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Topological *-algebras with C^*-enveloping algebras II

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Abstract. Universal C^*-algebras $C^*(A)$ exist for certain topological *-algebras called algebras with a C^*-enveloping algebra. A Frechet *-algebra A has a C^*-enveloping algebra if and only if every operator representation of A maps A into bounded operators. This is proved by showing that every unbounded operator representation π, continuous in the uniform topology, of a topological *-algebra A, which is an inverse limit of Banach *-algebras, is a direct sum of bounded operator representations, thereby factoring through the enveloping pro-C^*-algebra $E(A)$ of A. Given a C^*-dynamical system (G, A, α), any topological *-algebra B containing $C_c(G, A)$ as a dense *-subalgebra and contained in the crossed product C^*-algebra $C^*(G, A, \alpha)$ satisfies $E(B) = C^*(G, A, \alpha)$. If $G = \mathbb{R}$, if B is an α-invariant dense Frechet *-subalgebra of A such that $E(B) = A$, and if the action α on B is m-tempered, smooth and by continuous *-automorphisms: then the smooth Schwartz crossed product $S(\mathbb{R}, B, \alpha)$ satisfies $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$. When G is a Lie group, the C^∞-elements $C^\infty(A)$, the analytic elements $C^\omega(A)$ as well as the entire analytic elements $C^{e\omega}(A)$ carry natural topologies making them algebras with a C^*-enveloping algebra. Given a non-unital C^*-algebra A, an inductive system of ideals I_α is constructed satisfying $A = C^*\text{-ind}\lim I_\alpha$; and the locally convex inductive limit $\text{ind}\lim I_\alpha$ is an m-convex algebra with the C^*-enveloping algebra A and containing the Pedersen ideal K_A of A. Given generators G with weakly Banach admissible relations R, we construct universal topological *-algebra $A(G, R)$ and show that it has a C^*-enveloping algebra if and only if (G, R) is C^*-admissible.

Keywords. Frechet *-algebra; topological *-algebra; C^*-enveloping algebra; unbounded operator representation; O^*-algebra; smooth Frechet algebra crossed product; Pedersen ideal of a C^*-algebra; groupoid C^*-algebra; universal algebra on generators with relations.

1. Statements of the results

In [5], a functor E has been considered that associates C^*-algebras $E(A)$ with certain topological *-algebras A, called algebras with a C^*-enveloping algebra. By a classic construction due to Gelfand and Naimark, a Banach *-algebra A admits a C^*-enveloping algebra $C^*(A) = E(A)$ ([14], 2.7, p. 47). By ([15], Theorem 2.1), a complete locally m-convex *-algebra has a C^*-enveloping algebra if and only if it admits a greatest continuous C^*-seminorm. The following extrinsic characterization of such algebras has been motivated by the simple observation that any *-homomorphism from a Banach *-algebra into the *-algebra of linear operators on an inner product space maps the algebra into bounded operators.

Theorem 1.1. *Let A be a Frechet *-algebra. Then A is an algebra with a C^*-enveloping algebra if and only if every *-representation of A is a bounded operator representation.*

The above theorem is false without the assumption that A is metrizable (see Remark 4.4). By a *-*representation* $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A [37] is meant a homomorphism π from A into linear operators (not necessarily bounded) all defined on a common dense invariant subspace $\mathcal{D}(\pi)$ of a Hilbert space H such that for all x in A, $\pi(x^*) \subset \pi(x)^*$. In the general theory of *-algebras, following Palmer [24], A is called a BG^*-*algebra* if every *-homomorphism from A into linear operators on a pre-Hilbert space maps A into bounded operators. The absence of a complete algebra norm on a non-Banach *-algebra A indicates that A may contain elements that fail to be bounded in any natural sense. Hence an appropriate framework for the representation theory of A is that of unbounded operator representations. However, this natural point of view was developed rather late, following [30, 20]. Prior to (and later, in spite of) this, bounded operator representations of A have been investigated in detail, especially when A is a locally m-convex *-algebra, i.e., $A = \text{proj}\ \lim A_\alpha$, the inverse limit (also called the projective limit) of Banach *-algebras [9, 15], (see [16] for a summary of bounded operator representations of A). In fact, such an A, when *-semisimple, admits sufficiently many continuous irreducible bounded operator representations [9]. Then the enveloping pro-C^*-algebra (projective limit of C^*-algebras) $E(A)$ of A, discussed in [10], [19] and [15], turns out to be $E(A) = \text{proj}\ \lim\ E(A_\alpha)$, $E(A_\alpha) = C^*(A_\alpha)$ being the enveloping C^*-algebra of the Banach *-algebra A_α ([15], Theorem 4.3). Thus A has a C^*-*enveloping algebra* if $E(A)$ is a C^*-algebra. By the construction, $E(A)$ is universal for norm-continuous bounded operator representations of A. Theorem 1.2, to be used to prove Theorem 1.1, shows desirably that $E(A)$ is also universal for representations into unbounded operators. The uniform topology ([37], p. 77, 78) on an unbounded operator algebra is defined at the end of this section.

