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Quotient-bounded elements in locally convex algebras. II

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Abstract. Consideration of quotient-bounded elements in a locally convex GB^*-algebra leads to the study of proper GB^*-algebras viz those that admit nontrivial quotient-bounded elements. The construction and structure of such algebras are discussed. A representation theorem for a proper GB^*-algebra representing it as an algebra of unbounded Hilbert space operators is obtained in a form that unifies the well-known Gelfand–Naimark representation theorem for C^*-algebra and two other representation theorems for b^*-algebras (also called LMC*-algebras), one representing *a* b^*-algebra as an algebra of quotient bounded operators and the other as a weakly unbounded operator algebra. A number of examples are discussed to illustrate quotient-bounded operators. An algebra of unbounded operators constructed out of noncommutative L^p-spaces on a regular probability gauge space and the convolution algebra of periodic distributions are analyzed in detail; whereas unbounded Hilbert algebras and L^w-integral of a measurable field of C^*-algebras are discussed briefly.

Keywords. Generalized B*-algebras; unbounded representations; quotient-bounded elements; universally bounded elements; unbounded Hilbert algebras; locally multiplicative convex (LMC) algebras.

1. Introduction

The following two generalizations of abstract C^*-algebras at the level of topological algebras have been investigated in the literature.

1.1. *Definition* ([3], [19], [23])

A *b^*-algebra* (also called an LMC*-algebra) is a complete topological algebra A with a continuous involution $x \to x^*$ such that

(i) A is a locally m-convex algebra i.e. its topology is determined by a separating family $P = (p_\alpha)$ of sub-multiplicative seminorms, and

(ii) for each α, $p_\alpha(x^*x) = p_\alpha(x)^2$ for all $x \in A$.

A commutative b^*-algebra A with 1 admits a faithful representation as the algebra of all continuous complex valued functions on a completely regular space [23], whereas in the noncommutative case, there are two such representation theorems viz one due to Inoue [19] (see also [12]) representing A faithfully as a weakly unbounded algebra of unbounded Hilbert space operators, and the other due to Giles *et al* [17] wherein A is shown to be isomorphic to an algebra of quotient-bounded operators on a locally convex space.

1.2 *Definition* ([2], [14])

Let A be a locally convex algebra with 1 and with a continuous involution. An element $x \in A$ is called *(Allan) bounded* if for some $\lambda \neq 0$, $\{(\lambda^{-1}x)^n | n = 1, 2, \ldots\}$ is bounded. The algebra A is called *symmetric* if for each $x \in A$, $(1 + x^*x)^{-1}$ exists and is bounded. Let $\mathscr{B}^*$ be the collection of all $B \subset A$ such that B is absolutely convex, $B^2 \subset B$, $B^* = B$ and B is closed and bounded. A locally convex* algebra with 1 is called a *Generalized B^*-algebra* (*GB^*-algebra*) if

(i) A is symmetric,

(ii) the collection $\mathscr{B}^*$ has a greatest member B_0 under inclusion, called the unit ball of A; and

(iii) the *subalgebra $A(B_0) = \{\lambda x | \lambda \in ¢,\ x \in B_0\}$ is a Banach algebra with the Minkowski functional $\|\cdot\|_{B_0}$. (It turns out to be a C^*-algebra.)

Allan [2] has shown that such an algebra A, if commutative, is isomorphic to an algebra of continuous functions on a compact Hausdorff space taking values in the extended complex place $¢^* = ¢ \cup \{\infty\}$ such that $f^{-1}(\infty)$ is atmost a nowhere dense set. Dixon [14] has shown that a (locally convex) GB^*-algebra admits a faithful representation as a certain algebra of closed operators (called an *extended C^*-algebra*) all defined on a common dense subspace of a Hilbert space.

Now as shown in [2], a b^*-algebra is a GB^*-algebra. Thus it is natural to synthesize and unify the above Gelfand–Naimark type representation theorems. Here we present GB^*-representation theorems in forms that quickly reduces to the b^*-theorems mentioned above when the given algebra is a b^*-algebra. This we do for a class of GB^*-algebras that arise naturally in the study of quotient-bounded and universally-bounded elements introduced in [9].

1.3. *Definition*

Let A be a locally convex algebra with 1 (and with a separately continuous multiplication). Let $P(A)$ denote the collection of all calibrations P on A viz families P of seminorms determining the topology of A. Let $P = (p_\alpha | \alpha \in \Delta)$ be in $P(A)$. An element $a \in A$ is called *(left) P-quotient bounded* if for each α, there is a real constant $M_{\alpha,a}$ depending on α and a such that $p_\alpha(ax) \leqslant M_{\alpha,a} p_\alpha(x)$ holds for all $x \in A$. Further, it is called *(left) P-universally bounded* if the $M_{\alpha,a}$, for all α, have an upper bound depending only on a (written M_a).

Note that in a calibrated hypocontinuous locally convex algebra A, a universally-bounded element is (Allan) bounded; whereas a quotient-bounded element need not be; e.g. let A be an LMC algebra with an m-calibration P. Then each $a \in A$ is P-quotient bounded; on the other hand, $a \in A$ is (Allan) bounded iff a is Q-universally bounded for some m-calibration Q on A.

Given a locally convex algebra A with a calibration P, let B_P (respectively Q_P) be the set of all P-universally bounded (respectively P-quotient bounded) elements. In [9], it is shown that Q_P is a subalgebra of A and B_P is a subalgebra of Q_P with $1 \in B_P$. A natural locally m-convex (LMC) topology t_P is defined on Q_P by the m-calibration $\{q_\alpha | \alpha \in \Delta\}$ where $q_\alpha(a) = \sup\{p_\alpha(ax) | p_\alpha(x) \leqslant 1\}$ $(a \in Q_P)$, whereas $p(a) = \sup_\alpha q_\alpha(a)$ defines an algebra norm on B_P. If A is complete, then each of (B_P, p) and (Q_P, t_P) is complete. Further, if A is a hypocontinuous locally convex GB^*-algebra with unit ball B_0, and P is a GB^*-calibration (so that A admits a P-hermitian decomposition [32, Theorem 8.15]), then

(i) (Q_P, t_P) is a b^*-algebra having $\{q_\alpha\}$ as a b^*-calibration, and

(ii) (B_P, p) is a C^*-algebra with $B_P = \{\lambda x | \lambda \in \mathbb{C}, x \in B_0\} = A(B_0)$ and $p(x) = \inf \{\lambda > 0 | x \in \lambda B_0\} = \|x\|_{B_0}$.

Thus when A is a b^*-algebra with a b^*-calibration P, then $Q_P = A$. However, in general Q_P may be trivial. In fact, in [9], we have an example in which there is no b^*-algebra properly lying between $A(B_0)$ and A. This motivates the following.

1.4 *Definition*

A locally convex GB^*-algebra A is called *proper* if there is a GB^*-calibration P on A (called a *proper GB^*-calibration*) such that Q_P is nontrivial in the sense that $B_P \neq Q_P \neq A$. It is called *pure* if $A(B_0) \neq A$.

In § 2, we discuss a certain method of constructing proper GB^*-algebras and a decomposition theorem for such algebras. The functional representation theorem [14, Theorem 4.6] and the operator representation theorem [14, Theorem 7.1] are suitably modified for proper GB^*-algebras in such a way that in the latter case, the quotient-bounded elements of A are mapped to quotient-bounded operators on a locally convex space [17]. Our theorem can be regarded as a GB^*-analogue of a representation theorem for b^*-algebras [17, Theorem 8]. In § 3, we consider a couple of examples to illustrate the concepts. After briefly mentioning Arens algebra L^ω on a finite measure space and its σ-finite variant, we discuss in detail its now commutative analogue on a regular probability gauge space. We also have the Hardy–Arens algebra H^ω on the disc and its noncommutative analogue determined by a one-parameter group of *automorphisms on a van Newmann algebra. Q_p and B_p are also computed for unbounded Hilbert algebras recently studied by Inoue [21]. We also introduce L^p- and L^ω-direct integrals of a measurable field of C^*-algebras. The construction of these examples is analogous. Finally we analyze in detail the convolution algebra of periodic distributions on the circle.

In the sequel, we shall frequently need the following ([4], [5]).

1.5 *Theorem*

(a) Let A be a locally convex GB^*-algebra with unit ball B_0. Then for each $x \in A$, $x_n = x([1 + 1/n]x^*x)^{-1}$ is in $A(B_0)$ for $n = 1, 2, \ldots$ and $x_n \to x$.

(b) Let A be a complete hypocontinuous locally convex *algebra with 1. If there exists a *subalgebra B of A containing 1 such that

(i) B is a B^*-algebra under some norm $\|\cdot\|$, and

(ii) the inclusion $B \to A$ is a continuous injection (with $\|\cdot\|$ on B) with sequentially dense range, then A is a GB^*-algebra.

2. Main results

2.1 *Proposition*

Let (A_α) be an inverse system of an infinite family of pure complete hypocontinuous locally convex GB^*-algebras. Then the following hold.

(i) The inverse limit $A = \overset{\lim}{\leftarrow} A_\alpha$ is a pure complete hypocontinuous locally convex GB^*-algebra.

(ii) For each α, let P_α be a GB^*-calibration on A_α. Let P be the natural calibration on A. Then P is a GB^*-calibration on A. If $B_{P^\alpha} = Q_{P^\alpha}$ for each α and if A contains a P-quotient-bounded hermitian element that is not (Allan) bounded, then P is a proper GB^*-calibration.

In the course of the proof, we compute Q_P and B_P for A. It also follows that the product of an infinite family of pure complete hypocontinuous GB^*-algebras A_α admitting improper GB^*-calibrations (i.e. $Q_{P^\alpha} = B_{P^\alpha}$) is a proper GB^*-algebra.

Proof. Let $\{(A_\alpha, t_\alpha) | \alpha \in \Delta\}$ be an infinite family of complete hypocontinuous pure locally convex GB^*-algebras. Let B_α be the unit ball of A_α and 1_α the identity of A_α. Let $B = \prod_\alpha A_\alpha$ be the cartesian product of A_α's with the product topology t. Then it is a routine verification that (B, t) is a complete locally convex GB^*-algebra with unit ball $B'_0 = \Pi B_\alpha$.

A simple argument also shows that the product of hypocontinuous locally convex algebras is hypocontinuous.

