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Topological algebras with C^*-enveloping algebras

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Abstract. Let A be a complete topological *algebra which is an inverse limit of Banach *algebras. The (unique) enveloping algebra $\mathscr{E}(A)$ of A, providing a solution of the universal problem for continuous representations of A into bounded Hilbert space operators, is known to be an inverse limit of C^*-algebras. It is shown that $\mathscr{E}(A)$ is a C^*-algebra iff A admits greatest continuous C^*-seminorm iff the continuous states (respectively, continuous extreme states) constitute an equicontinuous set. A Q-algebra (i.e., one whose quasiregular elements form an open set) A has C^*-enveloping algebra. There exists (i) a Frechet algebra with C^*-enveloping algebra that is not a Q-algebra under any topology and (ii) a non-Q spectrally bounded algebra with C^*-enveloping algebra. A hermitian algebra with C^*-enveloping algebra turns out to be a Q-algebra. The property of having C^*-enveloping algebra is preserved by projective tensor products and completed quotients, but not by taking closed subalgebras. Several examples of topological algebras with C^*-enveloping algebras are discussed. These include several pointwise algebras of functions including well-known test function spaces of distribution theory, abstract Segal algebras and concrete convolution algebras of harmonic analysis, certain algebras of analytic functions (with Hadamard product) and Köthe sequence algebras of infinite type. The enveloping C^*-algebra of a hermitian topological algebra with an orthogonal basis is isomorphic to the C^*-algebra c_0 of all null sequences.

Keywords. C^*-enveloping algebra; Q-algebra; hermitian algebra; Segal algebra; Köthe sequence space.

1. Introduction

A complete locally m-convex *algebra A is a topological *algebra that is an inverse limit of Banach* algebras. In representation theory of such algebras, the enveloping algebra $(\mathscr{E}(A), \tau)$ of A has been introduced in [10], [21], [16] which provides a solution of the universal problem for continuous *representations of A into bounded Hilbert space operators. This corresponds to the construction of the enveloping C^*-algebra of a Banach* algebra [13, § 2.7.2, p. 48]. The algebra $(\mathscr{E}(A), \tau)$ is a pro-C^*-algebra [27] in the sense that it is an inverse limit of C^*-algebras. This paper is concerned with those A for which $(\mathscr{E}(A), \tau)$ is a C^*-algebra. In fact, in [16], [17], A is called a *bQ-algebra* if $(\mathscr{E}(A), \tau)$ is a barreled space that is a Q-algebra (a topological algebra A is a *Q-algebra* [26] if the set A_{-1} of all quasiregular elements of A is an open set). The barreled assumption turned out to be redundant; for a pro-C^*-algebra, which is a Q-algebra, is a C^*-algebra [18, Corollary 2.2], [27, Proposition 1.14].

Topological *algebras with C^*-enveloping algebras are important for a couple of reasons. Though non-normed, they are well-behaved. In the literature, bQ-condition has been assumed in several aspects like tensor products [17], hermitian K-theory [24] and representation theory [16]. In fact, the representation theory of such algebras is quite similar to that of Banach* algebras. Further, as exhibited in the present paper, there are several classes of examples of such algebras arising in function theory, Fourier series, abstract harmonic analysis, complex analysis and nuclear spaces, in particular, sequence spaces. In what follows, we briefly describe the contents of the present paper.

In [17], the question of completely specifying the class of bQ-algebras was discussed. We show that a complete lmc-*algebra A has C^*-enveloping algebra (i.e., A is a bQ-algebra) iff A admits greatest continuous C^*-seminorm $p_\infty(\cdot)$ iff the continuous states (respectively, continuous extreme states) constitute an equicontinuous set. This is used to show that the enveloping algebra of a Q lmc-*algebra is a C^*-algebra, but the converse does not hold. An lmc-*algebra A is spectrally bounded (sb) (respectively, *spectrally bounded (*sb)) if the spectrum of each $x \in A$ (respectively, the spectrum of each element of the form x^*x) is bounded. We discuss the examples exhibiting: (i) a Frechet algebra with C^*-enveloping algebra, which is not sb, and which fails to be a Q-algebra under any topology and (ii) a non-Q sb algebra with C^*-enveloping algebra. However, if A is hermitian and having C^*-enveloping algebra, then A is a Q-algebra. Further, it is also shown that if A is *sb, then A admits greatest C^*-seminorm $|\cdot|$ (not necessarily continuous); and such an A is hermitian iff $|\cdot| = s(\cdot)$ iff $s(\cdot)$ is a C^*-seminorm. Here $s(x) = r(x^*x)^{1/2}$ $(x \in A)$, $r(\cdot)$ denoting the spectral radius. Thus if A is *sb, then A is hermitian and has C^*-enveloping algebra iff $s(\cdot)$ is a continuous C^*-seminorm (in which case, $s(\cdot) = p_\infty(\cdot)$). We also show that if A is Frechet, then (i) A is sb iff A is a Q-algebra and (ii) if A is *sb, then A has C^*-enveloping algebra. Projective tensor products and complete quotients of algebras with C^*-enveloping algebras are algebras with C^*-enveloping algebras; but the enveloping algebra of a closed *subalgebra of an algebra with C^*-enveloping algebra need not be a C^*-algebra. We have also discussed several classes of algebras with C^*-enveloping algebras. Notable among these, besides pointwise algebras of functions (including the algebra $C^\infty(X)$ of smooth functions on a compact manifold) are the various test function spaces of distribution theory, topological Segal algebras [11] of harmonic analysis (in particular, certain convolution group algebras of locally compact groups) and Köthe G_∞-sequence algebras [22] (of significance in the theory of nuclear and Schwartz spaces). This also incorporates certain topological algebras with orthogonal bases [15], [20]; and via Fourier expansion and Taylor expansion, algebras of smooth periodic functions (convolution product) and of analytic functions (Hadamard product). The enveloping algebra, of an lmc-*algebra with hermitian orthogonal basis and having C^*-enveloping algebra, is *isomorphic to the C^*-algebra c_0 of all null scalar sequences. Let us note that the class of topological *algebras with C^*-enveloping algebras also include the Frechet algebra of C^∞-elements of automorphic action of Lie group on a C^*-algebra, certain Ψ^*-algebras of pseudo-differential operators and the algebra of local observables of quantum field theory. These will be discussed in a subsequent paper.

Preliminaries and notations

A locally m-convex *algebra (lmc-*algebra) [25], [26], [9], [10] is a linear associative involutive algebra A with complex scalars and with a Hausdorff locally convex topology t on it which is determined by a separating directed family $P=(p_\alpha:\alpha\in\Delta)$ of seminorms satisfying, for all α and for all x, y; $p_\alpha(xy)\leqslant p_\alpha(x)p_\alpha(y)$ (submultiplicativity) and $p_\alpha(x^*)=p_\alpha(x)$ (*-invariance). Let (e_γ) be a bounded approximate identity (bai) for A, i.e. $(e_\gamma)\subset A$ is a net such that (a) for each $x\in A$, $e_\gamma x\to x$, $xe_\gamma\to x$ and (b) for each α, $p_\alpha(e_\gamma)\leqslant 1$ for all γ. One can take P to be the collection $K(A)$ of all continuous *invariant submultiplicative seminorms p satisfying $p(e_\gamma)\leqslant 1$ for all γ. A pro-C^*-algebra [27], [28], [6], [7] (also called an l.m.c. C^*-algebra [16] or a locally C^*-algebra [21]) is a complete lmc-*algebra A in which each p_α is a C^*-seminorm, i.e., each p_α additionally satisfies $p_\alpha(x^*x)=p_\alpha(x)^2$ for all $x\in A$. Given an lmc-*algebra A and $p\in K(A)$, let $N_p=\{x\in A:p(x)=0\}$, and A_p be the Banach *algebra obtained by completing the quotient *algebra A/N_p in the norm $\|x_p\|_p=p(x)$, $x_p=x+N_p$. For a $p_\alpha\in P$, let $A_\alpha=A_{p_\alpha}$. If A is complete, then $A=\varprojlim_{\alpha\in\Delta}A_\alpha=\varprojlim_{p\in K(A)}A_p$, an inverse limit of Banach* algebras [26, Theorem 5.1]. Similarly, a pro-C^*-algebra is an inverse limit of C^*-algebras. An lmc-*algebra A is hermitian if for each $h=h^*$ in A, the spectrum $\mathrm{sp}(h)\subset\mathbb{R}$.

