|^2} \right) \leq \sum_{i=1}^{\infty} \left( \frac{\langle \pi(x_i)\xi, \xi \rangle \langle \sigma(y_i)\eta, \eta \rangle}{\| \xi \|^2 \| \eta \|^2} \right) \]

\[ = f^\pi_\xi \otimes g^\sigma_\eta(z) \leq \gamma(z). \]

Thus, for all \( z \in A \otimes B \), \( |\langle (\pi \otimes \sigma)(z)\xi \otimes \eta, \xi \otimes \eta \rangle| = \gamma(z) \| \xi \otimes \eta \|^2 \). It follows by the polarization identity, that for any \( \xi, \zeta \in H_\pi \) and \( \eta, \eta' \in K_\sigma \), \( |\langle \pi \otimes \sigma(z)\xi \otimes \eta, \zeta \otimes \eta' \rangle| \leq \gamma(z)|\langle \xi \otimes \eta, \zeta \otimes \eta' \rangle| \). Now taking \( \theta = \sum_{i=1}^{\infty} \xi_i \otimes \eta_i \in H_\pi \otimes K_\sigma \), and taking without loss of generality, \( (\eta_i) \) to be an orthonormal set in \( K_\sigma \), we obtain

\[ |\langle (\pi \otimes \sigma)(z)\theta, \theta \rangle| \leq \sum_{i,j} |\langle \pi \otimes \sigma(z)\xi_i \otimes \eta_i, \xi_j \otimes \eta_j \rangle| \]

\[ \leq \gamma(z) \sum_{i,j} |\langle \xi_i \otimes \eta_i, \xi_j \otimes \eta_j \rangle| \]

\[ = \gamma(z) \sum_{i=1}^{\infty} \| \xi_i \|^2 = \gamma(z) \| \theta \|^2. \] (5)
Since \( \pi \otimes \sigma(z) \) is a bounded operator on the completed Hilbert space tensor product \( H \otimes \mathbb{K} \), it follows that (5) holds for all \( \theta \) in \( H \otimes \mathbb{K} \). Thus, for \( \pi \in \mathcal{R}_p(A) \), \( \sigma \in \mathcal{R}_q(B) \), \( z \in A \otimes B \theta \in H \otimes \mathbb{K} \),

\[
\| \pi \otimes \sigma(z) \theta \| = \langle (\pi \otimes \sigma)(z^* z) \theta, \theta \rangle \\
\leq \gamma(z^* z) \| \theta \|^2 = \gamma(z) \| \theta \|^2.
\]  

(6)

Hence for all \( z \in A \otimes B \) (and so for all \( z \in A \hat{\otimes} B \)), \( \| \pi \otimes \sigma(z) \| \leq \gamma(z) \), \( \pi \otimes \sigma \in \mathcal{R}_p(A \hat{\otimes} \mathbb{K}, B) \).

Summarizing, we have shown that given \( p \in \mathcal{S}(A), q \in \mathcal{S}(B) \), there exists a \( \gamma \in \mathcal{S}(A \otimes B) \) such that \( \mathcal{R}_p(A) \otimes \mathcal{R}_q(B) \subseteq \mathcal{R}_\gamma(A \otimes B) \).

Now if \( z \to 0 \) in \( \tau \), it follows from above that for all \( p \in \mathcal{S}(A), q \in \mathcal{S}(B) \), \( t_{p,q}(z) \to 0 \). Hence

\[
\alpha \leq \tau.
\]  

(7)

Finally, assume that either \( A \) or \( B \), say \( A \), is abelian. Then for \( p \in \mathcal{S}(A), q \in \mathcal{S}(B) \), by standard C*-algebra result [27, p. 212], \( \lambda = \| \cdot \|_{\min} = \| \cdot \|_{\max} \) on \( A \otimes B \). Lemma 3.1 implies that \( A \hat{\otimes} B = A \hat{\otimes} B = A \hat{\otimes} B \) with \( \alpha = \varepsilon = \nu \); in fact, \( \varepsilon_{p,q} = t_{p,q} \). This completes the proof.

**Remarks 3.3.** (a) A closer examination of the proof reveals that we have, in fact, proved the following stronger assertions.

(i) Any admissible topology \( \tau \) on \( A \otimes B \) satisfies \( \varepsilon \leq \tau \leq \pi \).

(ii) In the derivation of inequality (6), if \( \gamma \) is not a C*-seminorm but a submultiplicative \( \ast \)-seminorm only, then too \( \| \pi \otimes \sigma(z) \theta \| \leq \gamma(z^* z) \| \theta \|^2 \leq \gamma(z^* \gamma(z) \| \theta \|^2 \) holds. Thus, for any admissible (not necessarily pro-C*) topology \( \tau \) on \( A \otimes B \), \( \alpha \leq \tau \).

(iii) In general, \( \nu < \pi \), since \( A \hat{\otimes} B \) is, under certain circumstances, the enveloping \( \ast \)-algebra [11] of the complete locally \( m \)-convex \( \ast \)-algebra \( A \hat{\otimes} \mathbb{K} \) [13].

(b) It is easily seen from [27] that for C*-algebras \( A \) and \( B \), any C*-topology on \( A \otimes B \) is automatically an admissible topology. This suggests: Let \( \tau \) be any pro-C*-topology on the tensor product \( A \otimes B \) of pro-C*-algebras \( A \) and \( B \). Is \( \tau \) necessarily an admissible topology?

Our next result shows that the conclusion \( \varepsilon = \alpha \) in abelian case of above theorem, in fact, characterizes commutativity of either \( A \) or \( B \). It gives a complete pro-C*-analogue of [27, Chapter IV, Theorem 4.14] and a recent result of Blecher [3]; viz. for C*-algebras \( A \) and \( B \), if \( A \hat{\otimes} B \) is a Banach algebra with norm \( \lambda \), then either \( A \) or \( B \) is abelian. We call the natural \( \varepsilon \)-calibration \( \Gamma = \{ \varepsilon_{p,q} \mid p \in \mathcal{S}(A), q \in \mathcal{S}(B) \} \) an \( \ast \)-calibration if each \( \varepsilon_{p,q} \) satisfies \( \varepsilon_{p,q}(w) \leq \varepsilon_{p,q}(u) \varepsilon_{p,q}(v) \) for all \( u, v \) in \( A \otimes B \). The given pro-C*-topologies on \( A \) and \( B \) are denoted by \( \tau_A \) and \( \tau_B \) respectively; and \( b(A) \) and \( b(B) \) carry C*-topologies unless stated otherwise.