Now assume that Δ is directed. For $\alpha \leqslant \beta$ in Δ, let $\pi_{\alpha\beta}$: $(A_\beta, t_\beta) \to (A_\alpha, t_\alpha)$ be onto continuous* algebra homomorphisms; and for each α, let π_α: $B \to A_\alpha$ be the natural projection. Suppose that A_α's with the maps $\pi_{\alpha\beta}$ form an inverse system. Let $(A, t) = \overset{\lim}{\leftarrow}(A_\alpha, t_\alpha) = \{x = (x_\alpha) \in \Pi A_\alpha | x_\alpha \in A_\alpha$ for all α and for $\alpha \leqslant \beta$, $\pi_{\alpha\beta}(x_\beta) = x_\alpha\}$ be the inverse limit. Then A, being closed in B, is a complete hypocontinuous locally convex GB^*-algebra with unit ball $B_0 = A \cap B'_0$ so that the C^*-algebra that underlies A is $(A(B_0), \|\cdot\|_{B_0})$ which is a norm closed *subalgebra of $A(B'_0) = \{\lambda x | \lambda \in ¢, x \in B'_0\}$. The C^*-norm on $A(B'_0)$ is $\|x\|_{B_0} = \inf\{\lambda > 0 | x \in \lambda B'_0\}$. Further for $\alpha \leqslant \beta$, $\pi_{\alpha\beta}(B_\beta) \subset B_\alpha$, and so $\pi_{\alpha\beta}(A(B_\beta)) \subset A(B_\alpha)$, $\|\pi_{\alpha\beta}(z)\|_\alpha \leqslant \|z\|_\beta$ for all $z \in A(B_\beta)$. It is easily seen that with $\pi_{\alpha\beta} | A(B_\beta)$, $\alpha \leqslant \beta$ in Δ as morphisms, $\{(A(B_\alpha), \|\cdot\|_\alpha) | \alpha \in \Delta\}$ forms an inverse system of C^*-algebras. Let (Q, τ) be the b^*-algebra obtained by taking the inverse limit of $(A(B_\alpha), \|\cdot\|_\alpha)$'s. Since each $A_\alpha \neq A(B_\alpha)$, $A \neq Q$. Also, the unit ball of (Q, τ), as a GB^*-algebra, is B_0; hence $A(B_0) = \{x \in Q \,|\, \sup \|\pi_\alpha(x)\|_\alpha < \infty\}$.

Now to prove (ii), for each α, let $P^\alpha = \{p^\alpha_\gamma | \gamma \in \Delta_\alpha\}$ (Δ_α some index set) be a GB^*-calibration for A_α. Hence by [32, Theorem 8.9], $(A(B_\alpha), \|\cdot\|_\alpha) = (B_{P^\alpha}, p^\alpha)$ viz the C^*-algebra of all P^α universally bounded elements of A_α. The natural calibration on A induced by P^α's is

$$P = \{p^\alpha_\gamma \circ \pi_\alpha | \gamma \in \Delta_\alpha, \alpha \in \Delta\}.$$

Claims: (a) $A(B_0) = B_P$, the P-universally bounded elements of A.

(b) $Q = Q_P$, the P-quotient bounded element of A.

(a) Given $a \in A(B_0)$, $\sup_\alpha \|a_\alpha\|_\alpha = \sup_\alpha \|\pi_\alpha(a)\|_\alpha < \infty$. Then for each $\alpha \in \Delta$, each $\gamma \in \Delta_\alpha$ and for each $x \in A$, $(p^\alpha_\gamma \circ \pi_\alpha)(ax) \leqslant (\sup_\alpha \|a_\alpha\|_\alpha)(p^\alpha_\gamma \circ \pi_\alpha)(x)$. Hence $a \in B_P$, $A(B_0) \subset B_P$. On the other hand, given $a \in B_P$, there is a constant M_a depending only on a such that $(p^\alpha_\gamma \circ \pi_\alpha)(ax) \leqslant M_a(p^\alpha_\gamma \circ \pi_\alpha)(x)$ for $\alpha \in \Delta$, $\gamma \in \Delta_\alpha$, $x \in A$. Hence $a_\alpha = \pi_\alpha(a) \in A(B_\alpha) = B_{P^\alpha}$ and $p^\alpha(a_\alpha) = \sup\{p^\alpha_\gamma(a_\alpha z) | p^\alpha_\gamma(z) \leqslant 1 \text{ for all } \gamma, \text{ for } z \in A_\alpha\} \leqslant M_a$. (Here p^α is the natural Banach algebra norm on B_{P^α}, the P^α-universally-bounded elements of A_α). Thus $\sup_\alpha p^\alpha(a_\alpha) < \infty$ and so $a \in A(B_0)$. Thus $B_P \subset A(B_0)$. (This also follows from the fact that B_0 is the greatest member of $\mathscr{B}^*(A)$.)

Note that in view of [32], this implies that P is a GB^*-calibration for A.

(b) That $Q \subset Q_P$ follows as in (a). Let $a \in Q_P$. Then for each $\alpha \in \Delta$, $\gamma \in \Delta_\alpha$, $p^\alpha_\gamma(a_\alpha x_\alpha) \leqslant M^\alpha_{\gamma,a}\, p^\alpha_\gamma(x_\alpha)$ for all $x_\alpha \in A_\alpha$, $x_\alpha = \pi_\alpha(x)$ for $x \in A$, $a_\alpha = \pi_\alpha(a)$. Hence $a_\alpha \in Q_{P^\alpha}$, P^α-quotient bounded elements of A_α. By hypothesis, $a_\alpha \in B_{P^\alpha} = A(B_\alpha)$, P^α being a GB^*-calibration. Thus $a_\alpha \in A(B_\alpha)$ for each α, and so $a \in Q$. Thus $Q = Q_P$.

Now each (Allan) bounded hermitian element of A is in $A(B_0)$ by [14, Lemma 2.6]; with the result, the stated condition implies that $B_P \neq Q_P$. This completes the proof.

In view of [3, Theorem 2.4] and [23, Theorem 5.1], it is immediate that a b^*-algebra can be expressed as an inverse limit of a family of C^*-algebras. This motivates our next result. We say that an inverse system $\{(A_\alpha, t_\alpha) | \alpha \in \Delta\}$ of calibrated locally convex GB^*-algebras with calibration P_α on A_α satisfies *property* (P) provided given x in $\overset{\lim}{\leftarrow} A_\alpha$ with associated thread $x \sim (x_\alpha)$, if x_α is P_α-quotient bounded in A_α for each α, then x_α is (Allan) bounded in A_α for each α.

2.2 *Theorem*

A complete metrizable proper locally convex GB^*-algebra is isomorphic to an inverse limit of a family satisfying (P) of pure metrizable calibrated complete locally convex GB^*-algebras.

Proof. Let (A, t) be a given GB^*-algebra with a proper GB^*-calibration $P = (p_\alpha | \alpha = 1, 2, \ldots)$. Let, for each α, $N_\alpha = \{a \in Q_P | q_\alpha(a) = 0\}$ which is a t_P closed* ideal of Q_P. Since (Q_P, t_P) is a b^*-algebra having $\{q_\alpha\}$ as a b^*-calibration, $\tilde{q}_\alpha(a + N_\alpha) = q_\alpha(a)$ $(a \in Q_P)$ defines a C^*-norm on $(Q_P)_\alpha = Q_P/N_\alpha$. By [3, Theorem 2.4], $((Q_P)_\alpha, \tilde{q}_\alpha)$ is complete. As in [23, Theorem 5.1], $(Q_P, t_P) = \overset{\lim}{\leftarrow} ((Q_P)_\alpha, \tilde{q}_\alpha)$, where $\{N_\alpha\}$ is directed as $\alpha \leqslant \beta$ if $N_\beta \subset N_\alpha$; and in this case, $\pi_{\alpha\beta}\colon (Q_P)_\beta \to (Q_P)_\alpha$ is $\pi_{\alpha\beta}(a + N_\beta) = a + N_\alpha (a \in Q_P)$. Let $\pi_\alpha(a) = a + N_\alpha$ be the natural map from Q_P onto $(Q_P)_\alpha$. Let t^α_q be the quotient topology on $(Q_P)_\alpha$ induced from the topology t. Let $A_\alpha = ((Q_P)_\alpha, t^\alpha_q)^\sim$, the completion of $((Q_P)_\alpha, t^\alpha_q)$. We denote the topology on A_α by t^α_q.

It is easily seen that $((Q_P)_\alpha, t^\alpha_q)$ is a locally convex *algebra, and that the algebraic structure on $(Q_P)_\alpha$ can be uniquely extended to A_α making A_α a complete locally convex *algebra with identity. On the other hand, let $\bar{N}_\alpha$ be the closure of N_α in (A, t). Since multiplication is jointly continuous in A, it is easily seen that N_α is an ideal in A; and $A/\bar{N}_\alpha$ with the quotient topology τ induced by t is a complete locally convex *algebra.

Claim. $(A/\bar{N}_\alpha, \tau)$ is homeomorphic to (A_α, t_q^α).

Indeed, let $\phi: (Q_P)_\alpha \to A/\bar{N}_\alpha$ be $\phi(a+N_\alpha) = a+\bar{N}_\alpha$. Since P is a GB^*-calibration, the Banach algebra (B_P, p) is identical to the C^*-algebra $(A(B_0), \|\cdot\|_{B_0})$ where B_0 is the unit ball of A. From Theorem 1.5(a), it is immediate that the range of ϕ is sequentially dense in $A/\bar{N}_\alpha$. Also, ϕ is one-to-one, and for any $p_\beta \in P$, $a \in Q_P$,

$$\ddot{p}_\beta(a+\bar{N}_\alpha) = \inf\{p_\beta(a+y) \mid y \in \bar{N}_\alpha\}$$
$$\leqslant \inf\{p_\beta(a+y) \mid y \in N_\alpha\} = \dot{p}_\beta(a+N_\alpha).$$

Hence ϕ is $t_q^\alpha - \tau$ continuous. Thus ϕ can be extended uniquely to a continuous one-to-one *homomorphism $\phi: A_\alpha \to A/\bar{N}_\alpha$. Further, ϕ is onto. For, given $x+\bar{N}_\alpha \in A/\bar{N}_\alpha$, Theorem (1.5) (a) gives a sequence (x_n) in $A(B_0)$ such that $x_n \xrightarrow{t} x$ in A, $x_n + \bar{N}_\alpha \to x + \bar{N}_\alpha$; and $(x_n + N_\alpha)$ being Cauchy in $(Q_P)_\alpha$, converges to some $y \in A_\alpha$. Then $\phi(y) \overset{t_q^\alpha}{=} x + \bar{N}_\alpha$. Now the open mapping theorem implies that ϕ is a homeomorphism.

The natural quotient map $\pi_\alpha: A \to A/\bar{N}_\alpha$ is identified with the unique continuous extension of $\pi_\alpha: Q_P \to A_\alpha$ which is an algebra homomorphism. For $x \in A$, $\pi_\alpha(x)^* = \pi_\alpha(x^*)$ is the continuous involution on A_α which is the extension of the involution on $(Q_P)_\alpha$.

Now let τ_P^q be the quotient topology on $(Q_P)_\alpha$ induced by the b^*-topology τ_P in Q_P. It is determined by the calibration $\{\ddot{q}_\beta\}$ where $\ddot{q}_\beta(a+N_\alpha) = \inf\{q_\beta(a+y) \mid y \in N_\alpha\}$. Then $((Q_P)_\alpha, \tau_P^q$ is a Frechet b^*-algebra; and for $a+N_\alpha$ in $(Q_P)_\alpha$, $\ddot{q}_\beta(a+N_\alpha) \leqslant q_\beta(a)$ for all β and $\ddot{q}_\alpha(a+N_\alpha) = q_\alpha(a) = \tilde{q}_\alpha(a+N_\alpha)$. (This is because for $y \in N_\alpha$, $q_\alpha(a) = q_\alpha(a+y-y) \leqslant q_\alpha(a+y) + q_\alpha(y) = q_\alpha(a+y)$). Thus, by the open mapping theorem, τ_P^q coincides with the norm topology given by $\tilde{q}_\alpha$.