Theorem 1.2. *Let A be complete locally m-convex *-algebra. Let $(\pi, D(\pi), H)$ be a closed *-representation of A continuous in the uniform topology on $\pi(A)$. Then there exists a unique *-representation $(\sigma, D(\sigma), H_\sigma)$ of $E(A)$ such that the following hold.*

(1) *$H_\sigma = H$ and $D(\sigma) = D(\pi)$.*
(2) *As a representation of $E(A)$, σ is closed and continuous in the uniform topology on $\sigma(E(A))$.*
(3) *σ is an 'extension' of π to $E(A)$ in the sense that for all x in A, $(\sigma \circ j)(x) = \pi(x)$, $j : A \to E(A)$ being the natural map, $j(x) = x + \text{srad}\,(A)$, $\text{srad}\,(A)$ denoting the star radical of A.*
(4) *On the unbounded operator algebra $\pi(A)$, the uniform topology $\tau_D^{\pi(A)}$ is a (not necessarily complete) pro-C^*-topology which coincides with the relative uniform topology $\tau_D^{\sigma(E(A))}$ from $\sigma(E(A))$.*

COROLLARY 1.3

Let π be a closed irreducible *-representation of a complete locally m-convex *-algebra A continuous in the uniform topology on $\pi(A)$. (In particular, let A be Frechet and π be irreducible). Then π maps A into bounded operators.

AO^*-algebras (abstract O^*-algebras) [36, 37] provide the unbounded operator algebra analogues of C^*-algebras. Starting with a topological (not necessarily m-convex)

*-algebra A, one can construct an enveloping AO^*-algebra $O(A)$ universal for *-representations continuous in the uniform topology, and declare A to have a C^*-*enveloping algebra* if the uniform topology on $O(A)$ is normable. On the other hand, by modifying the construction in [15], the pro-C^*-algebra $E(A)$ can also be considered as the universal object for norm-continuous bounded operator *-representations of more general locally convex, non-m-convex, *-algebras A. In general, the completion of $O(A)$ differs from $E(A)$. For a barrelled A, $O(A)$ is normable implies that $E(A)$ is a C^*-algebra, but the converse does not hold. In the present context, the following shows that both the approaches are consistent in the metrizable case.

Theorem 1.4. *Let A be a Frechet *-algebra. Then the pro-C^*-algebra $E(A)$ is the completion of the AO^*-algebra $O(A)$. Thus $O(A)$ is normable if and only if A is an algebra with a C^*-enveloping algebra.*

There are several situations in C^*-algebra theory in which topological *-algebras arise naturally [27]. Enveloping C^*-algebras provide a standard method of constructing C^*-algebras; and frequently, lurking behind such a construction is a topological *-algebra B such that $E(B) = A$. Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. The crossed product C^*-algebra $C^*(G, A, \alpha)$ is the enveloping C^*-algebra of the L^1-crossed product Banach *-algebra $L^1(G, A, \alpha)$. If B is a topological *-algebra such that $C_c(G, A) \subseteq B \subseteq C^*(G, A, \alpha)$ and $C_c(G, A)$ is dense in B, then $E(B) = C^*(G, A, \alpha)$. Let G be a Lie group. Then the *-subalgebra $C^\infty(A)$ of C^∞-elements of A is a Frechet *-algebra with an appropriate topology such that $E(C^\infty(A)) = A$. The *-algebras $C^\omega(A)$ and $C^{e\omega}(A)$ consisting of analytic elements and entire elements of A are shown to carry natural topologies making them algebras with C^*-enveloping algebras. We also consider the smooth crossed product [29, 34]. For simplicity, we take $G = \mathbb{R}$, and prove the following.

Theorem 1.5. *Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. Suppose that B is a dense Frechet *-subalgebra of A satisfying the following.*

(a) *A has a bounded approximate identity contained in B and which is a bounded approximate identity for B.*
(b) *$E(B) = A$.*
(c) *B is α-invariant; and the action α of $\mathbb{R}$ on B is smooth, m-tempered and by continuous *-automorphisms of B.*

Then the smooth Schwartz crossed product $S(\mathbb{R}, B, \alpha)$ is a Frechet *-algebra with a C^*-enveloping algebra, and $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$. Further, if the action of $\mathbb{R}$ on B is isometric (*see* § 5)*, then the L^1-crossed product $L^1(\mathbb{R}, B, \alpha)$ is also a Frechet *-algebra with a C^*-enveloping algebra, and $E(L^1(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$.*

It follows that $E(S(\mathbb{R}, C^\infty(A), \alpha) = C^*(\mathbb{R}, A, \alpha)$. In particular, if α is a smooth action of $\mathbb{R}$ on a C^∞-manifold M, then $E(S(\mathbb{R}, C^\infty(M), \alpha) = C^*(\mathbb{R}, C(M), \alpha)$, the covariance C^*-algebra of the $\mathbb{R}$-space M.