**Theorem 3.4.** Let \( A \) and \( B \) be pro-C*-algebras. The following are equivalent.

1. Either \( A \) or \( B \) is abelian.
2. Any continuous pure state \( \omega \) on \( A \hat{\otimes} \mathbb{K} \) is of form \( \omega = \omega_1 \otimes \omega_2 \), \( \omega_1 \) and \( \omega_2 \) being continuous pure states on \( A \) and \( B \) respectively.
(3) $A \hat{\otimes}_\varepsilon B = A \hat{\otimes}_\alpha B$ with $\varepsilon = \alpha$ and the natural $\varepsilon$-calibration on $A \hat{\otimes}_\varepsilon B$ is an $m^*$-calibration.

(3') The natural $\varepsilon$-calibration on $A \hat{\otimes}_\varepsilon B$ is an $m^*$-calibration.

(4) $b(A) \hat{\otimes}_\varepsilon b(B) = (b(A) \hat{\otimes}_{\min} b(B)$ and $\lambda = \|\cdot\|_{\min}.$

(4') $b(A) \hat{\otimes}_\varepsilon b(B)$ is a Banach algebra.

(5) $A \hat{\otimes}_\varepsilon b(B) = A \hat{\otimes}_\alpha b(B)$ with $\varepsilon = \alpha$ and the natural $\varepsilon$-calibration on $A \hat{\otimes}_\varepsilon b(B)$ is an $m^*$-calibration.

(5') The natural $\varepsilon$-calibration on $A \hat{\otimes}_\varepsilon b(B)$ is an $m^*$-calibration.

(6) $b(A) \hat{\otimes}_\varepsilon B = b(A) \hat{\otimes}_\varepsilon B$ with $\varepsilon = \alpha$ and the natural $\varepsilon$-calibration on $b(A) \hat{\otimes}_\varepsilon B$ is an $m^*$-calibration.

(6') The natural $\varepsilon$-calibration on $b(A) \hat{\otimes}_\varepsilon B$ is an $m^*$-calibration.

Lemma 3.5. Let $R$ be a pro-$C^*$-algebra. Let $E$ be a subalgebra of $R$ containing the identity of $R$ such that $E$ is a Banach $*$-algebra with some norm $|.|$. Then $E \subset b(A)$ and on $E$, $\|\cdot\|_\infty \leq |.|$.

Proof. By continuity of involution in $(R, |.|)$, we assume $|x^*| = |x|$ for all $x \in E$. Let $B = \{x \in E | |x| \leq 1\}$. Then $B$ is an absolutely convex *idempotent containing the identity 1 of $R$. By standard Banach $*$-algebra arguments, every positive linear functional $f$ on $R$, restricted to $E$, is $|.|$-continuous satisfying $|f(x)| \leq f(1)|x|$. Since the dual $R^*$ of $R$ is a complex linear span of continuous positive functionals, $B$ turns out to be $\sigma(R, R^*)$-bounded, hence bounded in the topology of $R$. Now, in $R$, it is easy to verify that $K = \{x \in E | |x|_\infty \leq 1\}$ is the largest (under inclusion) bounded absolutely convex *idempotent. Thus $B \subset K, E \subset b(R)$ and $\|\cdot\|_\infty \leq |.|$ on $E$.

Proof of theorem. (1)$\Rightarrow$(2). First of all, for any $A$ and $B$, not necessarily abelian, let $K(A \hat{\otimes}_\varepsilon B)$ be the set of all continuous states on $A \hat{\otimes}_\varepsilon B$. For each $j = (p, q) \in S(A) \times S(B), \quad$ Let $U_j = \{z \in A \hat{\otimes}_\varepsilon B | f_{p,q}(z) \leq 1\}$, $R_j$ be the $C^*$-algebra $A \hat{\otimes}_\varepsilon B/kert_{p,q}$ with the $C^*$-norm induced by $f_{p,q}$. Then $R_j$ is *isomorphic to the $C^*$-algebra $A_p \hat{\otimes}_{\min} B_q$. Let $K_j(A \hat{\otimes}_\varepsilon B) = \{f \in K(A \hat{\otimes}_\varepsilon B) | f \text{ is bounded on } U_j\}$. Then from [4], the following hold.

(a) $K(A \hat{\otimes}_\varepsilon B) = U_j K_j(A \hat{\otimes}_\varepsilon B).

(b) For the sets of extreme points, $E(K(A \hat{\otimes}_\varepsilon B)) = U_j E(K_j(A \hat{\otimes}_\varepsilon B)).

(c) $K_j(A \hat{\otimes}_\varepsilon B)$ is in bijective correspondence with $K_j(R_j)$ under the map $f \in K_j(A \hat{\otimes}_\varepsilon B) \rightarrow f_j: z \rightarrow f(z)$ where for $z \in A \otimes B, z \in R_j$ is $z_j = z + kert_{p,q}$, and this correspondence preserves the weak* topologies.

Now assume (1) say $B$ is abelian. For all $q$ in $S(B)$, the $C^*$-algebras $B_q$ are abelian. Let $\omega$ be a continuous pure state on $A \hat{\otimes}_\varepsilon B$, so that, by [4, Corollary 4.3], $\omega$ is an extreme point of $K(A \hat{\otimes}_\varepsilon B).$ By above, there exists a $j = (p, q) \in S(A) \times S(B)$ such that $\omega_j$ is a pure state on $R_j = A_p \hat{\otimes}_{\min} B_q$. The $C^*$-algebra result [27, p. 211] gives pure states $\omega_1$ of $A_p$ and $\omega_2$ of $B_q$ satisfying $\omega_j = \omega_1 \otimes \omega_2$. Again from above and [4, Theorem 4.3], $\omega_1(x) = \omega'_1 (x_p), \omega_2(y) = \omega'_2 (y_q)$ define continuous pure states on $A$ and $B$ satisfying $\omega = \omega_1 \otimes \omega_2$.

(2)$\Rightarrow$(1). This can be shown exactly as in [27, Chapter IV, Theorem 4.14] by using the facts [11] that a pro-$C^*$-algebra $R$ admits sufficiently many bounded continuous irreducible representations. In fact, for each $x \in S(R)$, an irreducible representation $\pi$ of the $C^*$-algebra $R_x$ gives an irreducible representation $\pi$ of $R$ as $\pi(z) = \pi(x)z$.