Further, $N_\alpha \cap A(B_0)$ is a norm closed *ideal of the C^*-algebra $A(B_0)$. Consider the quotient C^*-algebra $C = A(B_0)/N_\alpha \cap A(B_0)$ with the quotient norm $\ddot{p}(a+N_\alpha \cap A(B_0)) = \inf\{p(a+y) \mid y \in N_\alpha \cap A(B_0)\}$. (Note that P being a GB^*-calibration on A, $p = \|\cdot\|_{B_0}$ on $A(B_0) = B_P$). Then we can identify the C^*-algebras $(C, \ddot{p})$ and $((Q_P)_\alpha, \tilde{q}_\alpha)$. Indeed, the embedding id : $C \to (Q_P)_\alpha$ is a well-defined isomorphism of C into $(Q_P)_\alpha$. Hence it is an isometry and id(C) is a $\tilde{q}_\alpha$-closed *subalgebra of $(Q_P)_\alpha$. On the other hand, id(C) is dense in $(Q_P)_\alpha$. For, let $a+N_\alpha \in (Q_P)_\alpha$ with $a \in Q_P$. By Theorem 1.5 (a) (applied to Q_P as a GB^*-algebra), we get a sequence (a_n) in $A(B_0)$ such that for each β, $q_\beta(a_n - a) \to 0$. Hence in particular, $(\tilde{q}_\alpha(a_n + N_\alpha) - (a+N_\alpha)) \to 0$. Thus it follows that

$$A(B_0)/N_\alpha \cap A(B_0) \simeq (Q_P)_\alpha \text{ and } \ddot{p} = \tilde{q}_\alpha.$$

From above and Theorem 1.5 (b), it follows that (A_α, t_q^α) is a complete metrizable locally convex GB^*-algebra with the underlying C^*-algebra $((Q_P)_\alpha, \tilde{q}_\alpha)$.

Now for $\alpha \leqslant \beta$, consider $\pi_{\alpha\beta}$: $((Q_P)_\beta, t_q^\beta) \to ((Q_P)_\alpha, t_q^\alpha)$. Since for each $\delta \in \mathbf{N}$, inf $\{p_\delta(a+y) \mid y \in N_\alpha\} \leqslant \inf\{p_\delta(x+y) \mid y \in N_\beta\}$, $\pi_{\alpha\beta}$ is a continuous homomorphism, and in view of Theorem 1.5(a), extends to a continuous surjective *homomorphism $\tilde{\pi}_{\alpha\beta}$: $(A_\beta, t_q^\beta) \to (A_\alpha, t_q^\alpha)$. Given $x \in A$, if x_n is as in Theorem 1.5(a), then $\pi_\beta(x_n) = \pi_\beta(x)(1+(1/n)\pi_\beta(x)^*\pi_\beta(x))^{-1} \to \pi_\beta(x)$ and so $\pi_{\alpha\beta}(\pi_\beta(x) = \pi_\alpha(x)$. Thus $\{(A_\alpha, t_q^\alpha), \pi_{\alpha\beta} | \alpha \leqslant \beta\}$ forms an inverse system. Let $B = \overleftarrow{\lim} (A_\alpha, t_q^\alpha)$ be the inverse limit. Define π: $A \to B$ as $\pi(x) = (\pi_\alpha(x))$. Then π is a topological *isomorphism of A onto B.

Finally, identifying A and B, let $a = (a_\alpha)$ be such that for each α, $a_\alpha = a + \bar{N}_\alpha$ is P^α-quotient bounded in A^α for all each α, where P^α is the natural quotient calibration on A^α for t_q^α viz $P^\alpha = \{\ddot{p}_\beta \mid \beta \in \mathbf{N}\}$ where $\ddot{p}_\beta(x + \bar{N}_\alpha) = \inf\{p_\beta(a + y) \mid y \in \bar{N}_\alpha\}$. Then for each fixed α, there is a constant $M_{\beta,a}$ such that

$$\ddot{p}_\beta((a + \bar{N}_\alpha)(x + \bar{N}_\alpha)) \leqslant M_{\beta,a}\ddot{p}_\beta(x + \bar{N}_\alpha)$$
$$\leqslant M_{\beta,a}p_\beta(x) \text{ for all } x \in A.$$

In particular, $\ddot{p}_\alpha(ax + \bar{N}_\alpha) \leqslant M_{\alpha,a}p_\alpha(x)$ for all $x \in A$. Now for each $y \in \bar{N}_\alpha$, $p_\alpha(ax) = p_\alpha(ax + y - y) \leqslant p_\alpha(ax + y) + p_\alpha(y) = p_\alpha(ax + y)$ since $\bar{N}_\alpha \subset \ker p_\alpha$. Thus $p_\alpha(ax) \leqslant M_{\alpha,a}p_\alpha(x)$ for all $x \in A$. Hence $a \in Q_P$, and so $a_\alpha \in (Q_P)_\alpha$ for each α. Thus a_α is (Allan) bounded in A_α for each α. This completes the proof.

A *algebra of continuous functions [14, Definition 4.7] on a topological space M is a collection F of extended complex valued continuous functions on M such that each $f \in F$, $f^{-1}(\infty)$ is at most a set of first category and F is a *algebra under operations $f + y$, $\lambda f\,(\lambda \in ¢), fg$ which consists in defining these operations pointwise on the dense set where values involved are finite and then extending the resulting functions to be continuous extended complex valued functions on M. Let A be a commutative GB^*-algebra with unit ball B_0. By a theorem due to Allan [14, Theorem 4.6], the Gelfand representation of $A(B_0)$ onto $C(X)$ (where X is the compact Hausdorff maximal ideal space of $A(B_0)$) extends uniquely to a *isomorphism of A onto a *algebra of extended complex valued continuous functions on X. On the otherhand, Michael [23] and Apostol [3] have discussed the realization of a commutative b^*-algebra as the algebra of all continuous complex-valued functions on a completely regular space. We synthesize both these results in the following.

2.3 *Theorem*

Let A be a proper commutative locally convex GB^*-algebra with a proper GB^*-calibration P. Then there exists a real-compact completely regular Hausdorff space Z and a mapping π of A into extended complex-valued continuous functions on Z such that

(a) π is a *isomorphism of A onto a *algebra of functions on Z containing $C(Z)$, the algebra of all continuous complex-valued functions on Z;

(b) π represents B_P *isomorphically onto $C_b(Z)$, the C^*-algebra of all bounded continuous complex-valued functions on Z; and

(c) π represents Q_P*isomorphically onto $C(Z)$.

Proof. Let X (respectively Z) be the usual weakly topological space of all non-zero continuous multiplicative functionals (respectively all non-zero multiplicative functionals) on Q_P. Then as in [3, Corollary 3.4], X is a completely regular Hausdorff space and $X \subsetneq Z = vX \subsetneq M = \beta X$ are dense continuous embeddings. Here vX and βX are respectively the real-compactification and the Stone–Cech compactification of X, and M is the maximal ideal space of B_P. By results in [3], we have the algebraic *isomorphisms $Q_P \simeq C(X) \simeq C(Z)$ and $B_P \simeq C_b(X) \simeq C_b(Z) \simeq C(M)$ defined as $x \to \tilde{x}\,(x \in Q_P)$ where $\tilde{x}(\varphi) = \varphi(x)$ $(\varphi \in Z)$. If $x \to \hat{x}$ is the Gelfand representation of B_P

onto $C(M)$, then $\hat{x}|_Z = \tilde{x} = \phi(x)$ (say). By [14, Theorem 4.6], ϕ extends uniquely to a *isomorphism of A onto a *algebra of functions on M defined as $\hat{x}(\varphi) = \varphi'(x)$ where φ' is the unique $\mathbb{¢}^* = \mathbb{¢} \cup \{\infty\}$ valued 'partial homomorphism' that extends φ as in [2, Proposition 3.1]. Hence for each $a \in Q$, a agrees with the unique Stone extension [18, § 6.5] of $\tilde{a} \in C(Z) \simeq C(X)$ as a $¢^*$-valued continuous function on M. It follows that $\hat{a}|_Z = \tilde{a}$ $(a \in Q_P)$. The map ϕ on A defined as $\phi(a) = \hat{a}|_Z$ is the desired map.

Now we consider operator representations. A **algebra of closed operators* [14, Definition 7.1] on a Hilbert space H is a set $\mathscr{A}$ of closed operators in H that forms a *algebra under the operations $T, S \to (T+S)^-$, $(T, S) \to (TS)^-$ and $T \to T^*$. (Here $^-$ denotes the closure of the operator and * is the operator adjoint). It is called an *extended C^*-algebra* if $\mathscr{A} \cap \mathscr{B}(H)$ is a C^*-algebra and for each $T \in \mathscr{A}$, $(1 + T^*T)^{-1} \in \mathscr{A}$. ($\mathscr{B}(H)$ is the C^*-algebra of all bounded linear operators on H). It is said to have *a common dense domain* D if $D = \cap \{D(T) | T \in \mathscr{A}\}$ is dense in H where $D(T)$ denotes the domain of T. Such an algebra is an *EC^*-algebra* in the sense of Inoue [21].

In [20], Inoue introduced weakly unbounded operator algebras. Let $\{\mathscr{A}_\alpha | \alpha \in \Delta\}$ be a family of C^*-algebras, $\mathscr{A}_\alpha$ acting on a Hilbert space H_α. Let $H = \Sigma^\oplus H_\alpha$ be the Hilbert space direct sum and let $D = \{\xi = (\xi_\alpha) \in H | \xi_\alpha \in H_\alpha$ for all α, $\xi_\alpha = 0$ for all but finitely many α's$\}$. It is dense in H. A *algebra $\mathscr{A}$ of closable operators all defined on D is called a *weakly unbounded operator algebra* if $\mathscr{A}$ is a *subalgebra of $\Pi \mathscr{A}_\alpha = \{T = (T_\alpha) | T_\alpha \in \mathscr{A}_\alpha$ for all $\alpha\}$ such that $\mathscr{A}_b = \{\bar{T} | T \in \mathscr{A}$ is bounded$\} = \Sigma \mathscr{A}_\alpha$ the usual (1^∞) direct sum of C^*-algebras. Note that $\mathscr{A}$ is a pre b^*-algebra under a natural topology.

