On unbounded subnormal operators

ARVIND B PATEL and SUBHASH J BHATT
Department of Mathematics Sardar Patel University, Vallabh Vidyaganar 388 120 India

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Abstract. A minimal normal extension of unbounded subnormal operators is established and characterized and spectral inclusion theorem is proved. An inverse Cayley transform is constructed to obtain a closed unbounded subnormal operator from a bounded one. Two classes of unbounded subnormals viz analytic Toeplitz operators and Bergman operators are exhibited.

Keywords. Unbounded subnormal operator; Cayley transform; Toeplitz and Bergman operators; minimal normal extension.

1. Introduction

Recently there has been some interest in unbounded operators that admit normal extensions viz unbounded subnormal operators defined as follows:

DEFINITION 1.1

Let S be a linear operator (not necessarily bounded) defined in D(S), a dense subspace of a Hilbert space H. S is called a subnormal operator if it admits a normal extension (N, D(N), K) in the sense that there exists a Hilbert space K, containing H as a closed subspace (the norm induced by K on H is the given norm on H) and a normal operator N with domain D(N) in K such that Sh = Nh for all h ∈ D(S).

These operators appear to have been introduced in [12] following Foias [4]. An operator could be subnormal internally admitting a normal extension in H; or it could admit a normal extension in a larger space. As is well known, a symmetric operator always admits a self-adjoint extension in a larger space, contrarily a formally normal operator may fail to be subnormal ([2], [11]). Recently Sjoest and Szafrawie ([12], [13]) obtained a Halmos–Bram type characterization of unbounded subnormal operators.

Here we discuss the existence and characterization of minimal normal extension N of an unbounded subnormal S. This is followed by the spectral inclusion theorem σ(N) ⊆ σ(S). In §3, we set up a Cayley transform between a bounded subnormal and an unbounded one. We also exhibit two large classes of unbounded subnormals viz Bergman operators and analytic Toeplitz operators.

Let us recall [16, Ex. 5.39 p. 127] that given an operator T with domain D(T) in a Hilbert space H, a closed subspace M of H is invariant under T if T(D(T) ∩ M) ⊂ M. M is reducing under T if T(M ∩ D(T)) ⊂ M, T(M⊥ ∩ D(T)) ⊂ M⊥ and D(T) =
2. Minimal normal extension

**Definition 2.1**

A normal extension \((N, D(N), K)\) of a subnormal operator \((S, D(S), H)\) is a **minimal normal extension** (MNE) if for any normal extension \((N_1, D(N_1), K_1)\) of \(S, S \subset N_1 \subset N\) and \(K_1\) is reducing under \(N\) implies \(K_1 = K\) and \(N_1 = N\).

In [13, p. 51] a normal extension \(N\) in \(K\) of \(S (SD(S) \subset D(S))\) is called 'minimal' if \(D = \{N^*N^jx : x \in D(S), i, j = 0, 1, 2, \ldots\}\) is linearly dense in \(K\). The second half of the following theorem shows that it is in fact a MNE. The class of \(C^\infty\)-**vectors** for an operator \(T\) in \(H\) is \(C^\infty(T) = \bigcap_{n=1}^\infty D(T^n)\).

**Theorem 2.2.** (a) A subnormal operator admits a minimal normal extension.

(b) Let \(S\) be a subnormal operator with dense domain \(D(S)\) in a Hilbert space \(H\). Let \((N, D(N), K)\) be a normal extension of \(S\). Let \(D\) be the linear span of \(\{N^*N^jx : i, j = 1, 2, \ldots; x \in C^\infty(S)\}\).

(i) If \(D\) is dense in \(K\), then \(N\) is a MNE

(ii) If \(N\) is a MNE and \(D(N) = D + (D(N) \cap D^-1)\), then \(D\) is dense in \(K\).