(1)$\Rightarrow$(3). This is already shown in Theorem 3.2 wherein $\epsilon_{p,q} = f_{p,q}$ has been observed, i.e., the natural $\varepsilon$-calibration consists of $C^*$-seminorms. Also, (3)$\Rightarrow$(3'), (4)$\Rightarrow$(4'), (5)$\Rightarrow$(5') and (6)$\Rightarrow$(6') are trivial. Assume (3'). In view of Lemma 3.1, for each $(p, q)$
in $S(A) \times S(B)$, $A_p \hat{\otimes} B_q$ becomes a Banach algebra with the injective cross-norm $\lambda$. Hence by [3, Corollary 4.14, p. 123] either $A_p$ or $B_q$ is abelian; and so $C^\ast$-algebra arguments [27, Theorem 4.14, p. 211] gives $A_p \hat{\otimes} B_q = A_p \hat{\otimes}_{\min} B_q$ with $\lambda = \| \cdot \|_{\min}$. This gives (3). Similarly it follows that (4') $\Rightarrow$ (4), (5') $\Rightarrow$ (5) and (6') $\Rightarrow$ (6).

(3) $\Rightarrow$ (1). Observe the following.

(i) $b(A \hat{\otimes} B)$ is a $C^\ast$-algebra with $C^\ast$-norm $\| z \|_{\infty, A \hat{\otimes} B} = \sup \{ t_{p,q} \| z \|_{p,q} \mid p \in S(A), q \in S(B) \}$. 

(ii) Since the $\varepsilon$-calibration is an $m^\ast$-calibration, $A \hat{\otimes} B$ is a complete locally $m$-convex *algebra; and its bounded part, defined as $b(A \hat{\otimes} B) = \{ z \in A \hat{\otimes} B \mid \sup_\varepsilon \varepsilon_{p,q}(z) < \infty \}$ is a Banach *algebra with norm $\| z \|_{\infty, A \hat{\otimes} B} = \sup \{ \varepsilon_{p,q}(z) \mid p \in S(A), q \in S(B) \} = \| z \|_{\infty, A \hat{\otimes} B}$. Now assume $A \hat{\otimes} B = A \hat{\otimes} B$. We assert that $b(A \hat{\otimes} B) = b(A \hat{\otimes} B)$. Indeed, for all $p, q$ in $S(A) \times S(B)$, $\varepsilon_{p,q} \leq t_{p,q}$ and so $\| z \|_{\infty, A \hat{\otimes} B} \leq \| z \|_{\infty, A \hat{\otimes} B}$ with the result $b(A \hat{\otimes} B) = b(A \hat{\otimes} B)$.

Further, above (ii) in the light of Lemma 3.5, applied to the pro-$C^\ast$-algebra $A \hat{\otimes} B$, implies that $b(A \hat{\otimes} B) = b(A \hat{\otimes} B)$ and $\| \cdot \|_{\infty, A \hat{\otimes} B} = \| \cdot \|_{\infty, A \hat{\otimes} B}$.

Further, $b(A \otimes b(B) = b(A \hat{\otimes} B)$ and the norm $\| \cdot \|_{\min}$ on $b(A \otimes b(B)$ is [27, p. 207]

\[ \| \cdot \|_{\min} = \sup \{ \| \pi \otimes \sigma(z) \| \mid \pi \in S(A), \sigma \in S(B) \} \geq \| \cdot \|_{\infty, A \hat{\otimes} B} \] by definition of $\| \cdot \|_{\infty, A \hat{\otimes} B}$.

But $\| \cdot \|_{\min}$ is the smallest among all $C^\ast$-norms on $b(A \otimes b(B)$ [27, Proposition 4.19, p. 216]. Hence $\| \cdot \|_{\min} = \| \cdot \|_{\infty, A \hat{\otimes} B} = \| \cdot \|_{\infty, A \hat{\otimes} B}$ on $b(A \hat{\otimes} B)$ and so on $b(A \hat{\otimes} B)$. Finally, the $\lambda$-norm on $b(A \otimes b(B)$ is [27, p. 188], with $z = \Sigma f_i \otimes \chi_i$, $\lambda(z) = \sup \{ |\Sigma f_i(x_1)g_i(y_1)| \mid f \in b(A)^*, g \in b(B)^*, \| f \|_{\infty, A \hat{\otimes} B}, \| g \|_{\infty, A \hat{\otimes} B} \} \leq \| z \|_{\infty, A \hat{\otimes} B}$ by definition of $\| \cdot \|_{\infty, A \hat{\otimes} B}$.

Thus, on $b(A \otimes b(B)$, $\| \cdot \|_{\infty, A \hat{\otimes} B} \leq \lambda(\cdot) \leq \| \cdot \|_{\min} \leq \lambda(\cdot) \leq \| \cdot \|_{\infty, A \hat{\otimes} B}$ with the result that $\lambda(\cdot) = \| \cdot \|_{\min}$.

It follows [27, Theorem 4.14, p. 211] that either $b(A)$ or $b(B)$ is abelian. But $b(A)$ is dense in $A$, $b(B)$ is dense in $B$ and the multiplication in a pro-$C^\ast$-algebra is jointly continuous. It follows that either $A$ or $B$ is abelian. (1) $\Rightarrow$ (4) is a consequence of $C^\ast$-algebra theory together with density arguments as above. Similarly one gets (1) $\Rightarrow$ (5) and (1) $\Rightarrow$ (6) from (1) $\Rightarrow$ (3). This completes the proof.

**Remark 3.6.** In the above theorem, to conclude (3) $\Rightarrow$ (1) and similar other implications, the hypothesis that the $\varepsilon$-calibration is an $m^\ast$-calibration cannot be omitted, even for $C^\ast$-algebras. Take $A$ to be a non-abelian finite dimensional $C^\ast$-algebra. Then for all $C^\ast$-algebras $B$, $A \hat{\otimes} B = A \hat{\otimes} B$ and $\varepsilon = \pi$, i.e., $\lambda$ and $\gamma$, and so $\lambda$ and $\| \cdot \|_{\min}$ are equivalent. On the other hand $\lambda = \| \cdot \|_{\min}$ forces $A$ or $B$ to be abelian.

**Proposition 3.7.**

Let $\phi_1 : A_1 \to B_1$, $\phi_2 : A_2 \to B_2$ be continuous completely positive maps between pro-$C^\ast$-algebras. Then

$$\phi_1 \otimes \phi_2 : A_1 \otimes A_2 \to B_1 \otimes B_2,$$

is an $\alpha - \alpha$ continuous (respectively $\nu - \nu$ continuous) completely positive map that extends as a completely positive map $\phi_1 \otimes \phi_2 : A_1 \hat{\otimes} A_2 \to B_1 \hat{\otimes} B_2$ (respectively, $\phi_1 \otimes \phi_2 : A_1 \hat{\otimes} A_2 \to B_1 \hat{\otimes} B_2$).