Giles *et al* [17] have introduced quotient-bounded and universally-bounded operators on a locally convex space. Let X be a locally convex (Hausdorff) space with a calibration $\Gamma = (p_\alpha | \alpha \in \Delta)$. A linear operator $T: X \to X$ is called *quotient-bounded* if for each $\alpha \in \Delta$, there is a scalar $k_\alpha \geqslant 0$ such that $p_\alpha(Tx) \leqslant k_\alpha p_\alpha(x)$ $(x \in A)$. It is called *universally bounded* if these k_α's for all α have an upper bounded written k. Let $Q(X, \Gamma)$ (respectively $B(X, \Gamma)$) be the set of all quotient-bounded (respectively universally bounded) operators on X. Then $Q(X, \Gamma)$ forms an algebra and $B(X, \Gamma)$ is a subalgebra of $Q(X, \Gamma)$. The seminorms $q_\alpha(T) = \sup\{p_\alpha(Tx) | p_\alpha(x) \leqslant 1\}$ define a natural LMC topology on $Q(X, \Gamma)$, where as $p_\Gamma(T) = \sup_\alpha q_\alpha(T)$ defines an algebra norm on $B(X, \Gamma)$. If X is complete, then each of these is complete.

Dixon [14, Theorem 7.1] has faithfully represented a locally convex GB^*-algebra A as an extended C^*-algebra. Giles *et al* [17, Theorem 8] have proved that a b^*-algebra is *isomorphic to a *subalgebra of quotient-bounded operators on the product of a family of Hilbert spaces. The following representation theorem for proper GB^*-algebras synthesizes both these results. Note that in the course of the proof, we also show that a b^*-algebra is *isomorphic to a weakly-unbounded operator algebra. For basic notions of unbounded representations, we refer to [24].

2.4 *Theorem*

Let (A, t) be a proper complete bornological locally convex GB^*-algebra with a proper GB^*-calibration P. Then there exists a complete locally convex space X and a subspace

H of X such that H is a Hilbert space that is continuously and densely injected in X; and there exists a representation π of A into linear operators on X such that

(a) π represents A faithfully as an extended C^*-algebra with a common dense domain in H;

(b) π represents Q_P faithfully and continuously as a subalgebra Q of the algebra of quotient-bounded operators on X which, in turn, is isomorphic to a weakly-unbounded operator algebra on H; and

(c) π represents B_P faithfully as a subalgebra B of the algebra of universally bounded operators on X which, in turn, is isomorphic to a C^*-algebra of bounded operators on H.

Proof. For f in $P(A)$, the set of all positive linear functionals on A, let $N_f = \{x \in A \mid f(x^*x) = 0\}$, $N_f^{(1)} = N_f \cap Q_P$, $N_f^{(2)} = N_f \cap B_P$. Consider the inner product spaces $K_f = A/N_f$, $K_f^{(1)} = Q_P/N_f^{(1)}$, $K_f^{(2)} = B_P/N_f^{(2)}$ with the canonical inner product induced by f (e.g. on K_f, $\langle a+N_f, b+N_f \rangle = f(b^*a)$). Let $H_f, H_f^{(1)}, H_f^{(2)}$ be their respective Hilbert space completions.

Claim. $H_f = H_f^{(1)} = H_f^{(2)}$ modulo-unitary equivalence. Indeed, the natural injections $K_f^{(2)} \to K_f^{(1)} \to K_f$ define isometric embeddings of the preceding into the following. Hence they extend to linear isometries $u_1: H_f^{(2)} \to H_f^{(1)}$, $u_2: H_f^{(1)} \to H_f$. It suffices to prove that $u_2 \circ u_1$ is onto. For this, given $x \in A$, Theorem 1.5 (a) gives a sequence (x_u) in B_P such that $x_u \to x$. Now A, being complete and bornological, is barrelled; and hence hypocontinuous, and so the multiplication is sequentially jointly continuous. Thus $(x_u - x)^*(x_u - x) \to 0$. By [14, §8], each $f \in P(A)$ is t-continuous. Hence $x_u + N_f \to x + N_f$ in K_f. Thus $(u_2 \circ u_1)(K_f^{(2)})$ is dense in K_f. This gives $(u_2 \circ u_1)(H_f^{(2)}) = H_f$ establishing the claim.

Now $H^{(2)} \equiv \Sigma^{\oplus} H_f^{(2)} \simeq \Sigma^{\oplus} H_f^{(1)} \equiv H^{(1)} \simeq H \equiv \Sigma^{\oplus} H_f$. Let π_f (respectively $\pi_f^{(1)}, \pi_f^{(2)}$) be the representations of A (respectively Q_P, B_P) constructed from f on H_f (respectively $H_f^{(1)}, H_f^{(2)}$) by the GNS construction [24, Theorem 6.3]. These are in general unbounded representations with domains $D(\pi_f) = K_f$, $D(\pi_f^{(1)}) = K_f^{(1)}$, $D(\pi_f^{(2)}) = K_f^{(2)}$, and their closures [24, Lemma 2.6] are designated by $(\pi_f, D(\pi_f), H_f)^-$, $(\pi_f^{(1)}, D(\pi_f^{(1)}), H_f^{(1)})^-$ and $(\pi_f^{(2)}, D(\pi_f^{(2)}), H_f^{(2)})^-$. These are strongly cyclic representations respectively of A, Q_P and B_P having strongly cyclic vectors $\xi_f = 1 + N_f$, $\xi_f^{(1)} = 1 + N_f^{(1)}$, $\xi_f^{(2)} = 1 + N_f^{(2)}$ respectively. By forming their direct sum, consider the representations $(\pi, D(\pi), H) = \Sigma^{\oplus}(\pi_f, D(\pi_f), H_f)^-$ of A on H, $(\pi^{(1)}, D(\pi^{(1)}), H^{(1)}) = \Sigma^{\oplus}(\pi_f^{(1)}, D(\pi_f^{(1)}), H_f^{(1)})^-$ of Q_P on $H^{(1)}$ and $(\pi^{(2)}, D(\pi^{(2)}), H^{(2)}) = \Sigma^{\oplus}(\pi_f^{(2)}, D(\pi_f^{(2)}), H_f^{(2)})^-$ of B_P on $H^{(2)}$.

Further, Let $D(\pi_f'^{r}) = \{\pi_f(Q_P)\xi_f\}$, a subspace of K_f. Let H_f^r be the norm closure of $D(\pi_f'^{r})$ in H_f. Then

$$\pi_f'^{r} = \pi_f|_{D(\pi_f'^{r})}$$

defines a representation of Q_P on H_f^r with domain $D(\pi_f'^{r})$. Let $(\pi_f^r, D(\pi_f^r), H_f^r)$ be its closure (as representation of Q_P). It is a closed strongly cyclic representation of Q_P with strongly cyclic vector ξ_f^r. Now consider the following results due to Inoue [22, Proposition 3.12].

Lemma. (a) Every closed *representation of a locally convex GB^*-algebra is self-adjoint.

(b) If $(\pi_1, D(\pi_1), H_1)$ and $(\pi_2, D(\pi_2), H_2)$ be two strongly cyclic self-adjoint representations of a *algebra A, then π_1 and π_2 are unitarily equivalent if and only if $(\pi_1(x)\xi_1, \xi_1) = (\pi_2(x)\xi_2, \xi_2)$ $(x \in A)$ where ξ_1 and ξ_2 are strongly cyclic vectors for π_1 and π_2.

It follows from this that $(\pi^r_f, D(\pi^r_f), H^r_f)$ and $(\pi^{(1)}_f, D(\pi^{(1)}_f), H^{(1)}_f)^-$ are unitarily equivalent, hence so are $(\pi^{(r)}, D(\pi^{(r)}), H^{(r)}) = \Sigma^{\oplus}(\pi^{(r)}_f, D(\pi^{(r)}_f), H^{(r)}_f)$ and $(\pi^{(1)}, D(\pi^{(1)}), H^{(1)})$. In what follows, we shall identify unitarily equivalent spaces.

Now let $X = \Pi H_f$, product of the Hilbert space H_f's with the product topology that is determined by the calibration $\Gamma = \{p_f | f \in P(A)\}$ where $p_f(\eta) = \|\eta_f\|_f$ for $\eta = (\eta_f) \in X$, $\|\cdot\|_f$ denoting the norm on H_f. Then the identity maps $D = D(\pi) \subsetneq H \subsetneq X$ give dense continuous embeddings. Since each $f \in P(A)$ is t-continuous, its restriction $f|_{Q_p}$ is t_p-continuous; and so by a result due to Brooks [14, Theorem 6.1], it is admissible i.e. the GNS representation $\pi^{(1)}_f$ (and so π^r_f) of Q_P determined by f is bounded mapping Q_P into bounded operators. Note that admissibility of f is equivalent to the fact that for each $x \in Q_P$, $\sup\left\{\dfrac{f(y^*x^*xy)}{f(y^*y)} \,|\, y \in Q_P\right\} = M_{x,f}$ (say) $< \infty$. As in [14, Theorem 7.11]. π faithfully represents, as $x \to \pi(x)$: $D(\pi) \to D(\pi)$ by $\pi(x)\,((\eta_f)) = (\pi_f(x)\eta_f)$, the algebra A as an EC^*-algebra on $D(\pi)$ in H such that B_P is represented as $x \to \phi(x) = \overline{\pi(x)}$, as a C^*-algebra of bounded operators on H with $\phi(B_P) = \mathscr{B}(H) \cap \overline{\pi(A)}$.

Let $a \in Q_P$. We show that for each $f \in P(A)$, $p_f(\pi(a)x) \leqslant M_{a,f} p_f(x)$ for each $x \in X$ of the form $x = \{(b_g + N^{(1)}_g) | g \in P(A)\}$ with $b_g \in Q_P$ for each $g \in P(A)$. Indeed, $\pi(a)x = (ab_g + N^{(1)}_g) \in X$ and

$$\begin{aligned} p_f(\pi(a)x) &= \|ab_f + N^{(1)}_f\| \\ &= f(b^*_f a^* a b_f)^{1/2} \\ &= M^{1/2}_{a,f} f(b^*_f b_f)^{1/2} = M^{1/2}_{a,f}\, p_f(x). \end{aligned}$$

Now let $x \in X$ be of the form $x = (b_g + N_g)$ with $b_g \in A$ for all $g \in P(A)$. Then for each such g, by Theorem 1.5(a), $(b_g)_n = b_g(1 + (1/n)b^*_g b_g)^{-1} \in B_P \subset Q_P$ and $(b_g)_n \to b_g$. By the sequential joint continuity of multiplication and the continuity of f, we get $p_f(\pi(a)x) \leqslant M^{1/2}_{a,f}\, p_f(x)$. Thus $\pi(a)$ can be regarded as a Γ-quotient-bounded continuous linear operator defined on a dense subspace of X, and hence it admits a unique Γ-quotient-bounded continuous linear extension $\tilde{\pi}(a)$ on X. Thus $\tilde{\pi}(Q_P) \subset Q(X, \Gamma)$.