**Proof.** (a) Let \(\mathcal{E}\) be the class of all normal extensions \(\alpha = (N_\alpha, D(N_\alpha), K_\alpha)\) of a subnormal operator \(S\) in a Hilbert space \(H\) with domain \(D(S)\). \(\mathcal{E}\) is partially ordered by \(\alpha \leq \beta = (N_\beta, D(N_\beta), K_\beta)\) if \(N_\alpha \subset N_\beta\) and \(K_\alpha\) is a reducing subspace for \(N_\beta\). Note that for \(\alpha \leq \beta\), the restriction \(N_\beta|_{K_\alpha}\) of \(N_\beta\) on \(K_\alpha\) with domain \(D(N_\beta|_{K_\alpha}) = K_\alpha \cap D(N_\beta)\) is a normal operator in \(K_\alpha\) which is an extension in \(K_\alpha\) itself of the normal operator \(N_\alpha\).

Since a normal operator is maximally normal [10, p. 350], \(N_\alpha = N_\beta|_{K_\alpha}\) so that \(D(N_\alpha) = K_\alpha \cap D(N_\beta)\). We shall apply Zorn’s lemma to \(\mathcal{E}\).

Let \(\mathcal{E}\) be a chain in \(\mathcal{E}\). Let \(K = \cap \{K_\alpha | \alpha \in \mathcal{E}\}\), \(D = \cap \{D(N_\alpha) | \alpha \in \mathcal{E}\}\). For \(\alpha \in \mathcal{E}\), let \(P_\alpha^\perp : K \to K\) and for \(\gamma \leq \alpha\), \(P_\gamma^\perp : K_\gamma \to K_\gamma\) be orthogonal projections. Now, let \(\alpha \in \mathcal{E}\) be fixed. Since \(\mathcal{E}\) is a chain, \(K = \cap \{K_\alpha | \gamma \leq \alpha, \gamma \in \mathcal{E}\}\) and \(D = \cap \{D(N_\alpha) | \gamma \leq \alpha, \gamma \in \mathcal{E}\}\).

**Claim.** \(K\) is a reducing subspace for the normal operator \(N_\alpha\). For this, note that \(P_\alpha = \text{glb} \{P_\gamma | \gamma \in \mathcal{E}, \gamma \leq \alpha\}\), as in [15, p. 124]. Now consider the weak bounded commutant of \(N_\alpha\), viz \(\{N_\gamma^\vee | \gamma \in \mathcal{E}\}\). Let \(E = \text{gla} \{B(K_\gamma) : S N_\gamma \subset S N_\alpha\}\), \(B(K_\gamma)\) denoting the set of all bounded linear operators on \(K_\gamma\). By Fuglede–Putnam theorem for unbounded normal operators [10, p. 365], \(\{N_\gamma^\vee\} = \{S \in B(K_\gamma) : S N_\gamma \subset N_\alpha S, S N_\alpha \subset S N_\gamma\} = \{N_\alpha, N_\gamma^\vee\}\). If \(E\) is the spectral measure for the bounded normal operator \((1 + N_\alpha^2 N_\gamma^2)^{-1}\). For \(k = 0, 1, 2, \ldots\) let \(w_k(0) = (1/k + 1/k)\), and \(N_{\alpha k} = N_\alpha E(w_k)\) which are bounded normal operators. Then as shown in the proof of Theorem 2.1 in [8], \(\{N_{\alpha k} | k = 0, 1, 2, \ldots\}\) (usual commutant in \(B(K_\gamma)\) of a collection of bounded operators) is a von Neumann algebra. Now by [16, p. 128], reducing subspaces of \(N_\alpha\) correspond (via usual way of range projections) to projections in \(\{N_{\alpha k}\}\). Hence for \(\gamma \leq \alpha\), \(P_\gamma \in \{N_{\alpha k}\}\). Since projections in a von Neumann algebra form a complete lattice [15, p. 124], \(P_\gamma \in \{N_{\alpha k}\}\); and hence \(K\) is a reducing subspace for \(N_\alpha\).