**Proof.** We shall prove for the topology $\alpha$. The other assertion can be similarly established. In view of Lemma 3.1, it is sufficient to find coherent continuous completely positive maps $\psi_{p,q} : A_1 \hat{\otimes} A_2 \to (B_1)_q \hat{\otimes} (B_2)_q$ for $(q_1, q_2) \in S(B_1) \times S(B_2)$. By continuity of $\phi_1$ and $\phi_2$, there exist $p_1 \in S(A_1)$, $p_2 \in S(A_2)$ such that $\| (\tau_1 \circ \phi_1)(x_1) \|_{p_1} \leq M \| x_1 \|_{p_1}, \| (\tau_2 \circ \phi_2)(x_2) \|_{p_2} \leq M \| x_2 \|_{p_2}$ for $x_1 \in A_1$, $x_2 \in A_2$, $\pi_i : B_i \to (B_i)_{q_i}$, $i = 1, 2$ are the natural quotient maps. It follows that the maps $\pi_i \circ \phi_i$ factor through
the quotient C*-algebras \((A_1)_{p_1} \to (B_1)_{q_1}\). By the corresponding property of C*-algebras [27, p. 218], there is a completely positive map \(\vartheta: (A_1)_{p_1} \otimes_{\min} (A_2)_{p_2} \to (B_1)_{q_1} \otimes_{\min} (B_2)_{q_2}\). The desired map \(\psi_{p,q}\) is the composite map \(A_1 \otimes A_2 \to (A_1)_{p_1} \otimes_{\min} (A_2)_{p_2} \to (B_1)_{q_1} \otimes_{\min} (B_2)_{q_2}\). This completes the proof.

PROPOSITION 3.8

Let \(A\) and \(B\) be pro-C*-algebras. Let \(B^0\) (respectively \(B^*\)) denote the algebraic dual (respectively topological dual) of \(B\). For a linear functional \(\omega\) on \(A \otimes B\), let \(T_\omega: A \to B^0\) be defined by \(\langle y, T_\omega x \rangle = \langle x \otimes y, \omega \rangle\).

(1) \(\omega\) is a state on \(A \otimes B\) iff \(T_\omega\) is a complete state map (i.e., \(T_\omega\) is a completely positive map such that \(T_\omega(1)\) is a state on \(B\)).

(2) Further, \(\omega\) is continuous in the topology \(v\) (so that \(\omega\in (A \otimes N)B^*\)) iff \(T_\omega(A)\subseteq B^*\) and \(T_\omega: A \to B^*\) is continuous, where \(B^*\) carries the topology \(\tau_b\) of uniform convergence on all bounded subsets of \(B\).

Proof. (1) is exactly as in [27, Chapter IV, Proposition 4.6]. Further, let \(\omega\) be \(v\)-continuous. There exist \((p, q)\in S(A) \times S(B)\) and a scalar \(k > 0\) such that for all \(z\in \mathbb{C} \otimes A\), \(|\omega(z)| \leq kv_{p,q}(z)\). Hence for all \(x \in A\), \(y \in B\), \(|\langle y, T_\omega(x) \rangle| = |\langle x \otimes y, \omega \rangle| \leq kv_{p,q}(x \otimes y) = kp \otimes q(x \otimes y) = q(p(x)y)\). Hence for \(K \subseteq B\) bounded with \(q(y) = M_{k,K}(y)\subseteq K\), \(|\langle y, T_\omega(x) \rangle| \leq kM_{q,k}(x \otimes A)\) showing \(T_\omega A \subseteq B^*\) and \(T_\omega\) is continuous in \(\tau_b\). Conversely, let \(T_\omega: A \to B^*\) be \(\tau_b\) continuous. By general theory of topological tensor products [26], \(\omega\in (A \otimes N)B^*\). By the GNS construction [11, Theorem 3.4] on the complete locally m-convex *algebra \(A \otimes_\Delta B\), there exists a continuous bounded representation \(\pi_\omega: A \otimes_\Delta B \to B(H_x)\) on a Hilbert space \(H_x\) having a cyclic vector \(\xi_0\) such that \(\omega(z) = \langle \pi_\omega(z)\xi_0, \xi_0 \rangle\) and \(\|\pi_\omega(z)\| = (p \otimes q)(z)(z \in A \otimes B)\) for some \((p, q)\in S(A) \times S(B)\).

Thus \(\pi_\omega \in \mathcal{R}_{p,q}(A \otimes_\Delta B)\); and for all \(z\), \(|\langle z, \omega \rangle| = |\pi_\omega(z)\xi_0, \xi_0 \rangle| \leq \|\pi_\omega(z)\| \|\xi_0\|^2 \leq v_{p,q}(z) \|\xi_0\|^2\) showing that \(\omega\) is a continuous in the topology \(v\).

4. Nuclear pro-C*-algebras

Following a suggestion in [23], a pro-C*-algebra is called nuclear if for each \(p \in S(A)\), the C*-algebra \((\mathbb{C} \otimes N)A_p\) is a nuclear C*-algebra [16], [17] in the sense that for any C*-algebra \(B\), \(\|\cdot\|_{\min} = \|\cdot\|_{\max} \) on \(A \otimes N\). Thus a nuclear C*-algebra is a nuclear pro-C*-algebra; and a nuclear pro-C*-algebra is an inverse limit of nuclear C*-algebras. Commutative pro-C*-algebras, the matrix algebra \(M_n(A)\) over a nuclear pro-C*-algebra \(A\) and a pro-C*-algebra of type I (in the sense that all continuous bounded representations are of type I) are all nuclear pro-C*-algebras.

PROPOSITION 4.1

Let a pro-C*-algebra \(A = \lim_{\rightarrow \Delta} B_x\), an inverse limit of C*-algebras with the maps \(\pi_x: A \to B_x\) of the inverse system assumed surjective. Then \(A\) is a nuclear pro-C*-algebra iff each \(B_x\) is a nuclear C*-algebra.