Further let $a \in B_P$, $f \in P(A)$. Consider an $x \in X$ of the form $x = \{(b_g + N^{(2)}_g) | g \in P(A)\}$ with $b_g \in B_P = A(B_0)$ for each $g \in P(A)$. Then

$$\begin{aligned} p_f(\pi(a)x) &= f(b^*_f a^* a b_f) \\ &\leqslant r_{B_P}(a^*a)^{1/2} f(b^*_f b_f) \text{ by [10, Lemma 37.6]} \\ &\leqslant \|a\|_{B_0}\, p_f(x) \end{aligned}$$

and so $p_f(\pi(a)x) \leqslant \|a\|_{B_0} p_f(x)$ $(x \in X,\ f \in P(A))$. Thus $\pi(a) \in B(X,\Gamma)$ and $p_\Gamma(\pi(a)) \leqslant \|a\|_{B_0}$. Also π is faithful and the spectral radius of an element in a C^*-algebra $\mathscr{u}$, whether calculated in $\mathscr{u}$ or in any Banach algebra containing $\mathscr{u}$ is the same. Hence

$$\begin{aligned}\|a\|^2_{B_0} &= \|a^*a\|_{B_0} = r_{A(B_0)}(a^*a)\\ &= r_{B(X,\Gamma)}(\tilde{\pi}(a)^*\tilde{\pi}(a)) \leqslant p_\Gamma(\tilde{\pi}(a))^2.\end{aligned}$$

Thus $\|a\|_{B_0}(=p(a)) = p_\Gamma(\tilde{\pi}(a))$ and π isometrically maps B_P into $(B(X,\Gamma), p_\Gamma)$.

On the other hand, consider the natural LMC topology t_Γ on $Q(X,\Gamma)$ that is determined by the seminorms $T \to q_f(T) = \sup\{p_f(Tz) | z \in X, p_f(z) \leqslant 1\}$. To prove that π maps (Q_P, t_P) continuously into $(Q(X,\Gamma), t_\Gamma)$, it is sufficient to show that given $a \in Q_P$, $f \in P(A)$, there exists a finite set $\alpha_1, \ldots, \alpha_n$ in Δ such that $q_f(\pi(a)) \leqslant \max_{1 \leqslant i \leqslant n} q_{\alpha_i}(a)$ for each $a \in Q_P$. (The argument that we give for this is based on the idea in the proof of [11, Theorem 6.1]). Since $f|_{Q_P}$ is t_P-continuous, it is bounded on some neighbourhood U of o in (Q_P, t_P). This U can be assumed to be of the form $U = \bigcap_{i=1}^{n} U_{\alpha_i}$ for some $\alpha_1, \ldots, \alpha_n$ in Δ where $U_{\alpha_i} = \{y \in Q_P | q_{\alpha_i}(y) \leqslant 1\}$ for $i = 1, 2, \ldots, n$. Let $q = \max_{1 \leqslant i \leqslant n} q_{\alpha_i}$. Since $\{q_\alpha | \alpha \in \Delta\}$ is a b^*-calibration for (Q_P, t_P) [9, Theorem 3.1], $\ker q = \{x \in Q_P | q(x) = 0\}$ is a *ideal of Q_P and $Q_P/\ker q$ is a *algebra which is a normed algebra under the norm $\tilde{q}(y + \ker q) = q(y)$. Then f_q defined as $f_q(y + \ker q) = f(y)$ is a continuous positive linear functional on $(Q_P/\ker q,\ \tilde{q})$. Let h on $Q_P/\ker q$ be defined as $h(z + \ker q) = f_q((b + \ker q)^*(z + \ker q)(b + \ker q))$ for a fixed b in Q_P. Then h is a continuous positive linear functional satisfying $|h(z + \ker q)| \leqslant h(1 + \ker q)\tilde{q}(z + \ker q)$. Hence

$$\begin{aligned}f(b^*z^*zb) &= f_q((b + \ker q)^*\,(z + \ker q)^*(z + \ker q)(b + \ker q))\\ &= h((z + \ker q)^*(z + \ker q))\\ &= h(1 + \ker q)\tilde{q}((z + \ker q)^*(z + \ker q))\\ &= f(b^*b)(\tilde{q}(z + \ker q))^2\\ &= f(b^*b)(q(z))^2\end{aligned}$$

for all $z \in Q_P$. Hence

$$\begin{aligned}M^{1/2}_{z,f} &= \sup\left\{\frac{f(b^*z^*zb)^{1/2}}{f(b^*b)^{1/2}} \,|\, b \in Q_P\right\}\\ &\leqslant q(z).\end{aligned}$$

Hence $q_f(\tilde{\pi}(z)) \leqslant M^{1/2}_{z,f} \leqslant \max_{1 \leqslant i \leqslant n} q_{\alpha_i}(z)$ for all $z \in Q_P$, which means that π maps (Q_P, t_P) continuously into $(Q(X,\Gamma), t_\Gamma)$.

Note also that $\pi(Q_P)$ is *isomorphic to $\pi^r(Q_P)$. As π^r is a direct sum of bounded representations, it follows that $\pi^r(Q_P)$ is a weakly unbounded operator algebra. This completes the proof.

3. Examples

In this section, we analyse a few examples with a view of illustrating quotient-bounded elements and proper GB^*-algebras. First two of the following examples have been analyzed in detail in [6] and [7]. Since some of our examples are motivated by them, we have briefly included the relevant constructions.

3.1. *Arens' algebra $L^\omega(X)$ on a finite measure space*

Let (X, Σ, μ) be a finite measure space. Let $L^\omega(X) = \bigcap_{1 \leqslant p < \infty} L^p(X)$. It is a *algebra with pointwise operations. Let τ^ω be the locally convex topology on $L^\omega(X)$ defined by the calibration $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$. Then $(L^\omega(X), \tau^\omega)$ is a complete metrizable locally convex GB^*-algebra the underlying C^*-algebra being $L^\infty(X)$.

Assertions: [9, Example 4.1]

(1) $Q_P = B_P = L^\infty(X)$

(2) $L^\omega(X)$ is not a proper GB^*-algebra. In fact, there is no b^*-algebra containing $L^\infty(X)$ and properly contained in $L^\omega(X)$.

3.2. *Arens' algebra L^ω_{loc} on a σ-finite measure space*

On an infinite σ-finite, nonatomic measure space (X, Σ, μ) with $X = \bigcup_1^\infty X_n$, $X_n \in \Sigma$, $X_n \subset X_{n+1}$, $\mu(X_n) < \infty$ for all n; let $L^\omega_{\text{loc}}(X) = \bigcap_{1 \leqslant p < \infty} L^p_{\text{loc}}(X)$ with the locally convex topology defined by

$$P = \{\|\cdot\|_{k,p} \mid k, p \in \mathbf{N}\} \text{ where } \|f\|_{k,p} = \left(\int_{X_k} |f|^p \, d\mu)^{1/p} \right).$$

Then $L^\omega_{\text{loc}}(X)$ is a locally convex GB^*-algebra with unit ball $B_0 = \{f \in L^\infty(X) \mid \|f\|_\infty \leqslant 1\}$.

Assertions: [9, Example 4.3]

(1) $B_P = L^\infty(X)$ and $Q_P = L^\infty_{\text{loc}}(X)$, the topology t_P being determined by $\|f\|_{\infty,k} = \text{ess. sup.}_{x \in X_k} |f(x)|$.

(2) $L^\omega_{\text{loc}}(X)$ is a proper GB^*-algebra.

3.3. *Hardy–Arens algebra $H^\omega(U)$*

Let $U = \{z \in ¢ \mid |z| < 1\}$. Let $H^p(U)$ $(1 \leqslant p < \infty)$ be the usual Hardy space of function on U. Let $H^\omega(U) = \bigcap_{1 \leqslant p < \infty} H^p(U)$. It is a locally convex *algebra with pointwise operations (the involution being $f^*(z) = \overline{f(\bar{z})}$) and the topology determined by the calibration $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$. It properly contains the Banach algebra $H^\infty(U)$.

The algebra H^ω is not symmetric, hence not GB^*; however $Q_P = B_P = H^\infty(U)$.

Let $\Gamma = \{z \in ¢ \mid |z| = 1\}$. As usual, $H^p(U)$ can be identified with a subalgebra of the convolution algebra $L^p(\Gamma)$. Then $H^\omega(U)$ is a closed subalgebra of the convolution algebra $L^\omega(\Gamma)$. In this case $(L^\omega(\Gamma), \tau^\omega)$ is a complete LMC algebra, and $Q_P = L^\omega(\Gamma)$, $B_P = L^\infty(\Gamma)$. Similarly for $H^\omega(U)$, $Q_P = H^\omega(U)$, $B_P = H^\infty(U)$.

For $1 \leqslant p < \infty$, we can also consider the Banach algebra $AC_p[0, 1] = \{f \in C[0, 1] \mid f'$ exists a.e. and $f' \in L^p[0, 1]\}$ with norm $\|f\|_{(p)} = \|f\|_\infty + \|f'\|_p$; and construct the algebra $AC_\omega[0, 1]$.

Now we shall consider non-commutative analogues of these, which we analyze in more details.

3.4. *Non-commutative Arens' algebra* L^ω $(\mathscr{A}, \varphi)$ *on a regular probability gauge space*

Let Γ be a regular probability gauge space [31] viz a triple $\Gamma = (H, \mathscr{A}, \varphi)$ where H is a complex Hilbert space, $\mathscr{A}$ a von Neumann algebra acting on H and φ a faithful normal tracial state on $\mathscr{A}$. For $A \in \mathscr{A}$, let $\|A\|_p = (\varphi(|A|^p))^{1/p}(1 \leqslant p < \infty)$ where $|A| = (A^*A)^{1/2}$. It is norm on $\mathscr{A}$. Let $L^p(\mathscr{A}, \varphi)$ be the completion of $(\mathscr{A}, \|\cdot\|_p)$. For $p = \infty$, let $L^\infty(\mathscr{A}, \varphi) = \mathscr{A}$ with the operator norm. These spaces can be identified as subsets of the set of closed operators in H affiliated with $\mathscr{A}$. For $1 \leqslant r \leqslant p$, $L^\infty(\mathscr{A}, \varphi) \subset L^p(\mathscr{A}, \varphi) \subset L^r(\mathscr{A}, \varphi) \subset L^1(\mathscr{A}, \varphi)$. By using non-commutative Holder's inequality, it can be shown that $L^\omega(\mathscr{A}, \varphi) = \bigcap_{1 \leqslant p < \infty} L^p(\mathscr{A}, \varphi)$ is a *algebra with identity with strong operations $(S+T)^-$, $(\lambda T)^-$, $(ST)^-$ and the operator adjoint as the involution. The following summarizes some aspects of the structure of $L^\omega(\mathscr{A}, \varphi)$. Let τ^ω be the topology on L^ω defined by $P = \{\|\cdot\|_p \mid 1 \leqslant p < \infty\}$.

3.4a *Proposition*: (a) $(L^\omega(\mathscr{A}, \varphi), \tau^\omega)$ is a complete metrizable locally convex GB^*-algebra with unit ball $B_0 = \{A \in \mathscr{A} \mid \|A\| \leqslant 1\}$.

(b) The positive cone in $L^\omega(\mathscr{A}, \varphi)$ is normal and $\tau^\omega = T$, the largest locally convex GB^*-topology on $L^\omega(\mathscr{A}, \varphi)$ [14, § 6]

$= \pi(T)$ the largest locally convex GB^*-topology with normal positive cone [8].