For a locally compact Hausdorff space X, let $\mathcal{K}$ be the directed set consisting of all compact subsets of X. For $K \in \mathcal{K}$, let $C_K(X) = \{f \in C_c(X) : \operatorname{supp} f \subseteq K\}$, $C_c(X)$ denoting the compactly supported continuous functions on X. It is well known that $\{C_K(X) : K \in \mathcal{K}\}$ forms an inductive system; and $C_0(X) = C^*\text{-ind}\lim C_K(X)$ (C^*-inductive limit), $C_c(X) = \text{ind}\lim C_K(X)$ (locally convex inductive limit). Further, $C_c(X)$ with the locally convex inductive limit topology is a complete locally m-convex Q-algebra and $E(C_c(X)) = C_0(X)$. The following provides a non-commutative analogue of this. We refer to the last paragrapgh in this section for the relevant definitions pertaining to topological algebras.

Theorem 1.6. *Let A be a non-unital C^*-algebra. Let K_A denote its Pedersen ideal. For $a \in K_A^+$, let I_a denote the closed two sided ideal of A generated by aa^*. Let $K_A^{nc} = \mathrm{U}\{I_a : a \in K_A^+\}$. Then the following hold.*

(1) *$\{I_a : a \in K_A^+\}$ forms an inductive system, $A = C^* - \text{ind lim}\{I_a : a \in K_A^+\}$,* and *$K_A^{nc} = \text{ind lim}\{I_a : a \in K_A^+\}$.*
(2) *K_A^{nc} with the locally convex inductive limit topology t is a locally m-convex Q-algebra satisfying $E(K_A^{nc}) = E(K_A) = A$.*
(3) *If A has a countable bounded approximate identity, then (K_A^{nc}, t) is an LFQ-algebra.*

In general $K_A \neq K_A^{nc}$, though $K_A \subseteq K_A^{nc}$. Now K_A has been interpreted as a non-commutative analogue of $C_c(X)$. Then K_A^{nc} may be interpreted as continuous functions on a non-commutative space vanishing at infinity in 'commutative directions' and having compact supports in 'non-commutative directions'. This interpretation is suggested by the remarks preceeding ([28], Theorem 8).

The universal C^*-algebra $C^*(G,R)$ on a C^*-admissible set of generators G with relations R provides another method of constructing C^*-algebras. Motivated by some problems in C^*-algebras, Phillips introduced more general weakly C^*-admissible generators with relations (G,R) leading to the construction of the universal pro-C^*-algebra $C^*(G,R)$ on (G,R) [27]. In § 8, we construct a universal topological *-algebra $A(G,R)$ on (G,R) with weakly Banach admissible relations R, and prove the following.

Theorem 1.7. *Let (G,R) be weakly Banach admissible.*

(1) *$E(A(G,R)) = C^*(G,R)$.*
(2) *$A(G,R)$ has a C^*-enveloping algebra if and only if (G,R) is C^*-admissible.*

The paper is organized as follows. Proofs of Theorems 1.1, 1.2 and 1.4 are presented in § 3. The preliminary lemmas and constructions in the locally convex, non-m-convex set up more general than in [5], are discussed in § 2. Section 4 contains a couple of remarks including some corrections in [5]. The smooth crossed product is discussed in § 5 culminating in the proof of Theorem 1.5. Section 6 contains the proof of Theorem 1.6. This is followed by a brief discussion on the C^*-algebra of a groupoid in § 7. Universal C^*-algebras on generators with relations are discussed in § 8. In what follows, we briefly recall the relevant ideas in unbounded operator representations.