Proof. One way is obvious, since \(A = \lim_{\rightarrow \Delta} B_x\). Conversely, let \(A = \lim_{\rightarrow \Delta} B_x\) where each \(B_x\) is a nuclear C*-algebra. Then the family \(\{p_x\}_{x \in \Delta}\) of continuous C*-seminorms determines the topology of \(A\), where \(p_x(x) = \|\pi_x(x)\|_x\), \(\pi_x\) being the
*homomorphism from $A$ to the $C^*$-algebra $(B_x, \| \cdot \|_x)$. The $B_x$ is *isomorphic to the $C^*$-algebra $A_x = A/\ker p_x$ with $C^*$-norm $\| x + \ker p_x \|_{p_x} = p_x(x)$ [1, Theorem 2.4]. Thus $A_x$ is nuclear. Given $p \in S(A)$, by continuity, there exists an $a \in A$ such that $p \leq p_x$. Thus $\phi: A_x \to A_p$, $\phi(x + \ker p_x) = x + \ker p$ is a well defined continuous surjective *homomorphism, and $A_x$ is *isomorphic to the quotient $C^*$-algebra $A_y/\ker \phi$, which is nuclear, since $A_y$ is nuclear [6], [17]. This completes the proof.

For pro-$C^*$-algebras $A$ and $B$, the identity map $A \otimes B \to A \otimes B$ extends to a continuous surjective *homomorphism $\psi: A \otimes B \to A \otimes B$. The following shows that $A$ is nuclear iff $\psi$ is a homeomorphism for all $B$.

**Theorem 4.2.** For a pro-$C^*$-algebra $A$, the following are equivalent.

1. $A$ is nuclear.
2. For all pro-$C^*$-algebras $B$, $A \otimes \varepsilon B = A \otimes B$ with $\varepsilon = v$.
3. For all $C^*$-algebras $B$, $A \otimes \varepsilon B = A \otimes B$ with $\varepsilon = v$.
4. For all pro-$C^*$-algebras $B$ (respectively $C^*$-algebras $B$), there is only one admissible pro-$C^*$-topology on $A \otimes B$.

Further, if $A$ is a $\sigma$-$C^*$-algebra, then above are equivalent to any of the following.

5. For all $\sigma$-$C^*$-algebras $B$, $A \otimes \varepsilon B = A \otimes B$.
6. For all $C^*$-algebras $B$, $A \otimes \varepsilon B = A \otimes B$.
7. For all $\sigma$-$C^*$-algebras $B$ (respectively $C^*$-algebras $B$), the topology $v$ on $A \otimes B$ is faithful.

An admissible topology $\tau$ on $A \otimes B$ is faithful [12] if the map $i_\tau: A \otimes B \to A \otimes B \subset B(A^*, B^*)$, $i_\tau(z) = (x \otimes y)(z)$, $x \in A^*$, $y \in B^*$ is one–one. The following improves [12, Proposition 3.3].

**Lemma 4.3.** The injective tensorial pro-$C^*$-topology $\varepsilon$ on $A \otimes B$ is faithful.

**Proof.** The map $i_\varepsilon: A \otimes \varepsilon B \to A \otimes \varepsilon B$ is the unique continuous linear extension of the identity map $i: A \otimes B \to A \otimes B$, $i(z) = z$. Let $z \in A \otimes B$ with $i_\varepsilon(z) = 0$. Let $(z_\lambda)$ be a net in $A \otimes B$, $z_\lambda = \Sigma_{i=1}^{k_\lambda} x_{i(\lambda)} \otimes y_{i(\lambda)} \rightarrow z$. Then for all $f \in A^*$, $g \in B^*$, $\lim_\lambda i_\varepsilon(z_\lambda)(f, g) = \lim_\lambda f \otimes g(z_\lambda) = f \otimes g(z) = 0$. To show $z = 0$, we show that $t_{p,q}(z) = 0$ for all $(p, q) \in S(A) \times S(B)$. Let $\pi \in R_p(A)$, $\sigma \in R_q(B)$, $\xi_1 \in H_\pi$, $\eta_1 \in H_\pi$, $\xi_2 \in H_\sigma$, $\eta_2 \in H_\sigma$. Define $f$ and $g$ by $f(x) = \langle \pi(x) \xi_1, \eta_1 \rangle$, $g(y) = \langle \sigma(y) \xi_2, \eta_2 \rangle$. Then

$$\langle \pi \otimes \sigma(z)(\xi_1 \otimes \xi_2), \eta_1 \otimes \eta_2 \rangle$$

$$= \lim_\lambda \left( \sum_{i=1}^{k_\lambda} \pi(x_{i(\lambda)}) \otimes \sigma(y_{i(\lambda)}) \xi_1 \otimes \xi_2, \eta_1 \otimes \eta_2 \right)$$

$$= \lim_\lambda \sum_{i=1}^{k_\lambda} f(x_{i(\lambda)}) g(y_{i(\lambda)}) = \lim_\lambda f \otimes g(z_\lambda) = 0.$$

Hence $\pi \otimes \sigma(z) = 0$ on the completed Hilbert space tensor product $H_\pi \otimes H_\sigma$. It follows that $t_{p,q}(z) = 0$.

**Lemma 4.4.** The following are equivalent for pro-$C^*$-algebras $A$ and $B$.

(a) $A \otimes \varepsilon B = A \otimes B$ with $\varepsilon = v$. 

(b) For all \((p, q) \in S(A) \times S(B)\), \(A_p \hat{\otimes}_{\min} B_q = A_p \hat{\otimes}_{\max} B_q\).

(c) For all \((p, q) \in S(A) \times S(B)\), \(t_{p, q}(\cdot) = v_{p, q}(\cdot)\) on \(A \otimes B\).

Proof. (c) \(\iff\) (a) is a consequence of uniqueness of \(C^*\)-norm and Lemma 3.1. We show (a) \(\Rightarrow\) (b). Let \(p, q \in S(A), q \in S(B)\) and \(\pi_p: A \to A_p, \pi_q: B \to B_q\) be the quotient maps. Then \(\pi_p \otimes \pi_q: A \otimes B \to A_p \otimes_{\min} B_q\) defines a continuous \(*\)-homomorphism. By [27, Proposition 4.7, p. 207], the continuous \(*\)-homomorphisms \(\phi: A \to A_p \otimes_{\min} B_q = C\) (say), \(\psi: B \to C, \phi(x) = x_p \otimes 1_q, \psi(y) = 1_p \otimes y_q\) give continuous \(*\)-homomorphism \(\eta: A \otimes B \to C\) satisfying \(\eta(z) = (\pi_p \otimes \pi_q)(z)\) on \(A \otimes B\). By the assumption (a), \(\eta = \pi_p \otimes \pi_q\); and \(\pi_p\) and \(\pi_q\) being surjective, \(A_p \otimes_{\min} B_q = (\pi_p \otimes \pi_q)(A \otimes B) = \eta(A \otimes B) \subset A_p \otimes_{\max} B_q\). The continuous \(*\)-homomorphism \(k: A_p \otimes_{\min} B_q \to A_p \otimes_{\max} B_q\) so defined is the extension of the identity map. It follows that for all \(z \in A_p \otimes B_q\), \(\|z\|_{\max} \leq \|z\|_{\min} \leq \|z\|_{\max}\), and (b) follows.