Now for \(\gamma \leq \alpha\), \(P_\gamma^\perp(D(N_\alpha)) = K \cap D(N_\alpha)\) since \(D(N_\alpha) = [K \cap D(N_\alpha)] + [K^\perp \cap D(N_\alpha)]\).
Hence \( P_k^x(D(N_x)) = K_x \cap D(N_x) = D(N_x) \). Thus \( P_k^x(D(N_x)) \subset \{D(N_x) \mid y \in E, y \leq z\} = D \).
This implies that \( D \) is dense in \( K \). For, given \( x \in D^* (\perp \text{ in } K) \), for all \( y \in D(N_x) \),
\[
\langle x, y \rangle = \langle P_k^x x, y \rangle = \langle x, P_k^x y \rangle = 0,
\]
hence \( x = 0 \). Define an operator \( N \) in \( K \) with domain \( D(N) = D \) as \( N_x = N_x x \). Then \( N \) is a well-defined closed operator. To show that \( N \) is normal, consider an operator \( C \) in \( K \) with domain \( D(C) = D \) as \( Cx = N_x x \) (adjoint in \( K ) \). Then \( C \subset N^* \) (adjoint in \( K \)). Now given \( x \in D(N^* N) \), the functional \( y \in D \rightarrow \langle N^* x, N^* y \rangle = \langle C x, C y \rangle = \langle N^* x, N^* y \rangle = \langle N x, N y \rangle \) is continuous on \( D \) as \( N x \in D(N^* N) \). Thus \( N^* x \in D(N^* N) = D(N) \) is closed. Hence \( x \in D(N^* N) \) and \( D(N^* N) \subset D(N^* N) \). In fact, \( N^* N \subset N N^* \), and so \( N^* N = N N^* \) both being self-adjoint (as \( N \) is closed). (Note that normality of \( N \) also implies \( N = N^* N \).

The normal extension \((N, D(N), K)\) is a lower bound of \( \mathcal{E} \). Now Zorn's lemma completes the proof.

(b) (i) Let the linear span \( D \) of \( \{N^* N^i x \mid x \in C^\omega(S); i, j = 1, 2, \ldots\} \) be dense in \( K \). Let \( (N_0, D(N_0), K_0) \) be a normal extension of \( S \) such that \( (N_0, D(N_0), K_0) \subset (N, D(N), K) \) (partial order as in the proof of (a)). Let \( x \in C^\omega(S) \). Then for all \( i = 1, 2, \ldots, S_i \in C^\omega(S) \subset C^\omega(S) \) that gives \( N^i x = S_i x \in C^\omega(S) \subset C^\omega(S) \cap K_0 \). Now for any positive integer \( k \), by the normality of \( N^k \), \( D(N^k) = D(N^k) \) which implies that \( C^\omega(S) = C^\omega(S) \). Thus \( N^k \) is defined for all \( i, j = 1, 2, \ldots \). Further, since \( K_0 \) is invariant under \( N^k \), \( N^k x \in K_0 \). Thus \( D \subset K_0 \). Hence \( K_0 = K, N_0 = N \) showing that \( N \) is MNE.

(ii) Let \( (N, D(N), K) \) be a MNE of \( S \) satisfying the given condition. Let \( K_0 = D \) (closure in \( K \)). By definition of \( D \), \( N^* D \subset D \), \( N^* D \subset D \). These give \( D(N) \subset K_0 \). Further, the given condition is equivalent to \( D(N) = \{D(N) \cap K_0 \} + \{D(N) \cap K_0 \} \). We show that \( N(D(N) \cap K_0) \subset K_0 \). Let \( x \in D(N) \cap K_0 \). For all \( y \in D(N) \cap K_0 \), \( \langle N x, y \rangle = \langle N x, y \rangle = 0 \). As \( D(N) \cap K_0 \) is dense in \( K_0 \), \( N x \in K_0 \). Hence \( K_0 \) is reducing for \( N \). Then \( N |_{K_0} \) is a normal extension of \( S \) contained in \( N \). By the minimality of \( N, K_0 = K \). This completes the proof of the theorem.