(c) A subset $\mathscr{L}$ of L^ω $(\mathscr{A}, \varphi)$ is a closed left ideal of $L^\omega(\mathscr{A}, \varphi)$ if and only if $\mathscr{L} = L^\omega(\mathscr{A}, \varphi)K$ where K is a projection in $\mathscr{A}$. Further it is a closed maximal left ideal if and only if $1 - K$ is a minimal projection in $\mathscr{A}$.

(d) Every topologically primitive ideal $\mathscr{P}$ (i.e. quotient of a closed maximal left ideal) is of the form $\mathscr{P} = L^\omega(\mathscr{A}, \varphi)(1 - C(K))$ where $C(K)$ is the central support of K in $\mathscr{A}$.

Proof. (a) The topology τ^ω is also determined by the calibration $P = \{\|\cdot\|_n \mid n = 1, 2, \ldots\}$, hence is metrizable. It is also complete. Theorem 1.5(b) implies that $L^\omega(\mathscr{A}, \varphi)$ is a GB^*-algebra with the underlying C^*-algebra $\mathscr{A}$.

(b) For A, B in $L^p(\mathscr{A}, \varphi)$, $0 \leqslant A \leqslant B$ gives $\|A\|_p \leqslant \|B\|_p$ in view of [33, Proposition 2.5 (iv)]. Hence by [29, Ch. V, § 3.1], the positive cone in $L^\omega(\mathscr{A}, \varphi)$ is normal. Thus every continuous hermitian linear functional in L^ω is a difference of two positive functionals. This with the automatic continuity of a positive functional on a complete metrizable *algebra with 1 [13, p. 178] implies that the dual of $L^\omega(\mathscr{A}, \varphi)$ is $(L^\omega(\mathscr{A}, \varphi))^P$, the complex linear span of all positive functionals. Since $L^\omega(\mathscr{A}, \varphi)$ is Mackey, it follows from [14, §8] that $\tau^\omega = T$, and hence $\tau^\omega = \pi(T)$.

In fact, the dual of $L^\omega(\mathscr{A}, \varphi)$ is identified with $\bigcup_{1<q\leqslant\infty} L^q(\mathscr{A}, \varphi)$, and $L^\omega(\mathscr{A}, \varphi)$ is reflexive.

(c) and (d) can be verified either by a modification of the standard W^*-algebra techniques [28, Proposition 1.10.1] or else by applying [5, Corollary 3].

Assertions: (a) Taking $P = \{\|\cdot\|_n \mid n = 1, 2, \dots\}$, $Q_P = B_P = \mathscr{A}$

(b) $L^\omega(\mathscr{A}, \varphi)$ is not a proper GB^*-algebra.

Proof. (a) It suffices to show that $Q_P \subset \mathscr{A}$. For, then $Q_P = B_P = \mathscr{A}$, since $\mathscr{A} \subset B_P$; and further as P is countable, the b^*-topology on Q_P is also metrizable, and the open mapping theorem shows that the topologies on these algebras are identical.

Let $A \in Q_P$. Then for some scalar $q_1(A)$, $\varphi(|AX|) \leqslant q_1(A)\varphi(|X|)$ for all $X \in L^\omega(\mathscr{A}, \varphi)$. As $L^\omega(\mathscr{A}, \varphi)$ is dense in $L^1(\mathscr{A}, \varphi)$, this holds for all $X \in L^1(\mathscr{A}, \varphi)$. Also, for each $Y \in L^1(\mathscr{A}, \varphi)$,

$$\varphi(|Y|) = \|Y\|_1 = \sup\{|\varphi(YS)| \mid S \in L^\infty(\mathscr{A}, \varphi), YS \in L^1(\mathscr{A}, \varphi)\ \|S\|_\infty \leqslant 1\}$$
$$\geqslant |\varphi(Y)|.$$

Hence $|\varphi(AX)| \leqslant q_1(A)\|X\|_1$ $(X \in L^1(\mathscr{A}, \varphi))$. Hence by the noncommutative Radon–Nikodym Theorem, $A \in L^\infty(\mathscr{A}, \varphi) = \mathscr{A}$.

(b) Suppose that there exists a *subalgebra Q of $L^\omega(\mathscr{A}, \varphi)$ admitting a b^*-topology t and containing $L^\infty(\mathscr{A}, \varphi)$ such that $(L^\infty(\mathscr{A}, \varphi), \|\cdot\|_\infty) \to (Q, t) \to (L^\omega(\mathscr{A}, \varphi), \tau^\omega)$ are continuous injections. Now φ being a continuous positive linear form on $L^\omega(\mathscr{A}, \varphi)$, $\psi = \varphi|_Q$ is t-continuous. Let (π_ψ, H_ψ) be the cyclic GNS representation of Q on a Hilbert space H_ψ associated with ψ. By [11, Theorem 6.1], ψ is admissible i.e.

$$\sup\left\{\frac{\psi(Y^*X^*XY)}{\psi(Y^*Y)} \mid Y \neq 0, Y \in Q\right\} < \infty;$$

π_ψ maps elements of Q_P as bounded operators on H_ψ with

$$\|\pi_\psi(X)\|^2 = \sup\left\{\frac{\psi(Y^*X^*XY)}{\psi(Y^*Y)} \mid Y \neq 0, Y \in Q\right\}$$

and $\pi_\psi: (Q, t) \to (\mathscr{B}(H_\psi), \|\cdot\|)$ is continuous. Let 1_ψ be the unit cyclic vector in H_ψ for π_ψ. Then $\psi(X) = (\pi_\psi(X)1_\psi, 1_\psi)$ $(X \in Q)$. Since φ is faithful, 1_ψ is separating for π_ψ; hence π_ψ is faithful. This forces $Q = \mathscr{A}$.

3.5 *Noncommutative Hardy–Arens algebra*

Let $\mathscr{A}$ be a finite von Neumann algebra acting on a Hilbert space H. Let $\{\alpha_t \mid t \in \mathbf{R}\}$ be a flow on $\mathscr{A}$ i.e. a σ-weakly continuous one-parameter group of *automorphisms on $\mathscr{A}$. Let φ be a faithful, α_t-invariant, normal tracial state on $\mathscr{A}$. By [27, Proposition 2.2], for each p, $1 \leqslant p < \infty$, $\{\alpha_t \mid t \in \mathbf{R}\}$ extends uniquely to a strongly continuous representation of $\mathbf{R}$ on isometries of $L^p(\mathscr{A}, \varphi)$, to be denoted by $\{\alpha_t \mid t \in \mathbf{R}\}$. This defines a representation of $L^1(\mathbf{R})$ on the Banach space $L^p(\mathscr{A}, \varphi)$ as $\alpha(f)T = \int_{-\infty}^{\infty} f(t)\alpha_t(T)\,dt$ $(T \in L^p(\mathscr{A}, \varphi),\ f \in L^1(\mathbf{R}))$. For $f \in L^1(\mathbf{R})$, let $Z(f) = \{t \in \mathbf{R} \mid \hat{f}(t) = 0\}$, $\hat{f}$ being the Fourier transform of f. Let $Sp_\alpha(T) = \cap\{Z(f) \mid f \in L^1(\mathbf{R}), \alpha(f)T = 0\}$. The associated noncommutative Hardy spaces are defined as: For $1 \leqslant p \leqslant \infty$, $H^p(\alpha) = \{T \in L^p(\mathscr{A}, \varphi) \mid Sp_\alpha(T) \subset [0, \infty)\}$. Then $H^\infty(\alpha)$ is a nonself-adjoint closed subalgebra of $\mathscr{A}$.

Let $H^\omega(\alpha) = \bigcap_{1 \leqslant p < \infty} H^p(\alpha)$, a complete locally convex algebra with relative topology induced by τ^ω on $L^\omega(\mathscr{A}, \varphi)$. Taking $P = \{\|\cdot\|_p \mid p \in \mathbf{N}\}$, we have $Q_P = B_P = H^\infty(\alpha)$.

3.6 *Maximal-unbounded Hilbert algebras* [21]

Let $\mathscr{D}$ be an inner product space that is a *algebra. Assume the following:

(a) $\langle \xi, \eta \rangle = \langle \eta^*, \xi^* \rangle$,

(b) $\langle \xi\eta, \nu \rangle = \langle \eta, \xi^*\nu \rangle$,

(c) Let π and π' on $\mathscr{D}$ be defined as $\pi(\xi)\eta = \pi'(\eta)\xi = \xi\eta$. Let H be the completion of $\mathscr{D}$. Let $\mathscr{D}_0 = \{\xi \in \mathscr{D} \mid \pi(\xi) \text{ is bounded}\}$. Then $\mathscr{D}_0^2$ is assumed to be dense in $\mathscr{D}$. Such a $\mathscr{D}$ is called an *unbounded Hilbert algebra.*

For each $x \in H$, let $\pi_0(x)\xi = \overline{\pi_0'(\xi)}x$, $\pi_0'(x)\xi = \overline{\pi_0(\xi)}x$ $(\xi \in \mathscr{D}_0)$. Here on $\mathscr{D}_0$, $\pi_0 = \pi|_{\mathscr{D}_0}, \pi_0' = \pi'|_{\mathscr{D}_0}$. Let $(\mathscr{D}_0)_b = \{x \in H \mid \pi_0(x) \in \mathscr{B}(H)\}$, a Hilbert algebra. Let $\mathscr{U}_0(\mathscr{D}_0)$ be the left von Neumann algebra of $\mathscr{D}_0$ and let φ be the natural semifinite trace on $\mathscr{U}_0(\mathscr{D}_0)^+$. Let $\mathscr{m}(\varphi)$ (respectively $\mathscr{m}(\mathscr{U})$) denote the extended W^*-algebra of operators in H that are φ-restrictedly measurable (respectively measurable with respect to $\mathscr{U}_0(\mathscr{D}_0)$). For $T \in \mathscr{m}(\varphi)$, let $|T| = (T^*T)^{1/2}$, $\mu(T) = \sup\{\varphi(\overline{\pi_0(\xi)}) \mid 0 \leqslant \pi_0(\xi) \leqslant T, \xi \in (\mathscr{D}_0)_b^2\}$. For $1 \leqslant p < \infty$, let $L^p(\varphi) = \{T \in \mathscr{m}(\mathscr{U}_0) \mid \|T\|_p = \mu(|T|^p)^{1/p} < \infty\}$. For $p \geqslant 2$, $L_2^p(\mathscr{D}_0) = \{x \in H \mid \overline{\pi_0(x)} \in L^p(\varphi)\}$ with $\|x\|_p = \|\pi_0(x)\|_p$. Let $L_2^\omega(\varphi) = \bigcap_{2 \leqslant p < \infty} L^p(\varphi)$, $L_2^\omega(\mathscr{D}_0) = \{x \in H \mid \overline{\pi_0(x)} \in L_2^\omega(\varphi)\} = \bigcap_{2 \leqslant p < \infty} L_2^p(\mathscr{D}_0) = \bigcap_{n=2}^{\infty} L_2^n(\mathscr{D}_0)$. Let $\|x\|_\infty = \|\pi_0(x)\|_\infty$ for x in $L_2^\infty(\mathscr{D}_0) = (\mathscr{D}_0)_b$.