Let A be a complete lmc-*algebra with a bai (e_γ). In representation theory of such algebras [10], [21], [18], the enveloping algebra $(\mathscr{E}(A),\tau)$ of a has been introduced as follows. Let $R(A)$ (respectively, $R'(A)$) be the set of all continuous (respectively, continuous topologically irreducible) *representations $\pi:A\to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. For a $p\in K(A)$, let $R_p(A)=\{\pi\in R(A)$: there exists $k>0$ such that $\|\pi(x)\|\leqslant kp(x)$ for all $x\}$, $R'_p(A)=R_p(A)\cap R'(A)$, $R_\alpha(A)=R_{p_\alpha}(A)$, $R'_\alpha(A)=R'_{p_\alpha}(A)$. Then $R(A)=\cup_\alpha R_\alpha(A)=\cup\{R_p(A):p\in K(A)\}$, $R'(A)=\cup_\alpha R'_\alpha(A)=\cup\{R'_p(A):p\in K(A)\}$. For $p\in K(A)$, $r_p(x)=\sup\{\|\pi(x)\|:\pi\in R_p(A)\}=\sup\{\|\pi(x)\|:\pi\in R'_p(A)\}$ [16, Lemma 4.1] $(x\in A)$ defines a continuous C^*-seminorm on A. Let $r_\alpha(\cdot)=r_{p_\alpha}(\cdot)$. The *radical of A is the *ideal $s\mathrm{rad}\,A=\cap_\alpha N(r_\alpha)=\cap\{N(r_p):p\in K(A)\}$, where $N(r_p)=\{x\in A:r_p(x)=0\}$. The algebra $(\mathscr{E}(A),\tau)$ is the Hausdorff completion of $(A,\{r_\alpha\})$ (equivalently, of $(A,\{r_p:p\in K(A)\})$), i.e. the completion of $A/s\mathrm{rad}\,A$ in the topology τ defined by C^*-seminorms $q_\alpha(x+s\mathrm{rad}\,A)=\inf\{r_\alpha(x+i):i\in s\mathrm{rad}\,A\}=r_\alpha(x)\,(x\in A)$. The pro-$C^*$-algebra $(\mathscr{E}(A),\tau)=\varprojlim\mathscr{E}(A_p)=\varprojlim\varepsilon(A_\alpha)$, where $\mathscr{E}(A_p)$ is the enveloping C^*-algebra of the Banach *algebra A_p [13, §2.7.2, p. 48]. Let $\phi:A\to\mathscr{E}(A)$ be $\phi(x)=x+s\mathrm{rad}\,A$. The algebra $(\mathscr{E}(A),\tau)$ satisfies the universal property that, given $\pi\in R(A)$ (respectively, $\pi\in R'(A)$) there exists a unique $\sigma\in R(\mathscr{E}(A))$ (respectively, $\sigma\in R'(\mathscr{E}(A))$) such that $\pi=\sigma\circ\phi$ [16, p. 69–70]. Further, it is easily seen that $(\mathscr{E}(A),\tau)$ is a unique (up to a homeomorphic *isomorphism) pro-C^*-algebra satisfying this universal property. Thus the following unambiguously makes sense.

DEFINITION

A complete lmc-*algebra A has C^*-enveloping algebra if $(\mathscr{E}(A),\tau)$ is a C^*-algebra.

2. Basic theory of algebras with C^*-enveloping algebras

Throughout the section, A denotes a complete lmc-*algebra with a bai. The following corresponds to the fact that a Banach* algebra admits greatest C^*-seminorm (automatically continuous) viz the Gelfand-Naimark pseudonorm [8, §39].

Theorem 2.1. *The algebra A has C^*-enveloping algebra iff A admits greatest continuous C^*-seminorm. In this case, if $p_\infty(\cdot)$ dentoes the greatest continuous C^*-seminorm on A,* then $p_\infty(\cdot) = sup_\alpha r_\alpha(x) = sup\{\|\pi(x)\|:\pi\in R(A)\} = sup\{\|\pi(x)\|:\pi\in R'(A)\}$ $(x\in A)$; *and* $(\mathscr{E}(A),\tau)$ *is the* C^**-algebra* $(A/N(p_\infty))^\frown$, *the completion of* $A/N(p_\infty)$ *in the norm* $\|x+N(p_\infty)\|_\infty = p_\infty(x)$, $N(p_\infty) = \{x\in A: p_\infty(x)=0\}$.

Proof. Observe that on $A/s\,\mathrm{rad}\,A$, $q_\alpha(x+s\,\mathrm{rad}\,A) = r_\alpha(x)$ for each $\alpha\in\Delta$. Indeed, for any $x\in A$, $q_\alpha(x+s\,\mathrm{rad}\,A) = \inf\{r_\alpha(x+i): i\in s\,\mathrm{rad}\,A\} = \inf_{i\in I}[\sup\{f((x+i)^*(x+i))^{1/2}: f\in P_\alpha(A)\}]$ using $r_\alpha(z) = \sup\{f(z^*z)^{1/2}: f\in P_\alpha(A)\}$ [16, Lemma 4.1], where $P_\alpha(A)$ denotes the set of all continuous positive linear functionals f on A such that $|f(u)| \leqslant p_\alpha(u)$ for all $u\in A$. Since $i\in s\,\mathrm{rad}\,A$, $r_\alpha(i)=0$; and so $f(i^*i)=0$ for all $f\in P_\alpha(A)$. Further, for all such f, by the Cauchy-Schwarz inequality, $f(i^*x) = 0 = f(x^*i)$ for all $x\in A$. Hence

$$\begin{aligned} q_\alpha(x+s\,\mathrm{rad}\,A) &= \inf[\sup\{f(x^*x)+f(i^*x)+f(x^*i)+f(i^*i): i\in I\}^{1/2}: f\in P_\alpha(A)] \\ &= \sup\{f(x^*x)^{1/2}: f\in P_\alpha(A)\} = r_\alpha(x). \end{aligned}$$

Now suppose that A has C^*-enveloping algebra, so that $(\mathscr{E}(A),\tau)$ is a C^*-algebra, the topology τ being determined by a C^*-norm $\|\cdot\|$. By [27, p. 165], for any $z\in\mathscr{E}(A)$, $\sup_\alpha q_\alpha(z) < \infty$, and $\|z\| = \sup_\alpha q_\alpha(z)$. Thus $p_\infty(x) = \|x+s\,\mathrm{rad}\,A\| = \sup_\alpha r_\alpha(x)$ $(x\in A)$ defines a C^*-seminorm on A; and there exists $k>0$ and $\alpha\in\Delta$ such that for all $x\in A$, $p_\infty(x) = \|x+s\,\mathrm{rad}\,A\| \leqslant kq_\alpha(x+s\,\mathrm{rad}\,A) = kr_\alpha(x) \leqslant kp_\alpha(x)$ using [16, p. 69]. Let p be any continuous C^*-seminorm on A, so that, for some $l>0$ and some $\beta\in\Delta$, $p(x)\leqslant lp_\beta(x)$ $(x\in A)$. Then $R_p(A)\subset R_\beta(A)$ and for all x, $r_p(x)\leqslant r_\beta(x)$. Identifying $R_p(A)$ and $R(A_p)$ canonically [16, Proposition 3.5] and using that A_p is a C^*-algebra; it follows that for each x, $p(x) = \|x+N(p)\|_p = \sup\{\|\pi(x+N(p))\|:\pi\in R(A_p)\} = \sup\{\|\pi(x)\|:\pi\in R_p(A)\} = r_p(x) \leqslant r_\beta(x) \leqslant p_\infty(x)$. Thus $p_\infty(\cdot)$ is the greatest continuous C^*-seminorm on A.