The following is a spectral inclusion theorem analogous to the one for bounded subnormal. Its proof is patterned along Halmos [5, p. 157].

**Theorem 2.3.** Let \( S \) be a subnormal operator in a Hilbert space \( H \) with domain \( D(S) \) and a minimal normal extension \( N \). Then \( \sigma(N) \subset \sigma(S) \).

**Proof.** Let \( \lambda \in \sigma(S) \). Then \( (\lambda - S)^{-1} \) is a bounded operator on \( H \). We can assume \( \lambda = 0 \) and \( ||S^{-1}|| = 1 \). Now for \( 0 < \epsilon < 1 \), consider \( E_\epsilon = \{x \in C^\omega(N) \mid ||N^* x|| < \epsilon ||x|| \} \) for \( n = 1, 2, \ldots \). For \( x \in E_\epsilon, y \in H \),
\[
|\langle x, y \rangle| = |\langle x, S^* S^{-1} y \rangle| = |\langle N^* x, S^* y \rangle| \leq \epsilon \|x\| \|y\| \text{ for all } n.
\]
As \( \epsilon < 1, \langle x, y \rangle = 0 \). Thus \( H \subset E_\epsilon^* (\perp \text{ in } K) \). Let \( N = \bigcup_{\epsilon} E_\epsilon(z) \) be the spectral theorem for \( N \). Then \( E_\epsilon = E_\epsilon K \) where \( A_\epsilon = \{z \in C \mid |z| \leq \epsilon \} \). Hence \( E_\epsilon \), and so \( E_\epsilon \) is a reducing subspace of \( N \). Now \( N |_{E_\epsilon} \) being normal, the minimality of \( N \) implies that \( E_\epsilon = K \). Hence \( E_\epsilon K = E_\epsilon = 0 \). Thus \( \phi = A_\epsilon \cap \text{supp } E = A_\epsilon \cap \sigma(N) \); and so \( 0 \in \sigma(N) \).
Notice that, in above notations, \( bdy \sigma(S) = \sigma_+(S) = \sigma_+(N) = \sigma(N) \) (\( \sigma_+ \) denotes the approximate point spectrum) and component of \( \mathcal{C} \setminus \sigma(N) \) is either contained in \( \sigma(S) \) or is disjoint from \( \sigma(S) \).

**COROLLARY 1**

Let \( S \) be a subnormal operator. Then

(i) \( \sigma(S) \neq \phi \).

(ii) \( S \) is bounded iff \( \sigma(S) \) is bounded.

(iii) \( S \) is essentially self-adjoint iff \( \sigma(S) \) is real.

**COROLLARY 2**

A symmetric operator has nonempty spectrum.

Remarks 2.4. (i) Let \( S \) be an operator in a Hilbert space \( H \). A vector \( x \in C^\infty(S) \) is an analytic vector for \( S \) if there exists a \( t > 0 \) such that

\[
\sum_{n=1}^{\infty} \frac{\|S^n x\| t^n}{n!} < \infty.
\]

Let \( A(S) \) be the collection of all analytic vectors for \( S \). If \( S \) is subnormal admitting a normal extension \( N \) such that \( D(S) = D(N) \cap H \), then \( A(S) \) is dense in \( H \). Indeed, in this case, \( A(S) = A(N) \cap H \). Hence taking the orthogonal complement in \( K \), \( A(S)^+ = H^\perp \) as \( A(N) \) is dense in \( K \).

(ii) A symmetric operator in \( H \) admitting a normal extension \( N \) in (possibly a larger space) \( K \) satisfying \( D(S) = D(N) \cap H \) is essentially self-adjoint. For, in view of (i), the well-known Nelson theorem [16, p. 261] applies.