The topology τ_2^ω is defined on $L_2^\omega(\mathscr{D}_0)$ by $\{\|\cdot\|_p \mid 2 \leqslant P < \infty\}$, or equivalently by $P = \{\|\cdot\|_n \mid n = 2, 3, \ldots\}$–making $L_2^\omega(\mathscr{D}_0)$ a complete metrizable locally convex *algebra which is a GB^*-algebra with $((\mathscr{D}_0)_b, \|\cdot\|_\infty)$ as the underlying C^*-algebra. The algebra $L_2^\omega(\mathscr{D}_0)$ is a maximal unbounded Hilbert algebra containing $\mathscr{D}_0$. (We assume that $\mathscr{D}_0$ contains identity 1.)

Assertions: (a) For $L_2^\omega(\mathscr{D}_0)$ and P as above, $Q_P = B_P = (\mathscr{D}_0)_b$.
(b) For $L_2^\omega(\varphi)$ with analogous P, $Q_P = B_P = \mathscr{U}_0(\mathscr{D}_0)$.

Proof. We supply details of (a). As in example (3.4), it is sufficient to show that $Q_P \subset B_P = (\mathscr{D}_0)_b$. Let $a \in (\mathscr{D}_0)_b$. Then $\overline{\pi_0(a)} \in \mathscr{B}(H)$; and for each $x \in L_2^\omega(\mathscr{D}_0)$, $\|ax\|_p = \|\overline{\pi_0(ax)}\|_p \leqslant \|\overline{\pi_0(a)}\| \, \|\overline{\pi(x)}\|_p = \|a\|_\infty \|x\|_p$ giving $a \in B_p$, $(\mathscr{D}_0)_b \subset B_p$, $p(a) \leqslant \|a\|_\infty$.

Let $a \in Q_P$. Then for each p, $2 \leqslant p < \infty$, $\|ax\|_p \leqslant a_p(a) \|x\|_p (x \in L_2^\omega(\mathscr{D}_0))$. Taking $p = 2$ and using the fact that $L_2^2(\mathscr{D}_0)$ is identified with H, it follows that $\overline{\pi_0(a)}$ is a norm bounded operator on $L_2^\omega(\mathscr{D}_0)$. Let π_2^ω be the left regular representation of $L_2^\omega(\mathscr{D}_0)$. Then $\pi_2^\omega(a)$ is a bounded operator. But by [21 (II), p. 422], the algebra $\mathscr{U}(L_2^\omega(\mathscr{D}_0))$ generated by $\pi_2^\omega(L_2^\omega(\mathscr{D}_0))$ and $\mathscr{U}_0(\mathscr{D}_0)|_{L_2^\omega(\mathscr{D}_0)}$ is an EW^*-algebra over $\mathscr{U}_0(\mathscr{D}_0)$. Hence $\overline{\pi(a)} \in \mathscr{U}_0(\mathscr{D}_0)$. Hence $a \in (\mathscr{D}_0)_b$. Thus $Q_P \subset (\mathscr{D}_0)_b$, $Q_P = B_P = (\mathscr{D}_0)_b$. The topologies also agree.

Remark. It would be interesting to investigate a noncommutative analogoue of (3.2).

3.7 *L^ω- and L^ω_{loc}-integrals of a field of C^*-algebras*

Below we briefly describe a construction of auxilliary nature which may be of some independent interest.

(A) Let (X, Σ, μ) be a nonatomic finite measure space. Let $t \to H(t)$ be a measurable field of separable Hilbert spaces with $H = \int_X^\oplus H(t) \mathrm{d}\mu(t)$. Let $t \to \mathscr{A}(t)$ be a measurable field of C^*-algebra with $\mathscr{A}(t)$ acting on $H(t)$. Let $\mathscr{A} = \int^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t)$ realized as a C^*-algebra on H. For each p, $1 \leqslant p < \infty$, define the 'L^p-integral' of $t \to \mathscr{A}(t)$ as

$L^p - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t) =$ collection of all measurable operator $T: t \to T(t) \in \mathscr{A}(t)$ such that

$$\|T\|_p = \left(\int_X \|T(t)\|^p \mathrm{d}\mu(t) \right)^{1/p} < \infty.$$

It is a Banach space. For $1 \leqslant r < p < \infty$,

$$L^r - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t) \supset L^p - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t) \supset \mathscr{A} = L^\infty - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t).$$

Let

$$L^\omega - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t) = \bigcap_{1 \leqslant p < \infty} L^p - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t).$$

With usual operations on operator fields, it is a *algebra with 1 which is a complete metrizable locally convex GB^*-algebra under the topology τ^ω defined by $P = \{\|\cdot\|_p \,|\, 1 \leqslant p < \infty\}$, the unit ball being $B_0 = \{T: t \to T(t) \text{ in } \mathscr{A} \,|\, \|T\| = \operatorname{ess\,sup} \|T(t)\| \leqslant 1\}$. We conjecture that $Q_P = B_P = \mathscr{A}$.

(B) Let X be a σ-finite measure space as in (3.2). Define

$L^p_{\text{loc}} - \int_X^\oplus \mathscr{A}(t)\, \mathrm{d}\mu(t) =$ collection of all measurable operator fields $T: t \to T(t) \in \mathscr{A}(t)$ such that for each $n = 1, 2, 3, \ldots$

$$\|T\|_{p,n} = \left(\int_{X_n} \|T(t)\|^p \mathrm{d}\mu(t) \right)^{1/p} < \infty.$$

It is a Frechet space with the topology τ^p_{loc} defined by $P_p = \{\|\cdot\|_{p,n} | n = 1, 2, \ldots\}$. Note that

$$L^\infty_{\mathrm{loc}} - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = \text{collection of all measurable operator fields } T: t \to T(t) \in \mathscr{A}(t)$$
$$\text{such that for } n = 1, 2, \ldots \ \|T\|_{\infty,n} = \operatorname*{ess\,sup}_{X_n} \|T(t)\| < \infty.$$

It is a b^*-algebra with a b^*-calibration $P_\infty = \{\|\cdot\|_{\infty,n} | n = 1, 2, \ldots\}$. Now we can formulate, as in (3.2),

$$L^\infty_{\mathrm{loc}} - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = \bigcap_{1 \leqslant p < \infty} L^p_{\mathrm{loc}} - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t) = A \text{ (say)}.$$

We conjecture that A is a GB^*-algebra, P being a GB^*-calibration and $B_P = \mathscr{A}$, $Q_P = L^\infty_{\mathrm{loc}} - \int_X^\oplus \mathscr{A}(t)\,\mathrm{d}\mu(t)$.

Remark. To the best of our knowledge, the above L^p-integrals of operator fields do not seem to have been investigated in the literature. We do not know whether their elements can be regarded as concrete operators in H.

3.8 *A convolution algebra of periodic distributions*

Let $T = \{z \in ¢ |\ |z| = 1\}$ be the unit circle. Let $D(T)$ be the cummutative convolution *algebra with identity of all distributions on T, the identity being the Dirac delta δ and the involution being $u \to u^*$ defined as $\langle u^*, \phi \rangle = \overline{\langle u, \phi^* \rangle}$, $\phi^*(z) = \overline{\phi(\bar{z})}$ ($\phi \in C^\infty(T)$, $z \in T$). Let $u \to \hat{u}$, $\hat{u}(n) = \langle u, \exp(-\mathrm{i}nt) \rangle$ ($n \in \mathscr{Z}$, the integers) be the Fourier–Schwartz transform that maps $D(T)$ *isomorphically onto the *algebra $A = \{(a_n)_{-\infty}^{\infty} | a_n \in ¢$, $a_n = 0(|n|^m)$ for some m, depending in general on $(a_n)\}$ with pointwise operations and the complex conjugation as the involution. Under this map, the *subalgebra PM(T) (pseudomeasures on T) of $D(T)$ is mapped onto $1^\infty(\mathscr{Z})$, the algebra of bounded sequences on $\mathscr{Z}$. The algebra PM(T) with norm $\|u\| = \|\hat{u}\|_\infty$ is a W^*-algebra.

3.8a. *Proposition.*

The algebra $D(T)$ with the weak topology $\sigma = \sigma(D(T), C^\infty(T))$ is a sequentially complete locally convex GB^*-algebra with unit ball $B_0 = \{u \in \mathrm{PM}(T) |\ \|u\| \leqslant 1\}$.

Proof. Using [15, § 12.6], it is a simple matter to verify that $(D(T), \sigma)$ is a locally convex algebra with continuous involution and jointly continuous multiplication.

Claim. PM$(T) = (D(T), \sigma)_0$, the set of (Allan) bounded elements of $(D(T), \sigma)$ [1, § 2]. Let $A(T)$ be the Wiener algebra viz the algebra of continuous functions on T having absolutely convergent Fourier series. Given $u \in \mathrm{PM}(T)$, $\|u\| \leqslant \lambda$; the set $S = \{(\lambda^{-1}u)^n | n = 1, 2, \ldots\}$ is σ (PM(T), $A(T)$) bounded and so is σ-bounded. (Note that PM(T) is the dual of $A(T)$). Conversely, if $u \in D(T)$ is a bounded element of $(D(T), \sigma)$, then there is a $\lambda \neq 0$ in ¢ such that for each $\phi \in C^\infty(T)$, there is a constant M_ϕ such that $|\langle (\lambda^{-1}u)^n, \phi \rangle| \leqslant M_\phi$ for all $n = 1, 2, \ldots$ Taking $\exp(-ikt)$ (for some $k \in \mathscr{Z}$) as ϕ, it follows that $|[\hat{u}(k)]/\lambda| \leqslant M_k^{1/n}$ for all n. Thus $|\hat{u}(k)| \leqslant \lambda$ for all $k \in \mathscr{Z}$, $\hat{u} \in l^\infty(\mathscr{Z})$ and so $u \in \mathrm{PM}(T)$. Thus $\mathrm{PM}(T) = (D(T), \sigma)_0$.

Now it is easily seen that $D(T)$ is symmetric and $B_0 = \{u \in \mathrm{PM}(T) \mid \|u\| \leqslant 1\}$ is the greatest member of $\mathscr{B}^*$. The sequential completeness of σ follows from [26, Theorem 2.8]. This completes the proof.

Note that $(D(T), \sigma)$ is not complete in view of [30, Theorem 32.2]. The algebra $D(T)$ admits two other important topologies.

(a) The topology σ' defined by the seminorms $p_n(u) = |\hat{u}(n)|$ for $n \in \mathscr{Z}$. Then $(D(T), \sigma')$ is a metrizable LMC*-algebra which is also a GB^*-algebra with unit ball B_0.

(b) The strong topology $\beta = \beta(D(T), C^\infty(T))$ of uniform convergence on all bounded subsets of the Frechet space $C^\infty(T)$.