For the basic theory of unbounded operator *-representations $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A, we refer to [37, 30]. Let A^1 denote the unitization of A. The graph topology $t_\pi = t_{\pi(A^1)}$ on $\mathcal{D}(\pi)$ is defined by seminorms $\xi \to ||\xi|| + ||\pi(x)\xi||$, where $x \in A$. The *closure* $\bar{\pi}$ of π is the *-representation $(\bar{\pi}, D(\bar{\pi}), H)$, where $D(\bar{\pi}) = \bigcap\{D(\overline{\pi(x)}) : x \in A^1\}$, $D(\overline{\pi(x)})$ being the domain of the closure $\overline{\pi(x)}$ of $\pi(x)$; and $\bar{\pi}(x) = \overline{\pi(x)}|_{D(\bar{\pi})}$ for all x in A^1. Throughout, π is assumed *non-degenerate*, i.e., the norm closure $(\pi(A)H)^- = H$ and the t_π-closure $\overline{(\pi(A)\mathcal{D}(\pi))}^{t_\pi} = \mathcal{D}(\bar{\pi})$. If $\pi = \bar{\pi}$, then π is *closed*. The *hermitian adjoint* π^* of π is the representation (not necessarily a *-representation) $(\pi^*, D(\pi^*), H)$, where $D(\pi^*) = \bigcap\{D(\pi(x)^* : x \in A^1\}$, and $\pi^*(x) = \pi(x^*)^*|_{D(\pi^*)}$ for all $x \in A^1$. If $\pi = \pi^*$, then π is *self-adjoint*. Further, π is *standard* if $\pi(x^*)^* = \overline{\pi(x)}$ for all x in A^1. If each $\pi(x)$ is a bounded operator, then π is *bounded*. If π is a direct sum of bounded representations, then π is *weakly unbounded*. An *O^*-algebra* is a collection $\mathcal{U}$ of linear operators T all defined on a dense subspace D of a Hilbert space H such that for all $T \in \mathcal{U}$, one has $TD \subseteq D$, and $T^*D \subseteq D$; and $\mathcal{U}$ is a *-algebra with the pointwise linear operations, composition as the multiplication, and $T \to T^+ := T^*|_D$ as the involution. Given a *-representation $(\pi, D(\pi), H)$ of a *-algebra A, the *uniform topology* [20], ([37], p. 77–78) $\tau_D = \tau_{D(\pi)}^{\pi(A)}$ on the O^*-algebra $\pi(A)$ is the locally convex topology defined by the seminorms $\{q_K : K$ is a bounded subset of $(D(\pi), t_\pi)\}$, where

$$q_K(\pi(x)) = \sup\{|\langle \pi(x)\xi, \eta\rangle| : \xi, \eta \text{ in } K\}.$$

A vector ξ in $D(\pi)$ is *strongly cyclic* [30] (called cyclic in [37]) if $D(\pi) = (\pi(A)\xi)^{-t_\pi}$ the closure of $(\pi(A)\xi)$ in $(D(\pi), t_\pi)$. By a *cyclic vector*, we mean ξ in $D(\pi)$ such that the norm closure $(\pi(A)\xi)^- = H$. For topological *-algebras, we refer to [21]. A *Q-algebra* is a topological algebra whose quasi-regular elements form an open set. An *LFQ-algebra* is a Q-algebra which is an LF-space [41]. The topology of a locally convex (respectively locally m-convex) *-algebras A is determined by the family $K(A)$ (respectively $K_s(A)$), or a separating subfamily $\mathcal{P}$ thereof, consisting of continuous *-seminorms (repsectively continuous submultiplicative *-seminorms) p. If A has a bounded approximate identity (e_i), then it is assumed that $p(e_i) \leq 1$ for all i and all p. A *pro-C^*-algebra* is a complete locally m-convex *-algebra whose topology is determined by a family of C^*-seminorms. A *Frechet *-algebra* (respectively *locally convex F^*-algebra*) is a complete metrizable locally m-convex (respectively locally convex) *-algebra. A *σ-C^*-algebra* means a Frechet pro-C^*-algebra. For pro-C^*-algebras, we refer to [26, 27].

2. Preliminary constructions and lemmas

Let A be a *-algebra, not necessarily having an identity element. Let f be a positive linear functional on A. Then f is *representable* if there exists a closed strongly cyclic *-representation $(\pi, D(\pi), H)$ of A having a strongly cyclic vector $\xi \in D(\pi)$ such that $f(x) = \langle \pi(x)\xi, \xi\rangle$ for all $x \in A$. If π can be chosen to be a bounded operator representation, then f is *boundedly representable*. The first half of the following is an unbounded representation theoretic analogue of ([39], Theorem 1), whereas the remaining half improves a part of ([39], Theorem 1) even in the bounded case. The proof exhibits the unbounded analogue of the GNS construction in the case of non-unital algebras. This provides a useful supplement to ([37], § 8.6). It is well-known that a representable functional is boundedly representable if and only if it is *admissible* in the sense that for each $x \in A$, there exists $k > 0$ such that $f(y^*x^*xy) \leq kf(y^*y)$ for all $y \in A$. In the following, Lemma 2.1(3) is very close to ([39], Theorem 1) in which a C^*-seminorm p is taken.

Lemma 2.1. *Let f be a positive linear functional on a *-algebra A. The following are equivalent.*

(1) *f is representable.*
(2) *There exists $m > 0$ such that $|f(x)|^2 \leq mf(x^*x)$ for all $x \in A$.*
Further, f is boundedly representable if and only if f satisfies (2) *above and the following.*
(3) *There exists a submultiplicative *-seminorm p on A and $M > 0$ such that $|f(x)| \leq Mp(x)$ for all $x \in A$.*

When A is a Banach *-algebra, Lemma 2.1 is given in ([7], Theorem 37.11, p. 199). In the framework of unbounded representation theory, it is discussed in [2]. There is a gap in the proof in ([7], Theorem 37.11) in that hermiticity of f has been implicitly used. Regrettably it remained unnoticed in [2]. This was rectified in [39] in the formalism of bounded representations. The following proof provides an analogous correction in the context of unbounded representations.