Conversely, let A admit greatest continuous C^*-seminorm, say $p_\infty(\cdot)$. There exist $\beta\in\Delta$, $k>0$ such that for all $x\in A$, $p_\infty(x)\leqslant kp_\beta(x)$. Hence, as above $p_\infty(x)\leqslant r_\beta(x)$ and so $p_\infty(x) = r_\beta(x)$ for all x. Since each $r_\alpha(\cdot)$ satisfies $r_\alpha(x)\leqslant p_\alpha(x)$ $(x\in A)$ [16, p. 69], $r_\alpha(x)\leqslant p_\infty(x)$ for all α, for all x. Thus $p_\infty(x) = \sup_\alpha r_\alpha(x)$ $(x\in A)$, $s\,\mathrm{rad}\,A = N(p_\infty)$, and for any x, $\|x+s\,\mathrm{rad}\,A\|_{p_\infty} = p_\infty(x) = \sup_\alpha q_\alpha(x+s\,\mathrm{rad}\,A) = r_\beta(x) = q_\beta(x+s\,\mathrm{rad}\,A)$. It follows that the topology τ on $\mathscr{E}(A)$ is determined by $\|\cdot\|_{p_\infty}$; and then $(\mathscr{E}(A),\tau)$ is a C^*-algebra. This completes the proof.

COROLLARY 2.2.

If A is a Q-algebra, then A has C^*-enveloping algebra.

Proof. Let A be a Q-algebra. By [26, Lemma E.3] A is sb; and [26, Proposition 13.5] implies that there exists an $\alpha_0\in\Delta$ and $k>0$ such that $r(x)\leqslant kp_{\alpha_0}(x)$ for all x. Let q be any continuous C^*-seminorm on A. There exists $p\in K(A)$ and $M>0$ such that $q(x)\leqslant Mp(x)$ for all x. Then for any $h=h^*$ in A and for $n=1,2,3,\ldots$, $q(h) = q(h^{2^n})^{1/2^n} \leqslant M^{1/2^n}p(h^{2^n})^{1/2^n}$. By the spectral radius formula [26, p. 22], $q(h)\leqslant \lim_{n\to\infty}\sup M^{1/2^n}\cdot p_\alpha(h^{2^n})^{1/2^n} \leqslant \sup_{p\in K(A)}\lim_{n\to\infty}\sup p(h^n)^{1/n} = r(h) \leqslant kp_{\alpha_0}(h)$. Hence for any $x\in A$, $q(x) = q(x^*x)^{1/2} \leqslant k^{1/2}p_{\alpha_0}(x)$. Thus $p_\infty(x) = \sup\{q(x): q$ is a continuous C^*-seminorm on $A\} \leqslant k^{1/2}p_{\alpha_0}(x)(x\in A)$, and $p_\infty(\cdot)$ is the greatest continuous C^*-seminorm.

Let A be commutative. Let $\mathscr{M}(A)$ be the Gelfand space (with weak* topology) of A consisting of all continuous multiplicative linear functionals on A. For $\phi\in\mathscr{M}(A)$, let $\phi^*\in\mathcal{M}(A)$ be defined as $\phi^*(x)=\overline{\phi(x^*)}$. The hermitian Gelfand space of A is $\mathcal{M}^*(A)=\{\phi\in\mathcal{M}(A):\phi=\phi^*\}$. The following can be shown, as in [8, Theorem 40.2, p. 220] using the machinery in [16].

Lemma 2.3. *Let A be commutative. Let $\pi\in R'(A)$. Then π is one dimensional, and there exists $\phi\in\mathcal{M}^*(A)$ such that $\pi(x)=\phi^*(x)1$ for all $x\in A$.*

Example 2.4. There exists a unital commutative Frechet *algebra B with C^*-enveloping algebra such that B is not sb and B fails to be a Q-algebra (under any topology). Let $U=\{z\in\mathbb{C}:-1<\operatorname{Re}z<1\}$. Let $C(\bar{U})$ be the algebra, with pointwise operations, of all continuous complex valued functions on $\bar{U}$ with compact open topology t. Let $B=\{f\in C(\bar{U}):f$ is analytic in $U\}$. Then (B,t) is a Frechet *algebra with involution $f\to f^*$, $f^*(z)=\overline{f(\bar{z})}$ $(z\in\bar{U})$. The topology t is defined by the family of seminorms $P=(p_n:n=0,1,2,\ldots)$, where $p_n(f)=\sup\{|f(z)|:z\in K_n\}$, $K_n=\{z\in\bar{U}:n\leqslant \operatorname{Im}z\leqslant n+1\}$. It is easily seen that $\phi\in\mathcal{M}(B)$ iff $\phi=\phi_z$ for some $z\in\bar{U}$, $\phi_z(f)=f(z)$; and $\mathcal{M}^*(B)=\{\phi_z:z$ is real; $-1\leqslant z\leqslant 1\}$. In view of Theorem 2.1 and Lemma 2.3, $p_\infty(f)=\sup_n r_{p_n}(f)=\sup\{|f(z)|:-1\leqslant z\leqslant 1\}<\infty$ $(f\in B)$ defines greatest continuous C^*-seminorm; and $(\mathscr{E}(B),\tau)$ is the supnorm C^*-algebra $C[-1,1]$ of all continuous functions on $[-1,1]$. By [26, Corollary 5.6], for any $f\in B$, the spectrum $\operatorname{sp}(f)=\{f(z):z\in\bar{U}\}$. Thus B is not sb; hence it fails to be a Q-algebra under any topology making it a topological algebra [26, Appendix E].

For $x\in A$, let the hermitian spectral radius of x be defined as $r^h(x)=\sup\{r(\pi(x)):\pi\in R'(A)\}$, $r(\pi(x))$ being the spectral radius of the operator $\pi(x)$ in the C^*-algebra $B(H_\pi)$.

COROLLARY 2.5.

The algebra A has C^*-enveloping algebra iff there exists $p\in K(A)$ and $k>0$ such that $r^h(x)\leqslant kp(x)$ for all $x\in A$.

Proof. If A has C^*-enveloping algebra, then there exists $p\in K(A)$ and $k>0$ such that $p_\infty(x)\leqslant kp(x)$ for all x. It follows that $r^h(x)\leqslant kp(x)$ for all x. Conversely, suppose that there exists $p\in K(A)$ and $k>0$ such that $r^h(x)\leqslant kp(x)$ for all x. Let q be any continuous C^*-seminorm on A. Let $\sigma\in R_q'(A)$. Then, for any $x\in A$, $\|\sigma(x)\|^2=\|\sigma(x^*x)\|\leqslant r(\sigma(x^*x))\leqslant r^h(x^*x)\leqslant kp(x^*x)\leqslant kp(x)^2$ giving $\sigma\in R_p'(A)$. Thus $R_q'(A)\subset R_p'(A)$, with the result, for each $x\in A$, $q(x)=r_q(x)\leqslant r_p(x)$. It follows that $r_p(\cdot)$ is the greatest continuous C^*-seminorm on A; and A has C^*-enveloping algebra.

COROLLARY 2.6.

Let A be a hermitian algebra with C^*-enveloping algebra. Then A is a Q-algebra.

Proof. By [26, Theorem 5.2], the hermiticity of A implies that, for each $q\in K(A)$, the Banach* algebra $(A_q,\|\cdot\|_q)$ is hermitian. Hence, by [8, Lemma 41.2], for each $z\in A_q$, the spectral radius in A_q, $r_{A_q}(z)\leqslant r_{A_q}(z^*z)^{1/2}=|z_q|_q$, where $|\cdot|_q$ denotes the Gelfand-Naimark pseudonorm on A_q. Then $m_q(x)=|x_q|_q(x\in A)$ defines a continuous C^*-seminorm on A. By Theorem 2.1, there exists greatest continuous C^*-seminorm $p_\infty(\cdot)$ on A. By [26, Corollary 5.3], for each $x\in A$, the spectral radius in A, $r(x)=\sup\{r_{A_q}(x_q):q\in K(A)\}\leqslant$

$\sup\{m_q(x): q \in K(A)\} \leqslant p_\infty(x)$. By continuity of $p_\infty(\cdot)$, there exists $p \in K(A)$ and $k > 0$ such that $r(x) \leqslant p_\infty(x) \leqslant kp(x)$ $(x \in A)$. It follows from [26, Proposition 13.5] (or [3, Theorem 14]) that A is a Q-algebra.