The importance of β is revealed by the following. Note that the positive elements of $D(T)$ are

$$D(T)^+ = \{u \in D(T) \mid \langle u, \phi * \phi^* \rangle \geqslant 0 \text{ for all } \phi \in C^\infty(T)\}.$$

3.8b *Proposition*

(i) The algebra $(D(T), \beta)$ is a complete locally convex GB^*-algebra with unit ball $B_0 = \{u \in \mathrm{PM}(T) \mid \|u\| \leqslant 1\}$.
(ii) Every positive linear functional f on $D(T)$ is β-continuous, and is of the form $f(u) = \langle u, \phi^* * \phi \rangle$ for some $\phi \in C^\infty(T)$.
(iii) The strong topology β

$= T$ the largest locally convex GB^*-topology
$= \tau\langle D(T), D(T)^P \rangle$, the Mackey topology of the duality $\langle D(T), D(T)^P \rangle$, $D(T)^P$ being the linear span of positive functionals on $D(T)$

and

$\sigma = \sigma(D(T), D(T)^P)$, the positive functional topology as discussed in [14, §8] and [2, § 5].

(iv) The positive cone in $D(T)$ is β-normal, and $\beta = \pi(T)$, the topology that $D(T)$ admits via its universal representation as an OP^*-algebra with uniform topology [8].

Proof. Since $C^\infty(T)$ is Frechet (with its usual topology), the completeness of $(D(T), \beta)$ follows from [30, Corollary 2 to Theorem 32.2]. The rest of (i) is a routine. Further, $(D(T), \beta)$ being the strong dual of a reflexive Frechet space, is bornological [29, Ch. IV, §6.6 Corollary 1], hence barrelled [29, Ch. IV §6.6]. By [14, Corollary 8.2], every positive functional on $D(T)$ is β-continuous. This gives (ii). Since $(D(T), \beta)' = (D(T))^P \simeq C^\infty(T)$ due to reflexivity of $C^\infty(T)$, $\beta = \tau(D(T), D(T)^P) =$ Dixon topology, the last equality being a consequence of the fact that in a commutative GB^*-algebra A, Dixon topology $= \tau(A, A^P)$. Finally $D(T)$ being a vector lattice, and $(D(T), \beta)' = D(T)^P$, the positive cone is β-normal by [29, Ch. V §6.4]. Since $\pi(T)$ is the largest locally convex GB^*-topology with normal positive cone, $\beta = \pi(T)$. This completes the proof.

Since $(\mathrm{PM}(T), \|\cdot\|)$ is a W^*-algebra with predual $A(T)$ and the identity map id:

$(\mathrm{PM}(T), \sigma(\mathrm{PM}(T), A(T))) \to (D(T), \sigma)$ is continuous, the fact that β and σ have the same dual, and hence they determine same sets of closed ideals, gives the following by applying [5, Corollary 3].

3.8c *Proposition*

(a) A subset I of $D(T)$ is a β-closed ideal of $D(T)$ if and only if $I = I_K$ for some non-empty subset K of $\mathscr{Z}$ where

$$I_K = \{u \in D(T) \mid \hat{u}(n) = 0 \text{ for all } n \in K\}$$

(b) A linear functional f on $D(T)$ is multiplicative if and only if $f = f_n$ for some $n \in \mathscr{Z}$ where $f_n(u) = \hat{u}(n)$.

Now let

$$P_\sigma = \{p_f \mid f \in C^\infty(T)\},\ p_f(u) = |\langle u, f\rangle|,$$

$$P_\beta = \{p_B \mid B \text{ is a } \sigma\text{-bounded subset of } C^\infty(T)\},$$

$$p_B(u) = \sup\{|\langle u, f\rangle| \mid f \in B\},$$

$$P_{\sigma'} = \{p_n \mid n \in \mathscr{Z}\},\ p_n(u) = |\hat{u}(n)|.$$

These are the natural calibrations for σ, β and σ' respectively.

Assertions: For the algebra $D(T)$,

(i) $B_{P_\sigma} = Q_{P_\sigma} = ¢\delta$

(ii) $B_{P_\beta} = Q_{P_\beta} = ¢\delta$

(iii) $B_{P_{\sigma'}} = \mathrm{PM}(T)$, $Q_{P_{\sigma'}} = D(T)$.

Proof. (i) Let $u \in Q_P$. Then for each $f \in C^\infty(T)$, there is a constant $M_{u,f}$ such that

$$|\langle u * v, f\rangle| \leqslant M_{u,f} |\langle v, f\rangle| \quad (v \in D(T)). \tag{1}$$

Since every distribution on T is of finite order, we can assume that the order of $u \leqslant m$. By [15, § 12.5.7], there is a continuous function ϕ on T such that $u = D^{m+2}\phi + \hat{u}(0)\delta$ where $D^{m+2}\phi$ is the $(n+2)$th distributional derivative of ϕ. Since $u \in Q_{P_\sigma}$, $\delta \in Q_{P_\sigma}$, it follows that $D^{m+2}\phi \in Q_P$. Applying (1) to D^{m+2} with $v = \delta$, we get $|\langle D^{m+2}\phi, f\rangle| \leqslant M_{\phi,f}|f(1)|$ for all $f \in C^\infty(T)$; i.e. $|\int \phi(t) f^{(m+2)}(t)\,dt| \leqslant M_{\phi,f}|f(1)|$ $(f \in C^\infty(T))$. Taking $f(t) = \exp(\mathrm{int}) - 1$ $(n \in \mathscr{Z})$, it follows that $\phi(n) = \int \phi(t) \exp(-\mathrm{int})\,dt = 0$ for all $n \in \mathscr{Z}$. Hence $\phi = 0$ a.e., and by continuity, $\phi(t) = 0$ for all t. Thus $D^{m+2}\phi = 0$, $u = \hat{u}(0)\delta$. Thus $Q_{P_\sigma} = B_{P_\sigma} = ¢\delta$. (That $B_{P_\sigma} = ¢\delta$ can also alternatively be seen as: for $u \in B_{P_\sigma}$, (1) holds with a constant M_u depending only on u. Since $C^\infty(T)$ is dense in $C(T)$, it follows that $u * v$ is a measure if v is a measure. This forces u to be a measure [15, §12.8.4]. Also, as above, taking $v = \delta$ in (1), $|\int f\,du| \leqslant M_u|f(1)|$ for all $f \in C^\infty(T)$, and so for all $f \in C(T)$. Hence $u = \lambda\delta$ for some $\lambda \in ¢$).

(ii) is immediate from (i) in view of $B_{P_\beta} \subset B_{P_\sigma}$, $Q_{P_\beta} \subset Q_{P_\sigma}$. This is because $P_\sigma \subset P_\beta$.

(iii) is a consequence of the submultiplicativity of seminorms in $P_{\sigma'}$.

Remark. More generally, one may look to the structure of distributions on a compact Lie group.

3.9 *Operators on the product of Hilbert spaces*

Let $\{H_\alpha | \alpha \in \Delta\}$ be a family of Hilbert spaces. Let $X = \prod_\alpha X_\alpha$, a locally convex space with calibration $\Gamma = \{p_\alpha | \alpha \in \Delta\}$, $p_\alpha(x) = \|x_\alpha\|$ where $x = (x_\alpha) \in X$. Let $H = \Sigma^{\oplus} H_\alpha = \{x = (x_\alpha) \in X | \Sigma \|x_\alpha\|^2 < \infty\}$ with inner product $\langle x, y \rangle = \Sigma \langle x_\alpha, y_\alpha \rangle$ for $x = (x_\alpha)$, $y = (y_\alpha)$ in H. Let $D = \{x = (x_\alpha) \in X) x_\alpha = 0$ for all but a finite number of $\alpha\}$. The inclusion $H \to X$ is continuous and D is a dense subspace of H and X.

Let $Q(X, \Gamma)$ and $B(X, \Gamma)$ be the algebras consisting of respectively the quotient bounded operators and the universally bounded operators on X. As shown in [9], $(Q(X, \Gamma), t_\Gamma)$ is a b^*-algebra and $(B(X, \Gamma), p_\Gamma)$ is a C^*-algebra. (Recall: The topology t_Γ is determined by the b^*-calibration $\{q_\alpha\}$, $q_\alpha(T) = \sup_\alpha \{p_\alpha(Tx) | p_\alpha(x) \leqslant 1\}$ and $p_\Gamma(T) = \sup_\alpha q_\alpha(T)$). Note $X/\ker p_\alpha$ is identified with H_α with the result, given $T \in Q(X, \Gamma)$, $T^\alpha x_\alpha = (Tx)_\alpha$ is a well-defined bounded linear operator on H_α, $q_\alpha(T) = \|T^\alpha\|$ and the involution on $Q(X, \Gamma)$ is defined by $(T^*x)_\alpha = (T^\alpha)^* x_\alpha$ for $x = (x_\alpha)$ in X.

Now let $L(X)$ be the algebra of all continuous linear operators on (X, Γ). Let $\mathscr{G}$ be either on the following

$\mathscr{G}_1$ = all finite subsets of X,
$\mathscr{G}_2$ = all compact convex subsets of X,
$\mathscr{G}_3$ = all compact subsets of X,
$\mathscr{G}_4$ = all bounded subsets of X.

Let $\mathscr{u}$ be a o-neighbourhood base for X. On $L(X)$, the topology $\tau_{\mathscr{G}}$ of uniform convergence on members of $\mathscr{G}$ is defined by taking a o-neighbourhood base consisting of sets of the form $U(B, V) = \{T \in L(X) | T(B) \subset V\}$ $(B \in \mathscr{G}, V \in \mathscr{u})$. It is given by the calibration $P_{\Gamma,\mathscr{G}} = \{p_{\alpha,B} | \alpha \in \Delta, B \in \mathscr{G}\}$ where $p_{\alpha,B}(T) = \sup\{p_\alpha(Tx) | x \in B\}$. The sets of quotient-bounded and universally-bounded elements of the calibrated locally convex algebra $(L(X), P_{\Gamma,\mathscr{G}})$ are denoted by $Q_{P_{\Gamma,\mathscr{G}}}$ and $B_{P_{\Gamma,\mathscr{G}}}$ respectively. The natural LMC topology $t_{P_{\Gamma,\mathscr{G}}}$ on $Q_{P_{\Gamma,\mathscr{G}}}$ is determined by the seminorms $q_{\alpha,B}(T) = \sup\{p_{\alpha,B}(TS) | p_{\alpha,B}(S) \leqslant 1\}$ and the norm on $B_{P_{\Gamma,\mathscr{G}}}$ is $p(T) \equiv p_{P_{\Gamma,\mathscr{G}}}(T) = \sup_{\alpha,B} q_{\alpha,B}(T)$. Then as in [9, Example 4.1], $Q_{P_{\Gamma,\mathscr{G}}}$ is topologically *isomorphic to $(Q(X, \Gamma), t_\Gamma)$ and $B_{P_{\Gamma,\mathscr{G}}}$ is isometrically *isomorphic to $(B(X, \Gamma), p_\Gamma)$ which in turn is isometrically *isomorphic to a C^*-subalgebra of $\mathscr{B}(H)$.

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