(iii) Normal extensions of an unbounded subnormal operator satisfying the above spectral inclusion (distinguished normal extensions) have been discussed recently in [6]. Thus a MNE is distinguished, though a distinguished extension need not be minimal. For example let \( N_1 \) be a MNE of \( S \) in \( K_1 \). Let \( N_2 \) be a normal operator in \( K_2 \) with \( \sigma(N_2) \subset \sigma(N_1) \). Take \( N = N_1 \oplus N_2 \) a normal extension of \( S \) in \( K_1 \oplus K_2 = K \). Then \( N_0 = N_{(e,0)} \), (where \( E \) is the spectral measure of \( N \)) is distinguished normal extension as in [6] which is not minimal.

(iv) Ōta [7] showed that if \( T \) is a densely defined closed operator in a Hilbert space \( H \) such that \( TD(T) \subset D(T^*) \), then \( T \) is bounded. This has the following implication.

**PROPOSITION**

Let \( S \) be a closed subnormal operator in \( H \) with dense domain \( D(S) \) such that \( SD(S) \subset D(S) \).

Then \( S \) is bounded.

This follows from \( D(S) \subset D(S^*) \) [14].

We are thankful to Prof. Ōta for bringing this to our notice.

(v) Ōta [7] has also another interesting result, viz if \( T \) is a densely defined closed operator in a Hilbert space \( H \) such that the range of \( T \) is contained in its domain and if \( T \) is unbounded, then the numerical range \( W(T) = \{ \langle Tx, x \rangle | x \in D(T), \|x\| = 1 \} \) is the entire complex plane. The following is an analogous result for spectrum.
PROPOSITION

Let $T$ be a densely defined closed operator in a Hilbert space $H$ such that $TD(T) \subseteq D(T)$. If $\sigma(T)$ is not the whole of complex plane, then $T$ is bounded.

Proof. If $\lambda \notin \sigma(T)$, then $S = (T - \lambda I)^{-1}$ is a bounded operator satisfying $S(T - \lambda I) \subseteq (T - \lambda I)S = 1$. Thus $(T - \lambda I)D(T) \subseteq D(T) = H$. Closed graph theorem shows that $T$ is bounded.

3. A Cayley transform

The problem of self-adjoint extension (within the space) of a symmetric operator is discussed via Cayley transform [15, Ch. 8] which provides a correspondence between certain partial isometries and symmetric operators that admit self-adjoint extensions. We extend this so as to associate an unbounded subnormal operator with a bounded one.

**Theorem 3.1.** Let $S$ be a bounded subnormal operator on $H$ with a bounded normal extension $N$ on $K$. If

(i) $1 - N$ is one-to-one and
(ii) $1 \in \sigma(N), \sigma(N) \setminus \{1\} \subseteq \{z \in \mathbb{C} : |z| < 1\}$

then $\overline{\psi(N)}$ is an unbounded closed subnormal operator where $\psi(N)$ is the normal operator in $K$ defined via the spectral theorem by the function $\psi(z) = i(1 + z)(1 - z)^{-1}$.

Proof. Define $N'$ in $K$ with domain $D(N') = R(1 - N)$ by $N'x = i(1 + N)(1 - N)^{-1}x$. Then $N'$ is densely defined.

Claim (a). $N' = \psi(N)$.

For, given $x \in D(N'), (1 - N)y = x$, and so

$$
\int |(1 + z)(1 - z)^{-1}|^2 \, dE_{x,y} = \int |(1 + z)^2(1 - z)^{-2}|(1 - z)(1 - z)^{-1} \, dE_{x,y} < \infty,
$$

and for all $u \in K$

$$
\langle N'x, u \rangle = i\langle (1 + N)y, u \rangle = i \int (1 + z)(1 - z)^{-1} \, dE_{x,u} = \langle \psi(N)x, u \rangle.
$$

Hence $N' \in \psi(N)$. As $\psi(N)$ is closed, $\overline{\psi(N)} \subseteq \psi(N)$. Now let $N_0 = N^*|_{D(\psi(N^*)\psi(N))}$.