Recall that $P_\alpha(A)$ is the set of all continuous positive linear functionals f on A such that $|f(x)| \leqslant p_\alpha(x)$ for all x. As in [16, Theorem 3.1], the bijective correspondence $f \to f_\alpha: x_\alpha \to f_\alpha(x_\alpha) = f(x)$ $(x \in A)$, identifies $P_\alpha(A)$ with the set $P(A_\alpha)$ of all positive linear functionals (automatically continuous) on the Banach* algebra A_α (having bai $((e_\gamma)_\alpha)$). Let $P_c(A) = \cup_\alpha P_\alpha(A)$. The following identifies $P_c(A)$ intrinsically.

Lemma 2.7. *Let f be a continuous positive linear functional on A. Then $f \in P_c(A)$ iff $|f(x)|^2 \leqslant f(x^*x)$ for all $x \in A$.*

Proof. Let there be an $\alpha \in \Delta$ such that $f \in P_\alpha(A)$, so that $|f(x)| \leqslant p_\alpha(x)$ $(x \in A)$. Then, for any $x \in A$, by the continuity of f and Cauchy-Schwarz inequality, $|f(x)|^2 = \lim |f(e_\gamma x)|^2 \leqslant (\overline{\lim}_\gamma f(e_\gamma e_\gamma^*)) f(x^*x) \leqslant (\overline{\lim}\, p_\alpha(e_\gamma e_\gamma^*)) f(x^*x) \leqslant (\overline{\lim}\, p_\alpha(e_\gamma)^2) f(x^*x) \leqslant f(x^*x)$. Conversely, assume that $|f(x)|^2 \leqslant f(x^*x)$ $(x \in A)$. Since f is continuous and $P = (p_\alpha: \alpha \in \Delta)$ is directed, there exists $k > 0$ and $\alpha \in \Delta$ such that $|f(x)| \leqslant kp_\alpha(x)$ $(x \in A)$. Thus for any x, $|f(x)^2| \leqslant f(x^*x) \leqslant kp_\alpha(x^*x) \leqslant kp_\alpha(x)^2$; hence by iterations, $|f(x)| \leqslant k^{1/2^n} p_\alpha(x)$ $(x \in A, n \in \mathbb{N})$. It follows that $f \in P_\alpha(A) \subset P_c(A)$.

For each α, let $B_\alpha(A)$ be the set of all nonzero extreme points of $P_\alpha(A)$. Let $B_c(A)$ be the nonzero extreme points of $P_c(A)$. As in [16], $B_c(A) = \cup_\alpha B_\alpha(A)$. Let $S_c(A)$ denote the set of all continuous C^*-seminorms on A. The following is immediate in view of [16, Lemma 4.1] and Theorem 2.1.

COROLLARY 2.8.

Let A have C^*-enveloping algebra. Then for each $x \in A$, $p_\infty(x) = \sup\{f(x^*x)^{1/2}: f \in P_c(A)\} = \sup\{f(x^*x)^{1/2}: f \in B_c(A)\} = \sup\{p(x): p \in S_c(A)\}$.

COROLLARY 2.9.

The algebra A has C^*-enveloping algebra iff $P_c(A)$ is equicontinuous iff $B_c(A)$ is equicontinuous.

Proof. Let A have C^*-enveloping algebra, so that by Theorem 2.1, the topology τ on $A/s\mathrm{rad}\,A$ is determined by the C^*-norm $\|x + s\mathrm{rad}\,A\|_{p_\infty} = p_\infty(x)$ $(x \in A)$; and there is $c > 0$ and $p \in K(A)$ such that $p_\infty(x) \leqslant cp(x)$ $(x \in A)$. Since the quotient topology t_q on $A/s\mathrm{rad}\,A$ induced by the topology t of A is finer than τ, it follows that for a given bai (e_γ) of A, there exists $k > 0$ such that $p_\infty(e_\gamma) = \|e_\gamma + s\mathrm{rad}\,A\|_{p_\infty} \leqslant k$ for all γ. Let $f \in P_c(A)$. By Corollary 2.8 and the Cauchy-Schwarz inequality, it follows that for any $x \in A$, $|f(x)| = \lim_\gamma |f(e_\gamma x)| \leqslant (\lim_\gamma \sup f(e_\gamma^* e_\gamma)^{1/2})(f(x^*x))^{1/2} \leqslant (\lim_\gamma \sup p_\infty \cdot (e_\gamma^* e_\gamma)^{1/2}) p_\infty(x) \leqslant kp_\infty(x) \leqslant kcp(x)$. Thus $P_c(A)$ (hence $B_c(A)$) is equicontinuous. Conversely, let $B_c(A)$ (or $P_c(A)$) be equicontinuous, so that, there exists $p \in K(A)$ and $k > 0$ such that $|f(x)| \leqslant kp(x)$ for all $x \in A$, $f \in B_c(A)$. Then the quantity $q(x) = \sup\{f(x^*x)^{1/2}: f \in B_c(A)\} < \infty$ $(x \in A)$ defines greatest continuous C^*-seminorm on A. Thus Theorem 2.1 applies.

COROLLARY 2.10.

Let A be commutative. Then A has C^*-enveloping algebra iff the hermitian Gelfand space

$\mathcal{M}^*(A)$ is equicontinuous. In this case, for each $x \in A$, $p_\infty(x) = \sup\{|f(x)| : f \in \mathcal{M}^*(A)\} = r^h(x)$.

Remark 2.11. There is an analogy between Q lmc algebras and lmc-*algebras with C^*-enveloping algebras. Corollaries 2.5 and 2.9 correspond to the fact that B is a Q-algebra iff there is a continuous submultiplicative seminorm p on B and $k > 0$ such that $r(x) \leqslant kp(x)$ $(x \in A)$ iff (in commutative case) $\mathcal{M}(A)$ is equicontinuous [3]. Analogous to Theorem 2.1, it holds that a commutative B is a Q-algebra iff B admits greatest continuous submultiplicative seminorm q with square property $q(x^2) = q(x)^2 (x \in A)$. The details will appear elsewhere.

Remark 2.12. It follows from [17, Theorem 4.1] that if A and B have C^*-enveloping algebras, then so does the completed projective tensor product $A \hat{\otimes}_\pi B$; and $\mathscr{E}(A \hat{\otimes}_\pi B) = \mathscr{E}(A) \hat{\otimes}_{\max} \mathscr{E}(B)$, the maximal C^*-tensor product. Also, if I is a closed regular *ideal of A, then A has C^*-enveloping algebra iff I and the completion of A/I have C^*-enveloping algebras.

Example 2.13. The purpose of this example is to exhibit that (a) there exists a commutative Frechet *algebra B and a closed *subalgebra D such that B has C^*-enveloping algebra but D fails to have C^*-enveloping algebra, (b) there exists a commutative non-Q-algebra B with C^*-enveloping algebra such that B is strongly spectrally bounded (ssb) [17], i.e. for some family $Q = (q_\delta) \subset K(B)$ determining the topology of B, $\sup_\delta q_\delta(x) < \infty$ for all $x \in B$. The example is a modification of [8, Example 16, p. 202]. Let C be a complete lmc-*algebra having identity with a $P = (p_\alpha) \subset K(C)$ determining its topology. Let $B = C \oplus C$ with the product topology defined by the seminorms $q_\alpha((x, y)) = \max(p_\alpha(x), p_\alpha(y))$. The involution $(x, y)^* = (y^*, x^*)$ makes B a complete unital lmc-*algebra on which $f(z^*z) = 0$ $(z \in B)$ for any positive linear functional f on B. Thus $R(B) = \{0\}$, $B = s\,\mathrm{rad}\,B$ and $\mathscr{E}(B) = (0)$, the trivial C^*-algebra.

(i) Let $D = \{(x, x) \in B : x \in C\}$, a closed *subalgebra of B, homeomorphically *isomorphic to A. Take C to be the Frechet *algebra $C(\mathbb{R})$ of all continuous functions on $\mathbb{R}$ with pointwise operations, complex conjugation and with the compact open topology k. The resulting algebra D does not have C^*-enveloping algebra.

(ii) Note that A is ssb (respectively, Q-algebra) iff B is ssb (respectively, Q-algebra). Take C to be the *algebra $C[0, 1]$ of all continuous functions of $[0, 1]$ with the topology of uniform convergence on all countable compact subsets of $[0, 1]$. The resulting algebra B is non-Q, ssb and having C^*-enveloping algebra.