Proof of theorem 4.2. Lemma 3.1 gives (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3). (2) \(\iff\) (4) is a consequence of Theorem 3.2. (3) \(\Rightarrow\) (1) follows from (2) \(\Rightarrow\) (1), which is a consequence of Lemma 4.4. The remaining assertions on \(\sigma\)-\(C^*\)-algebras follow from the open mapping theorem and Lemma 4.3.

It is shown in [23] that the functor \(b(\cdot): A \to b(A)\) from pro-\(C^*\)-algebras to \(C^*\)-algebras is not well behaved with respect to tensor products. Still, the following main result of this section holds giving an aesthetically pleasing nuclearity criterion.

Theorem 4.5. A pro-\(C^*\)-algebra \(A\) is nuclear iff \(b(A)\) is a nuclear \(C^*\)-algebra.

Lemma 4.6. Let \(A\) be a pro-\(C^*\)-algebra. Let \(p \in S(A), I_p = b(A) \cap \ker p\). The quotient \(C^*\)-algebra \(b(A)/I_p\) with the quotient \(C^*\)-norm \(\|\cdot\|_{\alpha, p}\) induced by the \(C^*\)-norm on \(b(A)\) is isometrically \(*\)-isomorphic to the \(C^*\)-algebra \(A_p\).

Proof. Let \(\pi_p: A \to A_p\) be \(\pi_p(x) = x + \ker p\) so that \(\pi_p(A) = A_p\). Since \(b(A)\) is dense in \(A\), [23, Proposition 1.11] implies that \(\pi_p(b(A)) = A_p\). The \(*\)-ideal \(I_p\) in \(b(A)\) is closed in \(\|\cdot\|_\alpha\) due to continuity of embedding \(b(A) \to A\). Since \(I_p = \ker \psi_p\) where \(\psi_p = \pi_p|_{b(A)}\), \(b(A)/I_p\) is \(*\)-isomorphic to \(A_p\).

Proof of theorem 4.5. Nuclearity of the \(C^*\)-algebra \(b(A)\) implies [6, Corollary 4] that \((I_p, \|\cdot\|_\alpha)\) and \((b(A)/I_p, \|\cdot\|_{\alpha, p})\) are nuclear \(C^*\)-algebras for each \(p \in S(A)\). By Lemma 4.6, \(A_p\) is nuclear for all \(p\), and hence \(A\) is nuclear. Conversely, let \(A\) be a nuclear pro-\(C^*\)-algebra. For each \(p \in S(A)\), the \(C^*\)-algebra \(A_p\) can be regarded as a \(C^*\)-algebra of operators on some Hilbert space \(H_p\) by taking a faithful representation \(\sigma_p\). Then \(\theta_p = \sigma_p \circ \psi_p\) gives a representation of \(b(A)\) on \(H_p\) such that \(\theta_p(b(A)) = \sigma_p(A_p)\) is a nuclear \(C^*\)-algebra. By [6], [17], the von Neumann algebra \([\theta_p(b(A))]^{\text{ce}}\) (second commutant, identified with the bidual \([\theta_p(b(A))]^{**}\)) is semidiscrete. By [27, Chapter IV, Lemma 2.2], \(\theta_p\) extends as a surjective normal homomorphism \(\tilde{\theta}_p: b(A)^{**} \to [\theta_p(b(A))]^{\text{ce}}\). Thus \([\tilde{\theta}_p|_{p \in S(A)}\) is a faithful family of normal representations of the von Neumann algebra \(b(A)^{**}\) such that \(\tilde{\theta}_p(b(A)^{**}) = \theta_p(b(A))^{\text{ce}}\) is semidiscrete. Hence by [10, Corollary 3.3], \(b(A)^{**}\) is semidiscrete, and so \(b(A)\) is a nuclear \(C^*\)-algebra [10, Theorem 6.4].

Remark 4.7. Lemma 4.6 depends only on the fact that \(b(A)\) is dense in \(A\) and is continuously embedded in \(A\). Thus, in above theorem, we have a slightly stronger
Lemma 4.9 Let $A$ be a pro-$C^*$-algebra. Let $x \in A$. For each $n = 1, 2, \ldots$, Let $x_n = x(1 + (1/n)x^*x)^{-1}$. Then each $x_n \in b(A)$ and $x_n \to x$ in $A$.

We claim that $\cup_n b(B_n)$ is dense in $A$. Take $x \in A$. Choose a sequence $(x_k)$ in $\cup_n B_n$, say $x_k \in B_{n_k}$ so that $x_k \to x$. We can assume $(n_k)$ to be non-decreasing. By above lemma, for each $k$ and each $n = 1, 2, \ldots$, $x_{k,n} = x_k(1 + (1/n)x^*_kx_k)^{-1} \in b(B_{n_k})$, $x_k = \lim_{n \to \infty} x_{k,n}$ in $B_n$. Then $x = \lim_k x_k = \lim_k \lim_{n \to \infty} x_{k,n}(1 + (1/n)x^*_kx_k)^{-1}$ in $A$ and $x_k(1 + (1/n)x^*_kx_k)^{-1} \in \cup_n b(B_n)$ for all $k, n$. Hence $\cup_n b(B_n)$ is dense in $A$. Now for each $n$, $b(B_n) \subset b(A)$, $\cup_n b(B_n) \subset b(A)$. Let $K$ be the closure of $\cup_n b(B_n)$ in the $C^*$-algebra $b(A)$. Then the $C^*$-algebra $K$ is continuously embedded in $A$ with dense range. By Theorem 4.5, each $b(B_n)$ is a nuclear $C^*$-algebra, with the result, [15, Proposition 11.3.12, p. 859] implies that $K$ is nuclear $C^*$-algebra. The conclusion follows from Remark 4.7.

Remark 4.10. We could not establish a more general result involving arbitrary pro-$C^*$-algebras with arbitrary inductive limit. However, the following particular case can be similarly established.