Proof. Suppose (1) holds with $f(x) = \langle \pi(x)\xi, \xi \rangle$ for all $x \in A$. Then for all x in A.

$$\begin{aligned} |f(x)|^2 &\leq ||\pi(x)\xi||^2 ||\xi||^2 \leq ||\xi||^2 \langle \pi(x)\xi, \pi(x)\xi \rangle \\ &= ||\xi||^2 \langle \pi(x)^* \pi(x)\xi, \xi \rangle = ||\xi||^2 \langle \pi(x^*)\pi(x)\xi, \xi \rangle \end{aligned}$$

as $\pi(A)\xi \subseteq D(\pi) = D(\pi(x)) = D(\pi(x^*))$ and $\pi(x^*) \subseteq \pi(x)^*$. Thus

$$|f(x)|^2 \leq ||\xi||^2 \langle \pi(x^* x)\xi, \xi \rangle = ||\xi||^2 f(x^* x)$$

for all $x \in A$, giving (2).

Conversely, assume (2). We adopt the GNS construction. Let $N_f = \{x \in A : f(x^* x) = 0\}$. By the Cauchy-Schwarz inequality, N_f is a left ideal of A. Let $X_f = A/N_f$, and $\lambda_f : A \to X_f$ be $\lambda_f(x) = x + N_f$. Then $\langle \lambda_f(x), \lambda_f(y) \rangle = f(y^* x)$ defines an inner product on X_f. Let H_f be the Hilbert space obtained by completing X_f. Let $\varphi : X_f \to \mathbb{C}$ be $\varphi(\lambda_f(x)) = f(x)$, a linear functional. Then for all $x \in A$,

$$|\varphi(\lambda_f(x))|^2 = |f(x)|^2 \leq mf(x^* x) = m\langle \lambda_f(x), \lambda_f(x) \rangle = m||\lambda_f(x)||^2.$$

Thus φ extends uniquely to H_f as a bounded linear functional; and by Riesz theorem, there exists a $\xi \in H_f$ such that for all $x \in A, f(x) = \varphi(\lambda_f(x)) = \langle \lambda_f(x), \xi \rangle$. Further, if m is the minimum possible constant in the assumed inequality, then $||\xi|| = m^{1/2}$. The idea of using Riesz theorem at this stage is borrowed from [39]. Define a *-representation $(\pi_0, D(\pi_0), H_f)$ of A by: $D(\pi_0) = X_f$; and for any x in A, $\pi_0(x)\lambda_f(y) = \lambda_f(xy)$ for all y in A. Let π be the closure of π_0. Then for all x, y in A,

$$\langle \lambda_f(x), \lambda_f(y) \rangle = f(y^* x) = \langle \lambda_f(y^* x), \xi \rangle = \langle \pi_0(y^*)\lambda_f(x), \xi \rangle. \tag{i}$$

Assertion 1. $X_f = \pi_0^*(A)\xi$.

Let $x \in A$. For all $y \in A$,

$$\begin{aligned} |\langle \pi_0(x)\lambda_f(y), \xi \rangle| &= |\langle \lambda_f(xy), \xi \rangle| = |f(xy)| \leq f(xx^*)^{1/2} f(y^* y)^{1/2} \\ &= f(xx^*)^{1/2} ||\lambda_f(y)|| \end{aligned}$$

showing that the linear functional $\lambda_f(y) \to \langle \pi_0(x)\lambda_f(y), \xi \rangle$ on $D(\pi_0)$ is $||\ \ ||$-continuous. Hence $\xi \in D(\pi_0(x)^*)$ for all $x \in A$. It follows, by the definition of $D(\pi_0^*)$, that $\xi \in D(\pi_0^*)$. Now (i) becomes $\langle \lambda_f(x), \lambda_f(y) \rangle = \langle \lambda_f(x), \pi_0(y^*)^* \xi \rangle$ for all $x \in A$. Since X_f is dense in H_f, we obtain $\lambda_f(y) = \pi_0(y^*)^* \xi = \pi_0^*(y)\xi$ for all y in A. Thus $X_f = \pi_0^*(A)\xi$.

Assertion 2. $\xi \in D(\pi)$.