Let f be a positive linear functional, not necessarily continuous, on A. The GNS construction $(\pi_f, D(\pi_f), H_f)$ defines a *homomorphism π_f of A into linear operators (not necessarily bounded) all defined on a dense invariant subspace $D(\pi_f)$ of a Hilbert space H_f as follows: Let $N_f = \{x \in A : f(x^*x) = 0\}$, $D(\pi_f) = A/N_f$ with inner product $\langle x + N_f, y + N_f \rangle = f(y^*x)$, H_f = the Hilbert space obtained by completing $D(\pi_f)$, and $\pi_f(x)(y + N_f) = xy + N_f$. Further, each $\pi_f(x)$ is a bounded operator (so that, by extension, $\pi_f(x) \in B(H_f)$), iff f is admissible i.e., for each x, sup $\{f(y^*x^*xy)/f(y^*y) : f(y^*y) \neq 0, y \in A\}^{1/2}$ $(= \|\pi_f(x)\|) < \infty$. Also, f is extendible if f can be extended as a positive linear functional on the *algebra A_e obtained by adjoining the identity to A. As a consequence of the presence of a bai on A, every continuous positive functional f on A is extendible and hence admissible by Propositions 3.2 and 3.3 of [2]. Let

$P(A)$ be the set of all positive linear functionals f on A satisfying $|f(x)|^2 \leqslant f(x^*x)$ $(x \in A)$. Let $AP(A) = \{f \in P(A): f \text{ is admissible}\}$, $S(A)$ be the set of all C^*-seminorms on A (not necessarily continuous). For $f \in AP(A)$, $p_f(x) = \|\pi_f(x)\|$ gives a $p_f \in S(A)$. For $x \in A$, define $l(x) = \sup\{p(x): p \in S(A)\}$, $u(x) = \sup\{p_f(x): f \in AP(A)\}$, $m(x) = \sup\{f(x^*x)^{1/2}: f \in AP(A)\}$, $s(x) = r(x^*x)^{1/2}$.

Theorem 2.14. (1) *If the algebra A is *sb, then A admits greatest C^*-seminorm $|\cdot|$ (say). In this case, $|x| = l(x) = u(x) = m(x)$ for all $x \in A$.*

(2) *Let A be *sb. The following are equivalent.*

(i) *A is hermitian.*

(ii) *$s(x) = |x|$ for all x.*

(iii) *$x \to s(x)$ is a C^*-seminorm on A.*

If, moreover, A is hermitian, then A is sb and $r(x) \leqslant s(x)$ $(x \in A)$.

(3) *Let A be *sb and have C^*-enveloping algebra. The following are equivalent.*

(i) *A is hermitian.*

(ii) *$s(x) = p_\infty(x)$ for all x.*

(iii) *$x \to s(x)$ is a continuous C^*-seminorm on A.*

If A is commutative and hermitian, then $r(x) = p_\infty(x)$ for all x.

Lemma 2.15. (1) *The algebra A is hermitian iff A is symmetric.*

(2) *Let $a = a^*$ in A with $r(a) < 1$. There exists $x = x^*$ in A with $r(x) < 1$ such that $2x - x^2 = a$.*

(3) *Let $f \in P(A)$, $b \in A$, $f_b(x) = f(b^*xb)$ $(x \in A)$. Then the following hold.*

(i) *$|f_b(k)| \leqslant r(k) f(b^*b)$ for all $k = k^*$ in A.*

(ii) *$|f_b(a)| \leqslant s(a) f(b^*b)$ for all a in A.*

(4) *Given $p \in S(A)$, $b \in A$, there exists $f \in AP(A)$ such that $|f(x)| \leqslant p(x)$ for all $x \in A$ and $f(b^*b) = p(b^*b)$.*

(5) *Let A be Frechet. Each $p \in S(A)$ is continuous.*

(6) *Let A be *sb. For all $p \in S(A)$, $p(x) \leqslant s(x)$ for all $x \in A$.*

(7) *Let A be *sb and hermitian. The following hold.*

(i) *$r(x) \leqslant s(x)$ for all $x \in A$.*

(ii) *$x \to s(x)$ is a C^*-seminorm on A.*

Proof of the lemma. We prove (2). Let $\mathscr{P}_m(A)$ denote the collection of all families $Q = (q_\delta) \subset K(A)$ such that Q determines the topology of A. Given such a Q, the *subalgebra $B_Q = \{x \in A: \sup_\delta q_\delta(x) < \infty\}$ is a Banach* algebra with the norm $q(x) = \sup_\delta q_\delta(x)$, $x \in B_Q$. Now let $a = a^* \in A$ and $r(a) < 1$. By [19, Theorem 4], there exists a $Q \in \mathscr{P}_m(A)$ such that $a \in B_Q$ and the spectral radius of a in B_Q, $r_{B_Q}(a) < 1$. Ford's square root lemma [8, Proposition 12.11] applied to (B_Q, q) gives $x = x^*$ in B_Q with $2x - x^2 = a$ and $r(x) = r_{B_Q}(x) < 1$. This gives (2). The assertion (5) is a consequence of the automatic continuity of the homomorphism $\pi: A \to A_p$ from a Frechet *algebra with a bai to a C^*-algebra. The remaining assertions can be proved by using inverse limit decomposition of A and using corresponding results for Banach* algebras from [8].

Proof of Theorem 2.14. Given $p \in S(A)$, $b \in A$, there exists $f \in AP(A)$ as in Lemma 2.15(4). Then $p_f(b)^2 = p_f(b^*b) = \|\pi_f(b^*b)\| = \sup\{f(x^*b^*bx): f(x^*x) = 1\} \geqslant f(b^*b) = p(b)^2$ showing $l(b) \leqslant u(b)$. Thus $u(x) = l(x)$ for all $x \in A$. Similarly,

$p(b)^2 = f(b^*b) \leqslant m(b)^2$, so that $l(x) \leqslant m(x)$ for all x. Let $f \in AP(A)$ and $\xi_f \in H_f$ be a topologically cyclic vector of norm 1 for π_f so that $f(x) = \langle \pi_f(x)\xi_f, \xi_f \rangle$ $(x \in A)$. Then $f(x^*x) = \|\pi_f(x)\xi_f\|^2 \leqslant p_f(x)^2 \leqslant u(x)^2$ implies $m(x) \leqslant u(x)$. Thus, for all x, $l(x) = m(x) = u(x) = |x|$ (say), which is the greatest C^*-seminorm, if it exists. Assuming A to be *sb, one gets $|x| \leqslant s(x)$ for all x, as follows: For $f \in AP(A)$, $x \in A$, the boundedness of $\pi_f(x)$ implies that

$$
\begin{aligned}
p_f(x) = \|\pi_f(x)\| &= \sup\left\{\frac{\|\pi_f(x)\eta\|}{\|\eta\|} \,|\, \eta \neq 0 \text{ in } H_f\right\} \\
&= \sup\left\{\frac{\|\pi_f(x)(y+N_f)\|}{\|y+N_f\|} \,|\, y \in A, f(y^*y) \neq 0\right\} \\
&= \sup\left\{\frac{f(y^*x^*xy)^{1/2}}{f(y^*y)^{1/2}} \,|\, y \in A, f(y^*y) \neq 0\right\} \\
&\leqslant r(x^*x)^{1/2} = s(x)
\end{aligned}
$$

by Lemma 2.15(3).

(2) follows from Lemma 2.15 and arguments similar to those in Banach* algebras [18, Theorem 41.11].