Then $G(N_0)$ is dense in $G(N^*)$, $G(\cdot)$ denoting the graph of the operator. Indeed, note that $G(N^*)$ is closed in $K \times K$. Let $(u, N^*u) \in G(N^*)$, $(u, N^*u) \perp G(N_0)$. Then for all $x \in D(\psi(N^*)\psi(N))$,

$$
0 = \langle (u, N^*u), (x, N^*x) \rangle
= \langle u, x \rangle + \langle N^*u, \psi(N)x \rangle (as \psi(N)^* \subseteq N^*)
= \langle u, x \rangle + \langle u, \psi(N)\psi(N)x \rangle.
$$
Here we have used the following that can be easily verified.

Lemma. Let $A$ and $B$ be densely defined linear operators in a Hilbert space with $B$ closed and $D(B) = D(B^*)$. If $A \subset B$, then for all $u \in D(A^*)$, $v \in D(B)$, $\langle A^*u, v \rangle = \langle u, By \rangle$.

Thus in (a), since $R(1 + \psi(N)^*\psi(N))$ is dense in $K$, $u = 0$. Then $G(N_0)$ is dense in $G(N^*)$. Now, let $y \in D(N^*)$. Then for some sequence $(y_i)$ in $D(\psi(N)^*\psi(N))$, $y_i \to y$ and $N_0y_i - N^*y = N^*y_i - N^*y \to 0$. Since $\|\psi(N)y_i - \psi(N)y_j\| = \|\psi(N)^*y_i - \psi(N)^*y_j\| = \|N^*y_i - N^*y_j\|$, $(\psi(N)y_i)$ converges to some $u \in K$. Since $y \in D(\psi(N))$, $y \in D(\psi(N))$. Thus $D(N^*) \subset D(\psi(N)) = D(\psi(N)^*)$; hence $D(N^*) = D(\psi(N)^*)$, $N^* = \psi(N)$ and so $\overline{N} = \psi(N)$.

Claim (b). $H$ is invariant under $\psi(N)$ (and $N^*$). For if, $x \in D(\psi(N)) \cap H$, $y \in H^\perp$, then since 1 is not an eigenvalue, $E(\{1\}) = 0$; and so

$$\langle \psi(N)x, y \rangle = i \int_{\sigma(N)} (1 + z)(1 - z)^{-1} \text{d}E_{x,y}$$

$$= i \int_{\sigma(N) \setminus \{1\}} (1 + z)(1 - z)^{-1} \text{d}E_{x,y}$$

$$= i \sum_{k} \int (1 + z)^k \text{d}E_{x,y}$$

$$= i \sum_{k} \langle (1 + N)^k x, y \rangle = 0$$

as $(1 + N)^k x \in H$. Thus $\psi(N)x \in H$. This establishes our claim (b).

It is easy to see that $\psi(N)|H$ with domain $D(\psi(N)|H) = D(\psi(N)) \cap H$ is a closed operator.

Remark 3.2. Note that if $R(1 - S)$ (range of $(1 - S)$) is dense in $H$, then $S' = i(1 + S)(1 - S)^{-1}$ with domain $D(S') = R(1 - S)$ and $S_0 = N^*|H$ and hence are sub-normals (not necessarily closed) in $H$.

4. Examples

4.1 Unbounded analytic Toeplitz operators

Let

$$U = \{ze^{\mathbb{C}}:|z| < 1\}, \quad \Gamma = \{ze^{\mathbb{C}}:|z| = 1\}.$$ 

Let $\phi$ be a measurable function on $\Gamma$ and $D_\phi = \{f \in H^2(U): \phi f \in L^2(\Gamma)\}$. Define $T_\phi$ in $H^2(U)$ with domain $D_\phi$ as $T_\phi f = P(\phi f)$, where $P:L^2(\Gamma) \to H^2(U)$ is the projection. The Toeplitz operator $T_\phi$ is an analytic Toeplitz operator if $\phi$ is analytic. Such a $T_\phi$ admits a normal extension $M_\phi$ with domain $D(M_\phi) = \{f \in L^2(\Gamma): \phi f \in L^2(\Gamma)\}$, $M_\phi f = \phi f$.