Since $\overline{\pi_0(x)} = \pi_0(x)^{**}$, we show that $\xi \in D(\pi_0(x)^{**})$ for all $x \in A$, i.e., for all x, the functional on $D(\pi_0(x)^*)$ given by $\eta \to \langle \pi_0(x)^* \eta, \xi \rangle$ is $||\ \ ||$-continuous. Fix an $x \in A$. Now $\xi \in D(\pi_0^*)$, hence $\xi \in D(\pi_0(x^*)^*)$ so that the functional g on $D(\pi_0(x^*)) = X_f$ defined by $g(\eta) = \langle \pi_0(x^*)\eta, \xi \rangle$ is $||\ \ ||$-continuous, and extends continuously to H_f. Now let $\psi \in D(\pi_0(x)^*)$. Let (η_k) be a sequence in X_f such that $\eta_k \to \psi$ in $||\ \ ||$. Then $\xi \in D(\pi_0^*)$ implies that for any $x \in A$,

$$\begin{aligned} \langle \pi_0(x)^* \psi, \xi \rangle &= \langle \psi, \pi_0(x^*)^* \xi \rangle = \langle \psi, \pi_0^*(x)\xi \rangle \\ &= \lim \langle \eta_k, \pi_0^*(x)\xi \rangle = \lim \langle \pi_0(x^*)\eta_k, \xi \rangle = g(\psi) \end{aligned}$$

showing that $\psi \to \langle \pi_0(x)^*\psi, \xi\rangle$ is $\|\ \|$-continuous on $D(\pi_0(x)^*)$. This proves the assertion 2.

Now by the proof of assertions 1 and 2 above, it follows that for any $x \in A$,

$$\begin{aligned} f(x) = \varphi(\lambda_f(x)) &= \langle \lambda_f(x), \xi\rangle = \langle \pi_0(x^*)^*\xi, \xi\rangle = \langle \pi_0^*(x)\xi, \xi\rangle \\ &= \langle \bar{\pi}_0(x)\xi, \xi\rangle = \langle \pi(x)\xi, \xi\rangle. \end{aligned}$$

Clearly ξ is a strongly cyclic vector for π. Thus (2) implies (1).

Now assume (2) and (3). Let $N_p = \{x \in A : p(x) = 0\}$, a *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $\|x_p\|_p = p(x)$ where $x_p = x + N_p$. By (3), $F(x_p) = f(x)$ gives a well-defined continuous positive functional on A_p. By standard Banach *-algebra theory, for all x, y in A,

$$\begin{aligned} \|\pi(x)\pi(y)\xi\|^2 &= \langle \pi(y^*x^*xy)\xi, \xi\rangle = f(y^*x^*xy) = F(y_p^*x_p^*x_py_p) \\ &\le \|x_p^*x_p\|F(y_p^*y_p) \le p(x)^2 f(y^*y) = p(x)^2\|\pi(y)\xi\|^2. \end{aligned}$$

Since $\pi(A)\xi$ is dense in H_f, π is a bounded operator representation.

COROLLARY 2.2

Let A be a *-algebra.

(1) *A positive functional f on A is representable if and only if f is extendable as a positive functional on the unitization A^1 of A.*
(2) *A representable positive functional on A satisfies $f(x^*) = f(x)^-$ for all x in A.*
(3) *Let A be a topological *-algebra having a bounded approximate identity. Then every continuous positive functional on A is representable.*

COROLLARY 2.3

Let A be a complete locally m-convex *-algebra with a bounded approximate identity (e_γ) satisfying $p(e_\gamma) \le 1$ for all ρ in a defining family of seminorms.

(1) *Let f be a continuous positive functional on A. Then f is boundedly representable and there exists $p \in K_s(A)$ such that $|f(x)| \le (\limsup f(e_\gamma e_\gamma^*))p(x)$ for all $x \in A$.*
(2) *Let $(\pi, \mathcal{D}(\pi), H)$ be a *-representation of A. Then each $\pi(e_\gamma)$ is a bounded operator and $\|\pi(e_\gamma)\| \le 1$ for all γ. Further, if π is strongly continuous (in particular, if π is continuous in the unifrom topology, which is the case if A is locally convex F^** ([37], *Theorem 3.6.8, p. 99*)), *then $\|\pi(e_\gamma)\xi - \xi\| \to 0$ for each ξ.*

Proof. (1) By continuity, there exist $p \in K_s(A)$ and $m > 0$ such that $|f(x)| \le mp(x)$ for all $x \in A$. Now Lemma 2.1 applies by Corollary 2.2(3). Let $l = \limsup f(e_\gamma e_\gamma^*)$, which is finite. Let $c = \sup\{|f(x)| : p(x) = 1\}$. Choose a sequence (x_n) in A such that $f(x_n) \to c$ and $p(x_n) = 1$ for all n. Then, by the Cauchy-Schwarz inequality,

$$|f(x_n)|^2 = \lim_\gamma |f(e_\gamma x_n)|^2 \le (\limsup f(e_\gamma e_\gamma^*))f(x_n^*x_n) \le lc,$$