(3) Let A be *sb and have a C^*-enveloping algebra. Then (ii) implies (i) follows from above (2). Conversely, if A is hermitian, then each A_α is hermitian, hence symmetric by the Shirali-Ford Theorem, and so by the Raikov symmetry criterion [29, Theorem 4.7.21], for all $x \in A$, $r(x^*x) = \sup_\alpha r_{A_\alpha}(x_\alpha^* x_\alpha) = \sup_\alpha[\sup\{f(x_\alpha^* x_\alpha): f \in B(A_\alpha)\}] = \sup_\alpha[\sup \{f(x^*x): f \in B_\alpha(A)\}] = \sup\{f(x^*x): f \in B_c(A)\} = p_\infty(x)^2$ by Corollary 2.8. This, with Lemma 2.15(6), gives $s(x) = p_\infty(x)$ for all x. That (ii) implies (iii) is immediate and (iii) implies (i) follows from (2). If A is commutative, then using [26, § 5], it is easily seen, as in [8, Theorem 35.3], that A is hermitian iff $\mathcal{M}(A) = \mathcal{M}^*(A)$; and if A is *sb, then this holds iff $r(x^*x) = r(x)^2$ for all x. Thus $x \to r(x) = s(x)$ is a C^*-seminorm dominated by a $p \in K(A)$, as A is also Q-algebra by Corollary 2.6. Thus $r(x) = p_\infty(x)$ for all x.

Remarks 2.16. (1) The pro-C^*-algebra $C[0,1]$ of continuous functions on $[0,1]$ with the topology of uniform convergence on all countable compact subsets of $[0,1]$ admits greatest C^*-seminorm, but fails to admit greatest continuous C^*-seminorm.

(2) It follows from Theorem 2.15 that a *sb Frechet *algebra has C^*-enveloping algebra. This is analogous to the result that a sb Frechet algebra is a Q-algebra [3, Theorem 1].

(3) The Frechet *algebra B of Example 2.4 has C^*-enveloping algebra, it admits greatest C^*-seminorm, but is not *sb (as is exhibited by the function $f(z) = z$ in B).

3. Examples: function algebras

Throughout this section, we consider *algebras of functions with pointwise operations and complex conjugation as the involution (except in Example 3.5).

3.1 Let X be a compact, second countable C^∞-manifold. Let $C^\infty(X)$ be the *algebra of all C^∞-functions on X with the topology of uniform convergence on X of functions and all their derivatives. It is a Frechet sb hermitian Q-algebra, and $\mathscr{E}(C^\infty(X)) = C(X)$, the supnorm C^*-algebra of all continuous functions on X. If A is a complete lmc-*algebra with C^*-enveloping algebra, then by [17, Corollary 4.3] $\mathscr{E}(C^\infty(X)\hat{\otimes}_\pi A) = \mathscr{E}(C^\infty(X))\hat{\otimes}_v\mathscr{E}(A) = C(X)\hat{\otimes}_\varepsilon\mathscr{E}(A) = C(X,\mathscr{E}(A))$, $\mathscr{E}(A)$-valued continuous functions on X.

3.2 Let $(B_k)_{k=1}^\infty$ be a sequence of commutative Frechet lmc-*algebras with identities. Let $A_n = B_1 \oplus \cdots \oplus B_n$ with the product topology and the natural involution. Then $(A_n)_1^\infty$ is an m-compatible sequence, and $A = \cup_{n=1}^\infty A_n$ is an involutive LF-algebra. It is a Q-algebra (hence has C^*-enveloping algebra), if each A_n is a Q-algebra (in particular, each B_k is a Banach* algebra) [26, Proposition 15.8]. In particular, the algebra $C_c(\mathbb{R}^n)$ of continuous function on $\mathbb{R}^n$ with compact supports and with the measure topology [26, Example 3.5], as well as the test function algebras $C_c^k(\mathbb{R}^n)$, $1 \leqslant k \leqslant \infty$ of C^k-functions with compact supports and with the Schwartz topologies are Q-algebras. One has $\mathscr{E}(C_c(\mathbb{R}^n)) = \mathscr{E}(C_c^k(\mathbb{R}^n)) = \mathscr{E}(C_c^\infty(\mathbb{R}^n)) = C_0(\mathbb{R}^n)$, the C^*-algebra of continuous functions on $\mathbb{R}^n$ vanishing at infinity. The Frechet *algebra $s(\mathbb{R}^n)$ of rapidly decreasing C^∞-functions on $\mathbb{R}^n$-with the Schwartz topology in an AE-algebra [12, p. 89], hence is lmc [12, Proposition 1], and $\mathscr{E}(s(\mathbb{R}^n)) = C_0(\mathbb{R}^n)$.

3.3 The constructions, analogous to the one due to Arens [1], also lead to several algebras with C^*-enveloping algebras. For $1 \leqslant p < \infty$, let $AC^p[0,1] = \{f\in C[0,1]$: the derivative f' exists a.e. and $f'\in L^p[0,1]\}$, a Banach* algebra with norm $|f|_p = \|f\|_\infty + (\int_0^1 |f'(t)|^p \mathrm{d}t)^{1/p}$, $\|\cdot\|_\infty$ being the supnorm on $C[0,1]$. Let $AC^\omega[0,1] = \cap_{1\leqslant p<\infty} AC^p[0,1]$, a Frechet *algebra, with topology defined by $f \to |f|_p$, $1 \leqslant p < \infty$; and $AC^\omega[0,1] = \varprojlim_p AC^p[0,1]$. Since $\mathscr{M}(AC^p[0,1]) = [0,1]$ by [29, p. 303]. $\mathscr{M}(AC^\omega[0,1]) = [0,1]$ by [26, Proposition 7.5]. The algebra $AC^\omega[0,1]$ is hermitian Q-algebra, and $\mathscr{E}(AC^\omega[0,1]) = C[0,1]$. One can also consider the Sobolev spaces $W_{p,k}[0,1] = \{f\in C^{k-1}[0,1]: f^{(k-1)}\in AC[0,1]$ and $f^{(k)}\in L^p[0,1]\}$, which are Banach* algebras with norms

$$\|f\|_{p,k} = \sup_{0\leqslant t\leqslant 1} \sum_{l=1}^{k} \frac{|f^{(l)}(t)|}{l!} + \left(\int_0^1 |f^{(k)}(t)|^p \mathrm{d}t\right)^{1/p},$$

and analogously construct the Sobolev-Arens algebras $W_{\omega,k}[0,1] = \cap_{1\leqslant p<\infty} W_{p,k}[0,1]$.

3.4 Let $C_b(\mathbb{R})$ be the C^*-algebra of all bounded continuous functions on $\mathbb{R}$. Let $BV_{\mathrm{loc}}C_b(\mathbb{R}) = \{f\in C_b(\mathbb{R}): f$ is of bounded variation on $[-n,n]$ for all $n = 1,2,3,\ldots\}$, a Frechet lmc *algebra having seminorms $p_n(f) = \|f\|_\infty + V_n(f)$, $V_n(f)$ denoting the total variation of f on $[-n,n]$. One has $\mathscr{E}(BV_{\mathrm{loc}}C_b(\mathbb{R})) = C_b(\mathbb{R})$.

3.5 Let $U = \{z\in\mathbb{C}\,|\,|z|<1\}$, $H(U)$ be the algebra, with pointwise operations, of all holomorphic functions on U. Let $A^n(U) = \{f\in H(U): f^{(k)}$ has continuous extension on $\bar{U}$ for all k, $0 \leqslant k \leqslant n\}$, a Banach* algebra with involution $f^*(z) = \overline{f(\bar{z})}$ and norm $\|f\|_n = \sup_{z\in\bar{U}} \Sigma_{k=0}^n (1/k!)|f^{(k)}(z)|$. The Frechet *algebra $A^\omega(U) = \cap_{n=0}^\infty A^n(U)$, with the topology defined by $f \to \|f\|_n$, $n = 0,1,2,\ldots$, is a non-hermitian Q-algebra with $\mathscr{E}(A^\omega(U)) = C[-1,1]$.

4. Segal Algebras

The following is a modification of the definition in [11] tailored for the present set up.

DEFINITION 4.1.

Let $(A, \|\cdot\|)$ be a Banach* algebra with a bai. A *subalgebra B of A is an A-Segal *algebra, if there exists a topology τ on B satisfying the following:

(a) B is a dense *ideal in A.
(b) (B,τ) is a complete lmc *algebra with a bai.
(c) The inclusion $(B,\tau)\to(A,\|\cdot\|)$ is continuous.
(d) The multiplication $(A,\|\cdot\|)\times(B,\tau)\to(B,\tau)$ is continuous.

PROPOSITION 4.2.