Thus, in this case, if $D_\phi$ is dense in $H^2(U)$, then $T_\phi$ is subnormal. Note that it is indeed if $\phi$ is bounded. We exhibit below a class of function $\phi$ for which $T_\phi$ is a nonunbounded subnormal operator.
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(i) \( \phi(x) = (1 - x)^{-1} \). Then \( D_\phi = R(1 - S) \) where \( S \) is the unilateral shift. Hence \( D_\phi = \ker(1 - S^*)^* = H^2(U) \). Also \( T_\phi \) is closed. For, if \( (f_n, T_\phi f_n) \to (f, g) \), then identifying \( H^2(U) \) with a closed subspace of \( L^2(\Gamma) \), there exists a subsequence \( (f_{n_k}) \) of \( (f_n) \) each of whose subsequence converges a.e. to \( f \) on \( \Gamma \). Since \( f_{n_k}(z)(z - 1)^{-1} \to g \in L^2(\Gamma) \), \( (z - 1)g = f \) a.e. Hence \( g = T_\phi f \) in \( H^2(U) \).

(ii) A similar argument can be made for \( \phi(z) = (z - \lambda_1)^{-n_1}(z - \lambda_2)^{-n_2} \cdots (z - \lambda_k)^{-n_k} \) with \( |\lambda_i| \geq 1, n_i = 1, 2, \ldots \).

(iii) As discussed in [6], functions \( \phi \in H^2(U) \) define unbounded analytic Toeplitz operators.

Unbounded Toeplitz operators also arise quite naturally in representation of certain topological algebras by unbounded operators.

Consider Arens algebra [1] \( L^p(\Gamma) = \bigcap_{1 \leq p < \infty} L^p(\Gamma) \neq L^\infty(\Gamma) \) with pointwise operations. It is a Frechet * algebra with the topology of \( L^p \)-convergence for each \( p, 1 \leq p < \infty \). The Hardy-Arens algebra \( H^\infty(U) = \bigcap_{1 \leq p < \infty} H^p(U) \neq H^\infty(U) \) [9, Ch. 17, Ex. 10] can be regarded as a closed subalgebra of \( L^p(\Gamma) \). For \( \phi \in H^\infty(U) \), \( D_\phi \) is dense in \( H^2(U) \) since \( H^\infty(U) \subseteq D_\phi \) and \( H^\infty(U) \) is dense in \( H^2(U) \). In fact, as in (i) above, \( T_\phi \) is closed. It is easily seen that \( \phi \to T_\phi \) is a representation of \( H^\infty(U) \) by unbounded subnormal operators in \( H^2(U) \) which is the restriction of the unbounded * representation \( \phi \to M_\phi \) of \( L^p(\Gamma) \) into normal operators in \( L^2(\Gamma) \).

4.2 Unbounded Bergman operators

Let \( G \) be a bounded domain in \( \mathbb{C} \). For \( 1 \leq p < \infty \), consider the Bergman spaces \( L^2_p(G) = \{ f \in L^p(G) : f \text{ is analytic on } G \} \) with \( \| \cdot \|_p \) norm. Let \( L^\infty_p(G) = \bigcap_{1 \leq p < \infty} L^p(G) \). For \( g \in L^\infty_p(G) \), define \( S_g \) in \( L^2_p(G) \) with domain \( D(S_g) = \{ f \in L^2_p(G) : \| g \| f \in L^2(G) \} \). As \( S_g f = \overline{g} f \).

Again \( S_g \) is densely defined if \( L^\infty_p(G) \) is dense in \( L^2_p(G) \). In particular, if \( G \) is a Caratheodory domain [3, Ch. 3] in which case \( L^2(G) = P^2(G) \), the \( L^2(G) \)-closure of polynomials. In this way, one gets a large class of unbounded subnormals.

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