**PROPOSITION**

Let $A$ be a pro-$C^*$-algebra. Suppose there exists a family $\{B_{\alpha} \mid \alpha \in \Delta\}$ of $C^*$-algebras such that

(i) each $B_{\alpha}$ is a closed *subalgebra of $A$ containing the identity of $A$,

(ii) each $B_{\alpha}$ is a nuclear $C^*$-algebra,

(iii) given $\alpha, \beta$ in $\Delta$, there exists a $\gamma \in \Delta$ such that $B_{\alpha} \cup B_{\beta} \subset B_{\gamma}$,

(iv) $\cup_{\alpha} B_{\alpha}$ is dense in $A$.

Then $A$ is a nuclear pro-$C^*$-algebra.

Finally, we aim to discuss the analog of a basic result of Lance [16, Theorem 3.4] using Proposition 3.8.

**Theorem 4.11.** Let $A$ be a pro-$C^*$-algebra. If $A$ is nuclear, then for any pro-$C^*$-algebra $B$, every continuous complete state map from $A$ to the strong dual $B^*$ can be approximated in simple weak* convergence by continuous complete state maps from $A$ to $B^*$ of finite rank.

**Lemma 4.12.** Let $A$ and $B$ be pro-$C^*$-algebras. Let $f$ be a continuous state on $A \otimes B$. Then the complete state map $T_f : A \to B^*$, $\langle y, T_f x \rangle = \langle x \otimes y, f \rangle$ can be approximated in simple weak* convergence by complete state maps of finite rank.

**Proof.** Let $(p, q) \in S(A) \times S(B)$. Letting the $C^*$-algebras $A_p$ and $B_q$ act faithfully on Hilbert spaces $H_p$ and $K_q$ respectively, $A_p \otimes_{\min} B_q$ acts faithfully on $H_p \otimes K_q$; and by $C^*$-theory, the state space $K(A_p \otimes_{\min} B_q)$ is the weak* closed convex hull of $D_p(0)$, $j(p, q)$, where $D_p(0) = \{\omega_1 | \xi = \sum \xi_i \otimes \eta_i \} \cup \{\omega_2 | \xi = \langle \xi, \xi \rangle\}$. Let $\pi : A \otimes_{\alpha} B \to A \otimes B / \ker \psi_{A,B} = A_p \otimes_{\min} B_q$ be the quotient map. Let $D_j = \{\omega_3 \psi_{A,B}(\omega_2 \in D_j(0))\}$, $D = \cup_j \{D_j | j \in S(A) \times S(B)\}$. Then in the light of statement (a), (b), (c) based on [4] in the proof of (1)⇒(2) of Theorem 3.4, $K_j(A \otimes B)$ is the weak* closed convex hull of $D_j$, and $c_{D} = c_{D} \cup \{c_{D} \cup S(A) \times S(B)\} = c_{D} \cup \{K_j(A \otimes B)\} = c_{D} K(A \otimes B) = K(A \otimes B)$, being weak* closed and convex [6].
The lemma follows from this, since for \( f \in D \), \( T_f \) is a continuous complete state map of finite rank as in Proposition 3.8.

**Proof of theorem 4.11.** Let \( T: A \to B^* \) be a continuous complete state map. By Proposition 3.8, there is an \( f \in K(A \otimes B) \) such that \( T = T_f \). By Theorem 4.2, \( \alpha = \nu \) and \( f \in \mathcal{C}D \) so that \( f \) is a weak* limit of convex combinations of members of \( D \). The assertion follows from Lemma 4.12.

It would be interesting to examine the converse of Theorem 4.11. Arguments in [16, Theorem 3.4] fail essentially because \( K(A \otimes B) \) need not be equicontinuous, nor does the equality \( A \otimes B = A \otimes \alpha B \) seem to imply \( \nu = \alpha \) automatically. However, it is possible to obtain a version of Theorem 4.11 that admits a converse, and that too can be regarded as a generalization of [16, Theorem 3.4].

**Theorem 4.13.** For pro-C*-algebra \( A \), the following are equivalent.

1. \( A \) is nuclear.
2. For every C*-algebra \( B \), for every continuous complete state map \( \phi: A \to B^* \) and for every \( p \in S(\phi) \), there exists \( K > 0 \) such that \( \|\phi(x)\| \leq Kp(x) \) for all \( x \), there exists a net \( (\phi_j) \) of continuous complete state maps \( \phi_j: A \to B^* \) of finite ranks such that
   a. \( \phi = \lim_j \phi_j \) in simple weak* convergence,
   b. \( p \in S(\phi_j) \) for all \( j \).

This can be proved by passing to the C*-algebra quotient \( A_p \) and applying corresponding result for C*-algebras.

5. An application: Multipliers of the Pedersen ideal of a C*-algebra

A *multiplier* on a *algebra \( A \) without identity is a pair \((l, r)\) consisting of linear maps \( l, r: A \to A \) such that for all \( x, y \) in \( A \), \( l(xy) = (l(x))y \), \( r(xy) = xr(y) = x\) and \( xl(y) = r(x)y \). The collection \( \Gamma(A) \) of all multipliers on \( A \) is a *algebra with operations: \((l_1 l_2, r_1 r_2) = (l_1 + l_2, r_1 + r_2), \lambda(l, r) = (\lambda_l, \lambda r), (l_1 r_1)(l_2 r_2) = (l_1 l_2, r_1 r_2) \) and \((l, r)^* = (r, l^*) \) where \( r^* = r(a^*) \), \( l^* = l(a^*) \). A is embedded as a two-sided *ideal in \( \Gamma(A) \) via the *isomorphism \( \mu: \mathcal{A} A \to \{l_a x = xa, r_a x = xa; \mu \) is onto iff \( A \) has identity. For a topological *algebra \( A \), let \( M(A) = \{ (l, r) \in \Gamma(A) \mid l \) and \( r \) are continuous \}. If \( A \) is a pro-C*-algebra, then \( M(A) = \Gamma(A)[30] \); and then \( M(A) \) is a pro-C*-algebra with seminorm topology \( \tau \) defined by the calibration \( \{ \|p\|_p \mid p \in S(A) \}, \|l, r\|_p = \sup \{p(l(a)) | p(a) \leq 1 \} [23] \). \( M(A) \) is a C*-algebra if \( A \) is a C*-algebra Corollary 4.8 (2) implies that if \( M(A) \) is nuclear, then \( A \) is nuclear; but the converse does not hold even if \( A \) is a C*-algebra. Take \( A = K(H) \), the C*-algebra of all compact operators on a Hilbert space \( H \). Being a C*-algebra of type I, \( K(H) \) is nuclear; but \( M(A) = B(H), (the C*-algebra of all bounded operators on \( H) \) is not nuclear, for the Calkin algebra \( B(H)/K(H) \) is known not to be nuclear.