as $p(x_n^*x_n) \le p(x_n)^2 = 1$. Hence $c^2 \le lc$, i.e. $c \le l$, and the assertion follows.
(2) Let $P = (p_\alpha)$ be a cofinal subset of $K_s(A)$ determining the topology of A. Let $A_p = \{x \in A : \sup_\alpha p_\alpha(x) < \infty\}$. Then A_p is a *-subalgebra of A containing each e_γ. As A is complete, A_p is a Banach *-algebra with norm $p(x) = \sup_\propto p_\propto(x)$. For any $\xi \in H$,

consider the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi \rangle$ on A. Then for all $x \in A$, $|\omega_\xi(x)|^2 \leq ||\xi||^2\, \omega_\xi(x^*x)$. By Lemma 2.1, ω_ξ is representable, hence extends as a positive functional ω on the unitization A^1 of A. In view of the inclusion map $(A_p)^1 \to A^1$, ω is a positive functional on $(A_p)^1$. By ([7], Corollary 37.9, p. 198), ω is continuous in the norm of $(A_p)^1$. It follows that ω_ξ restricted to A_p is continuous in the norm of A_p and $||\omega_\xi|| \leq ||\xi||^2$. For each γ,

$$||\pi(e_\gamma)\xi||^2 = \omega_\xi(e_\gamma e_\gamma^*) \leq ||\omega_\xi|| p(e_\gamma)^2 \leq ||\xi||^2$$

showing that $||\pi(e_\gamma)|| \leq 1$. Now suppose that π is strongly continuous. Let $\eta \in \mathcal{D}(\pi)$ and $\varepsilon > 0$. There exists $x \in A$ and $\eta' \in \mathcal{D}(\pi)$ such that $||\pi(x)\eta' - \eta|| \leq \varepsilon/3$. Since $e_\gamma x \to x$, there exists γ_0 such that for all $\gamma \geq \gamma_0$,

$$\begin{aligned} ||\eta - \pi(e_\gamma)\eta|| &\leq ||\eta - \pi(x)\eta'|| + ||\pi(x)\eta' - \pi(e_\gamma x)\eta'|| \\ &\quad + ||\pi(e_\gamma)||\ ||\pi(x)\eta' - \eta|| < \varepsilon \end{aligned}$$

showing that $\pi(e_\gamma)\eta \to \eta$ for each $\eta \in \mathcal{D}(\pi)$. This completes the proof of Corollary 2.3.

The enveloping pro-C^*-algebra $E(A)$

We construct the enveloping pro-C^*-algebra $E(A)$ for a locally convex *-algebra A with jointly continuous multiplication. This extends the consideration in [10, 15, 19] in which A is additionally assumed m-convex. The added generality will include several constructions relevant in C^*-algebra theory (like the C^*-algebra of a groupoid). Let $R(A)$ denote the set of all continuous bounded operator *-representations $\pi : A \to B(H_\pi)$ of A into the C^*-algebras $B(H_\pi)$ of all bounded linear operators on Hilbert spaces H_π. Let $R'(A) = \{\pi \in R(A) : \pi \text{ is topologically irreducible}\}$. For $p \in K(A)$, let

$$R_p(A) = \{\pi \in R(A) : \text{ for some } k > 0, ||\pi(x)|| \leq kp(x) \text{ for all } x\},$$

and $R'_p(A) = R_p(A) \cap R'(A)$. Then

$$R(A) = \bigcup \{R_p(A) : p \in K(A)\}, \quad R'(A) = \bigcup \{R'_p(A) : p \in K(A)\}.$$

Let $r_p(x) = \sup\{||\pi(x)|| : \pi \in R_p(A)\}$.

Lemma 2.4. *Let A be as above, $p \in K(A)$. Then $r_p(\ )$ is a continuous C^*-seminorm on A satisfying $r_p(x) \leq p(x^*x)^{1/2}$. If $p \in K_s(A)$, then $r_p(x) = \sup\{||\pi(x)|| : \pi \in R'_p(A)) \leq p(x)$ for all $x \in A$.*

Proof. Let $s_p(x) = p(x^*x)^{1/2}$. Let $h = h^* \in A$ and $\pi \in R_p(A)$. Then $||\pi(h^n)|| \leq kp(h^n)$ for all $n \in \mathbb{N}$. By standard Banach algebra arguments, the spectral radius satisfies

$$r(\pi(h)) = \liminf ||\pi(h^n)||^{1/n} = \inf ||\pi(h^n)||^{1/n} \leq \inf p(h^n)^{1/n} \leq p(h).$$