*If B is an A-Segal *algebra, then B is a Q-algebra; and $\mathscr{E}(B)=\mathscr{E}(A)$, the enveloping C^*-algebra of A.*

Proof. B being an ideal, $B_{-1}=A_{-1}\cap B$; which is open in (B,τ) by above (c). We show that for continuous (with respective topologies) topologically irreducible *representations, $R'(A)=R'(B)$ via restriction map. By (c), there exists $p_0\in K(B)(=K(B,\tau))$ such that $\|x\|\leqslant kp_0(x)(x\in B)$, with the result, $\pi\in R'(A)$ implies $\|\pi(x)\|\leqslant\|x\|\leqslant kp_0(x)\ (x\in B)$ and $\pi|_B\in R'(B)$ in view of (a). Let $\pi\in R'(B)$. Let $\xi\in H_\pi$ be any topologically cyclic vector for π so that $H_\pi=$ closed span of $\{\pi(B)\xi\}$. Let (e_γ) be a bai for B. Let $x\in A$, $y\in B$. Then $xy\in B$; and $e_\gamma y\to y$ in (B,τ). By (d), $xe_\gamma y\to xy$ in (B,τ); hence $\|\pi(xe_\gamma y)-\pi(xy)\|\to 0$. Since (e_γ) is τ-bounded, $\|\pi(xe_\gamma)\|\leqslant M_x<\infty$. Thus $\|\pi(xy)\xi\|=\lim_\gamma\|\pi(xe_\gamma)\pi(y)\xi\|\leqslant M_x\|\pi(y)\xi\|$. Thus the bounded linear operator $\tilde{\pi}(x):\pi(y)\xi\to\pi(xy)\xi$ defines $\tilde{\pi}\in R'(A)$, $\tilde{\pi}|_B=\pi$. Thus $R'(A)=R'(B)$. Let $P=(p_\alpha)\subset K(B,\tau)$ determine τ on B. Then, for any $y\in B$, $p_\infty(y)=\sup_\alpha r_\alpha(y)=\sup_\alpha\{\sup_{\pi\in R'\alpha(B)}\|\pi(y)\|\}=\sup\{\|\pi(y)\|:\pi\in R'(B)\}=\sup\{\|\pi(y)\|:\pi\in R'(A)\}\leqslant\|y\|\leqslant kp_0(y)$; and the greatest continuous C^*-seminorm $p_\infty(\cdot)$ on (B,τ) is the restriction of the Gelfand-Naimark pseudonorm (denoted by $p_\infty(\cdot)$ only). Finally, we show that $\mathscr{E}(B)=[B/(N(p_\infty)\cap B),\|\cdot\|_{p_\infty}]^{\sim}=(A/(N(p_\infty)),\|\cdot\|_{p_\infty})^{\sim}=\mathscr{E}(A)$. The map $\phi(x+(N(p_\infty)\cap B))=x+N(p_\infty)$ is a well defined *isomorphism of $B/(N(p_\infty)\cap B)$ into $A/N(p_\infty)$. Thus $\mathscr{E}(B)$ is a C^*-subalgebra of $\mathscr{E}(A)$. Let $z\in\mathscr{E}(A)$. There exist sequences (x_n) in A, (y_n) in B such that

$$\|x_n+N(p_\infty)-z\|_{p_\infty}<\frac{1}{2^{n+1}},\quad \|x_n-y_n\|<\frac{1}{2^{n+1}}.$$

Then,

$$\begin{aligned}\|y_n+N(p_\infty)-z\|_{p_\infty}&\leqslant\|x_n+N(p_\infty)-z\|_{p_\infty}+\|y_n-x_n\|_{p_\infty}\\&\leqslant\frac{1}{2^{n+1}}+\|y_n-x_n\|\\&<\frac{1}{2^n}\to 0\text{ as }n\to\infty\end{aligned}$$

showing that $\mathscr{E}(B)$ is dense in $\mathscr{E}(A)$.

Examples: Convolution algebras. For various group algebras on locally compact groups, we take convolution multiplication and the involution $f^*(g)=\Delta(g^{-1})\overline{f(g^{-1})}$, Δ being the modular function. Throughout, $\|\cdot\|_p$ denotes the usual norm on L^p-space.

4.3 For a locally compact abelian group G, take $A = L^1(G)$, $B = L^\omega(G) := \{f \in L^1(G)$: $f \in L^p(G)$ for all p, $1 < p < \infty\}$, the topology τ on B is determined by submultiplicative seminorms $p_k(f) = \|f\|_1 + \|f\|_k$, $k = 2, 3, 4, \ldots$. Then $\mathscr{E}(B) = C^*(G)$, the group C^*-algebra of G.

4.4 For $A = L^1(\mathbb{R})$, $B = \{f \in L^1(\mathbb{R}): f \in C^\infty(\mathbb{R})$ and the derivative $f^{(n)} \in L^1(\mathbb{R})$ for all $n = 1, 2, \ldots\}$. The topology τ on B is defined by $p_k(f) = \|f\|_1 + \|f^k\|_1$, $k = 0, 1, 2, \ldots$. That $p_k(f * g) \leqslant p_k(f) p_k(g)$ is a consequence of the identity $(f * g)^{(k)} = f^{(k)} * g = f * g^{(k)}$. Then B is an A-Segal *algebra.

4.5 Let G be a compact group. Let $(S, |\cdot|)$ be a Banach* algebra with a bai. For $1 \leqslant p < \infty$, $B^p(G, S)$ be the Banach* algebra of functions $f: G \to S$ with $|f|_p = [\int |f(g)|^p \mathrm{d}\mu]^{1/p} < \infty$.

Then $B^p(G, S)$ can be realized as a suitable completed tensor product $L^p(G) \hat{\otimes}_p S$, with the norm $\eta_p(\cdot)$ defined by taking a finite tensor $f = \Sigma x_i \otimes y_i$, as $\eta_p(f) = [\int |\Sigma x_i(g) y_i|^p \mathrm{d}\mu]^{1/p}$. By [23, Proposition 7.10], for all p, $\mathscr{E}(B^p(G, S)) = C^*(G) \hat{\otimes}_{\min} \mathscr{E}(S)$. Taking $A = B^1(G, S)$, $B = B^\omega(G, S) = \cap_{1 \leqslant p < \infty} B^p(G, S) = \lim_p B^p(G, S)$ with the topology of $\|\cdot\|_p$-convergence for each p, $\mathscr{E}(B^\omega(G, S)) = C^*(G) \hat{\otimes}_{\min} \mathscr{E}(S)$.

4.6 Let $(A, \|\cdot\|)$ be a commutative hermitian Banach* algebra with a bai. Let μ be a positive regular Borel measure on $\mathscr{M}(A) = \mathscr{M}^*(A)$. For $1 \leqslant p < \infty$, let $A_p(\mu) = \{x \in A$: $\hat{x} \in L^p(\mathscr{M}(A), \mu)\}$, $\hat{x}$ denoting the Gelfand transform of x. It is a Banach* algebra with norm $\|x\|_{A_p} = \|x\| + \|\hat{x}\|_p$. For the A-Segal *algebra $B = \cap_{1 \leqslant p < \infty} A_p(\mu)$, $\mathscr{E}(B) = \mathscr{E}(A)$. In particular, for a locally compact abelian group G with dual group $\hat{G}$ and Haar measure μ on $\hat{G}$, consider $A = L^1(G)$, $B = \{f \in L^2(G)$: Fourier transform $\hat{f}$ is in $L^p(\hat{G}, \mu)$, $1 \leqslant p < \infty\}$.