**Theorem 5.1.** Let \( A \) be a pro-C*-algebra. The C*-algebra \( b(M(A)) \) is isometrically *isomorphic to the C*-algebra \( M(b(A)) \). Thus \( M(A) \) is a nuclear pro-C*-algebra if \( b(A) \) and the generalized Calkin algebra \( M(b(A))/b(A) \) are nuclear C*-algebras.
Proof. In view of Corollary 4.8 (2), it is sufficient to show that \( b(M(A)) = M(b(A)) \) up to isomorphism. Let \((e_\varepsilon, \|e_\varepsilon\| \leq 1\) be an approximate identity for the \(C^*\)-algebra \(b(A)\). Then \((e_\varepsilon)\) is also an approximate identity for \(A\) [23]. We show that \(b(M(A)) \subset M(b(A))\). Let \((l, r) \in M(A)\) with \(\|l(l, r)\|_\infty = \sup \{ \|l(l, r)\|_p | p \in S(A) \} < \infty\). It is sufficient to show that \(l(b(A)) \subset b(A), r(b(A)) \subset b(A)\). For \(x \in b(A), p \in S(A), p(l(x)) = p(l(lim_{\varepsilon \to 0} e_\varepsilon x)) = p(lim_{\varepsilon \to 0} p(l(e_\varepsilon x))) = lim_{\varepsilon \to 0} p(l(e_\varepsilon x)) \leq lim_{\varepsilon \to 0} p(l(e_\varepsilon x)) p(x) \leq \|l(l, r)\|_\infty \|x\|_\infty\) showing \(l(x) \in b(A)\). Similarly \(r(x) \in b(A)\). This defines a \(\ast\)-homomorphism \(\Phi : b(M(A)) \to M(b(A)), \Phi(l, r) = (l|_{M(A)}, r|_{M(A)})\). Since \(b(A)\) is dense in \(A\), \(\Phi\) is one-one. We show that \(\Phi\) is surjective by establishing \(M(b(A)) \subset b(M(A))\) in the sense that given \((l, r) \in b(M(A))\) each of \(l\) and \(r\) extends uniquely as continuous linear maps \(L, R : A \to A\) such that \((L, R) \in M(A), \|L, R\|_\infty < \infty\). It is sufficient to show that each \(l, r: (b(A), \tau) \to b(A), \tau\) is continuous, where \(\tau\) is the relativization of the pro-\(C^*\)-topology from \(A\). Since \(l, r\) are continuous in the \(C^*\)-topology on \(b(A), M_1 = sup \|l(e_\varepsilon)\|_\infty, M_\varepsilon = sup \|r(e_\varepsilon)\|_\infty < \infty\). Take \(M = (M_1, M_\varepsilon)\). Let \(p \in S(A), a \in b(A)\). Then \(\|ae_\varepsilon - a\|_\infty \to 0, \|e_\varepsilon - a\|_\infty \to 0\); and \(p(l(a)) = lim p(l(lim_{\varepsilon \to 0} e_\varepsilon a))\) as \(\|\cdot\|_\infty\)-convergence implies \(\tau\)-convergence. Thus \(p(l(a)) = lim p(l(e_\varepsilon a)) \leq lim p(l(e_\varepsilon)) p(a) \leq sup p(l(e_\varepsilon)) p(a) \leq M(a)\) giving the desired continuity of \(l\) (and of \(r\), by a similar argument). This gives existence of continuous linear extensions \(L, R : A \to A\). That \((L, R) \in M(A)\) is a consequence of density of \(b(A)\) in \(A\) and joint continuity of multiplication in \((A, \tau)\), which also implies that for each \(x, y \in A, p(L(x) y) \leq M(p(x), p(y)) \leq M(p(x))\) for all \(p \in S(A)\). Hence \(\|L, R\|_\infty = sup_{p \in S(A)}\|L, R\|_p < \infty\) giving \(L, R) \in b(M(A))\). This completes the proof.

Now let \(A\) be a \(C^*\)-algebra. Let \(\mathcal{X}_A\) denote the Pedersen ideal of \(A\) [22, p. 175], [21]. Let \(X\) be the primitive ideal space of \(A\). Let \(\mathcal{F}\) be the collection of all compact closed subsets of \(X\). For an open subset \(U\) of \(X\), let \(I(U)\) be the closed ideal corresponding to \(U\). It is shown in [25, Theorem 7] that \(\Gamma(\mathcal{X}_A)\) is a pro-\(C^*\)-algebra realized as the multiplier algebra \(M(B)\) of the pro-\(C^*\)-algebra \(B = \lim_{\varepsilon \to 0} A/I(X - C)\); and \(b(\Gamma(\mathcal{X}_A)) = M(A) = B(H)\) [25, Theorem 2]. Theorem 4.1 gives the following.

**Corollary 5.2**

For a \(C^*\)-algebra \(A, \Gamma(\mathcal{X}_A)\) is a nuclear pro-\(C^*\)-algebra iff \(M(A)\) is a nuclear \(C^*\)-algebra iff \(A\) and \((M(A), A)\) are nuclear \(C^*\)-algebra.

Thus nuclearity of \(A\) is not sufficient to guarantee nuclearity of \(\Gamma(\mathcal{X}_A)\). In the case of \(A = K(H)\), one has: \(\mathcal{X}_A = \) all finite rank operators in \(B(H)\) and \(\Gamma(\mathcal{X}_A) = M(A) = B(H)\) [18, p. 30].

6. Nuclear pro-\(C^*\)-algebras and linear topological nuclearity

If a pro-\(C^*\)-algebra is nuclear as a locally convex space in the sense of [26, Chapter IV], then we call \(A\) linearly nuclear, in which case, for every locally convex space \(B, A \otimes \varepsilon B = A \otimes \pi B\) and \(\varepsilon = \pi\). Thus a linearly nuclear pro-\(C^*\)-algebra is a nuclear pro-\(C^*\)-algebra. An infinite dimensional nuclear \(C^*\)-algebra fails to be linearly nuclear in view of Dvoretzky-Roger Theorem [26, Corollary 3, p. 184].

**Theorem 6.1.** Let \(A\) be a \(\sigma\)-\(C^*\)-algebra. The following are equivalent.

1. \(A\) is linearly nuclear.

1. \(A\) is weak* sequentially closed.