5. Topological algebras with bases and Köthe sequence algebras

Let ω denote the *algebra of all scalar sequences $(a_n)_1^\infty$ with pointwise operations and complex conjugation. Let $Q \subset \omega$ be a Köthe power set, i.e. Q satisfies (i) for each $a \in Q$, $a_n \geqslant 0$ for all n; (ii) for each $a \in Q$, $b \in Q$, there exists $c \in Q$ such that $a_n \leqslant c_n$, $b_n \leqslant c_n$ for all n; and (iii) for each n, there exists $a \in Q$ satisfying $a_n > 0$. We assume $a_1 \neq 0$ for all $a \in Q$. Further, let Q satisfy G_∞-property, i.e. (iv) for each $a \in Q$, $a_n \leqslant a_{n+1}$ for all n; (v) for each $a \in Q$, there exists $d \in Q$ such that $a_n^2 \leqslant d_n$ for all n. Köthe space of infinite type ([22], [32, p. 203]) is the complete locally convex space $\Lambda_\infty(Q) = \{x = (x_n) \in \omega$: $p_\alpha(x) = \Sigma |x_n| a_n < \infty$ for all $a \in Q\}$ with the locally convex Köthe topology t defined by the family $\Gamma = (p_\alpha : a \in Q)$ of seminorms. It so turns out [4] that $(\Lambda_\infty(Q), t)$ is a complete lmc-*algebra which is a *subalgebra of $(\ell^1, \|\cdot\|_1)$ and which is a Q-algebra. (Note that a complete commutative continuous inverse Q-algebra is lmc [31].) Further, it is hermitian. For the present purpose, we note the following.

PROPOSITION 5.1.

(i) $\Lambda_\infty(Q)$ *is an* ℓ^1*-Segal *algebra.*
(ii) $\mathscr{E}(\Lambda_\infty(Q)) = c_0$, *the* C^**-algebra of all null sequences.*

(i) can be easily checked; whereas (ii) follows from a more general result to follow. The following important particular case of $\Lambda_\infty(Q)$ we shall need.

5.2 ($\Lambda_\infty[\theta_n]$): For an increasing sequence (θ_n) of positive numbers, $Q=\{(k^{\theta_n})_{n=1}^\infty : k=1,2,\ldots\}$ gives the algebra $\Lambda_\infty(Q)$ denoted by $\Lambda_\infty[\theta_n]$ called power series space of infinite type [32, p. 204]. The algebra $s=\{x:\omega:\Sigma_{n=1}^\infty n^k|x_n|<\infty \text{ for all } k=1,2,\ldots\}$ of rapidly decreasing sequences is $\Lambda_\infty[\theta_n]$ taking $\theta_n=\log n$, $n=1,2,\ldots$ [32, p. 205].

Recall [5, §1] that an orthogonal basis on a topological algebra A is a basis (f_n) for A such that $f_n f_m=\delta_{nm}f_n$ for all n, m in $\mathbb{N}$, δ_{nm} being the Kronecker delta. The algebra $\Lambda_\infty(Q)$ admits $f_n=(\delta_{\text{nm}})_{m=1}^\infty$ as an orthogonal basis.

PROPOSITION 5.3.

*Let A be a complete hermitian lmc-*algebra with C^*-enveloping algebra. Let A admit an orthogonal basis consisting of hermitian elements. Then A is a Q-algebra; and $\mathscr{E}(A)$ is *isomorphic to the C^*-algebra c_0.*

Proof. Let (f_n) be an orthogonal basis for A, $f_n^*=f_n$ for all n. Then (f_n) is a Schauder basis and A is commutative [5], [20]. Let ϕ_n be the coefficient functional defined by f_n viz expanding $x\in A$ as $x=\Sigma_n x_n f_n$, $x_n\in\mathbb{C}$, $\phi_n(x)=x_n$. Then the Gelfand space $\mathscr{M}(A)=\{\phi_n\}=\mathscr{M}^*(A)$ by hermiticity; and $\phi:A\to\omega$, $\phi(x)=(x_n)$ is a *isomorphism of A onto a *subalgebra K of ω; which is continuous for the topology of pointwise convergence on K [20, Corollary 1.3]. We identify A with K algebraically. By [26, Corollary 5.6], for all $x\in A$, the spectrum $sp(x)=\{\phi_n(x)\}=\{x_n\}$; and by Corollary 2.10 and Thorem 2.1, $p_\infty(x)=\sup_n|x_n|=\|x\|_\infty=r(x)<\infty$ for all x. Thus $A\subset\ell^\infty$. By Corollary 2.6, A is a Q-algebra. Further, A contains the set of all finitely many nonzero sequences. Also, A cannot have identity, otherwise A has to be ω with pointwise convergence [20, Theorem 2.1], which is not an algebra with C^*-enveloping algebra. It follows that $\mathscr{E}(A)$, which is the completion of $(A,\|\cdot\|_\infty)$, contains c_0. On the other hand, $p_\infty(\cdot)$ being a continuous C^*-seminorm on A (in the topology of A), $x^{(n)}=\Sigma_1^n x_k f_k\to x=\Sigma_1^\infty x_k f_k$ in A implies that $|x_{n+1}|\leqslant\sup_{k>n}|x_k|=p_\infty(x-x^{(n)})\to 0$ as $n\to\infty$. Thus $x=(x_n)\in c_0$, $\mathscr{E}(A)\subset c_0$ with the result, $\mathscr{E}(A)=c_0$.

Example 5.4. Let $A=C^\infty(\Gamma)$, the convolution algebra of all C^∞-functions on the circle Γ with involution $u^*(z)=u(z^{-1})$. By [14, p. 48], for any $u\in C^\infty(\Gamma)$, the Fourier series expansion $u=\Sigma_{-\infty}^\infty \hat{u}(n)e^{\text{int}}$ gives a sequence $(\hat{u}(n))\in s(\mathbb{Z})$, = two sided rapidly decreasing sequences. The map $\phi:C^\infty(\Gamma)\to s(\mathbb{Z})$, $\phi(u)=(\hat{u}(n))$ establishes a *isomorphism of $C^\infty(\Gamma)$ onto s, which is a homeomorphism for the (usual) Frechet C^∞-topology on $C^\infty(\Gamma)$ and Frechet Köthe topology on $s(\mathbb{Z})$ [30, Theorem 5.1]. Now, $s(\mathbb{Z})$ is a Q-algebra and $\mathscr{E}(s(\mathbb{Z}))=c_0$. Thus, via ϕ, $C^\infty(\Gamma)$ is a complete Q lmc-*algebra with $\mathscr{E}(C^\infty(\Gamma))=\{\mu\in \text{PM}(\Gamma):(\hat{\mu}(n))\in c_0\}$, where $\text{PM}(\Gamma)$ is the convolution algebra of all pseudo measures on Γ, isomorphic to ℓ^∞ via Fourier expansion [14, §12.11].

Example 5.5. For the open unit disc U, let $H^p(U)$ be the Hardy space, for $1<p<\infty$. The Banach space $(H^p(U),\|\cdot\|_p)$ is a Banach* algebra with Hadamard product

$$(f*g)(x)=(1/2\pi i)\textstyle\int_{|z|=r}f(z)g(xz^{-1})z^{-1}\text{d}z,\ |x|<r<1,$$

having involution $f^*(z)=\overline{f(\bar{z})}$. The sequence $e_n(z)=z^n$ is an orthogonal basis for $H^p(U)$ [20, Example 3]. Thus, the Hardy-Arens algebra $H^\omega(U)=\cap_{1<p<\infty}H^p(U)$ is a

Frechet lmc-*algebra with basis (e_n), the topology being the topology of $\|\cdot\|_p$-convergence for each p, $1 < p < \infty$. The coefficient functionals ϕ_n are $\phi_n(f) = f^{(n)}(0)/n!$, the nth Taylor coefficient of f (exactly as in [15, Example 3.2(ii)] for the *algebra $H(U)$); and for any $f \in H^\omega(U)$, $\mathrm{sp}(f) = \{\phi_n(f)\}$. It is easily seen that $H^\omega(U)$ is hermitian, and $p_\infty(f) = \sup_n(|f^{(n)}(0)|/n!) \leqslant \|f\|_p$ $(p > 1)$ is the greatest continuous C^*-seminorm.

Example 5.6. Let E be the Frechet space of all entire functions of one complex variable with compact open topology. It is a topological *algebra with Hadamard product and the involution $f^*(z) = \overline{f(\bar{z})}$, admitting orthogonal basis $e_n = z^n$, $n \in \mathbb{N}$. The mapping $\phi: E \to \omega$, $\phi(\Sigma x_n e_n) = (x_n)$ is a *isomorphism of E onto the sequence algebra $\Lambda_\infty[n]$. Also, ϕ is a homeomorphism for the respective topologies on E and $\Lambda_\infty[n]$ [32, p. 206]. Thus $\mathscr{E}(E)$ is *isomorphic to c_0.

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