Hence, for any $x \in A$,

$$||\pi(x)||^2 = ||\pi(x^*x)|| = r(\pi(x^*x)) \leq p(x^*x),$$

so that $r_p(x) \leq s_p(x)$. We use the joint continuity of multiplication to conclude the continuity of the C^*-seminorm $x \to r_p(x)$. Now suppose $p \in K_s(A)$. Then

$$r_p(x) \leq s_p(x) \leq (p(x^*)p(x))^{1/2} \leq p(x).$$

Further, let $N_p = \{x \in A : p(x) = 0\}$, a closed *-ideal in A. Let A_p be the Banach *-algebra obtained by completing A/N_p in the norm $||x + N_p||_p = p(x)$. Then $R_p(A)$ (respectively $R'_p(A)$) can be identified with $R(A_p)$ (respectively $R'(A_p)$). The assertion follows from the fact that for all $z \in A_p$,

$$\sup\{||\varphi(z)|| : \varphi \in R(A_p)\} = \sup\{||\varphi(z)|| : \varphi \in R'(A_p)\}$$

([14], 2.7, p. 47). This completes the proof of the lemma.

Define the *star radical* to be

$$\begin{aligned} \mathrm{srad}\,(A) &= \{x \in A : r_p(x) = 0 \text{ for all } p \in K(A)\} \\ &= \{x \in A : \pi(x) = 0 \text{ for all } \pi \in R(A)\}. \end{aligned}$$

For each $p \in K(A)$, $q_p(x + \mathrm{srad}\,(A)) = r_p(x)$ defines a continuous C^*-seminorm on the quotient locally convex *-algebra $A/\mathrm{srad}\,(A)$ with the quotient topology. Let τ be the Hausdorff topology on $A/\mathrm{srad}\,(A)$ defined by $\{q_p : p \in K(A)\}$. The *enveloping pro-C^*-algebra* $E(A)$ of A is the completion of $(A/\mathrm{srad}\,(A), \tau)$. When A is metrizable, $E(A)$ is metrizable. In view of Corollary 2.2, when A is m-convex, this coincides with the enveloping l.m.c. *-algebra defined in [10, 19, 15].

Lemma 2.5. *Let A be a locally convex *-algebra with jointly continuous multiplication.*

(a) *Let $\bar{A}$ be the completion of A. Then $E(\bar{A}) = E(A)$.*
(b) $E(A^1) = E(A)^1$.

Proof. Since A has jointly continuous multiplication, $\bar{A}$ is a complete locally convex *-algebra. The map $i : A/\mathrm{srad}\,(A) \to \bar{A}/\mathrm{srad}\,(\bar{A})$, where $i(x + \mathrm{srad}\,(A)) = x + \mathrm{srad}\,(\bar{A})$, is a well defined *-isomorphism into $E(\bar{A})$. Note that for any $p \in K(\bar{A})$, $R_p(A) = R_p(\bar{A})$ via the restriction (in fact, also $K(\bar{A}) = K(A)$), hence $\mathrm{srad}\,(A) = A \cap \mathrm{srad}\,(\bar{A})$. For any $p \in K(A)$, let $\tilde{p} \in K(\bar{A})$ be the unique extension of p. Then, for any $x \in A$,

$$q_p(x + \mathrm{srad}(A)) = r_p(x) = q_{\tilde{p}}(x + \mathrm{srad}(\bar{A}));$$

and for any $\tilde{p} \in K(\bar{A})$, $q_{\tilde{p}}(x + \mathrm{srad}(A)) = q_{\tilde{p}|A}(x + \mathrm{srad}(A))$. Thus i is a homeomorphism for the respective pro-C^*-topologies. On the other hand, i has dense range in $\bar{A}/\mathrm{srad}(\bar{A})$. Indeed, let $z \in \bar{A}$. Choose a net (x_i) in A such that $x_i \to z$ in the topology t of $\bar{A}$. Then

$$\begin{aligned} q_{\tilde{p}}(x_i - z + \mathrm{srad}(A)) = r_{\tilde{p}}(x_i - z) &= \sup\{||\pi(x_i - z)|| : \pi \in R_{\tilde{p}}(\bar{A})\} \\ &\le k\tilde{p}(x_i - z) \to 0 \end{aligned}$$

for all $\tilde{p} \in K(\bar{A})$. Thus $E(\bar{A})$, which is the completion of $\bar{A}/\mathrm{srad}(\bar{A})$, coincides with the completion $E(A)$ of $A/\mathrm{srad}(A)$. This completes the proof of (a). We omit the proof of (b).

A representation $(\pi, D(\pi), H)$ of A is *countably dominated* if there exists a countable subset B of A such that for any $x \in A$, there exists $b \in B$ and a scalar $k > 0$ such that $||\pi(x)\xi|| \le k||\pi(b)\xi||$ for all $\xi \in D(\pi)$ ([22], p. 419).

Lemma 2.6. (a) *Let A be a locally convex *-algebra. Let $j : A \to E(A)$, $j(x) = x + \mathrm{srad}(A)$.*

(1) *If $\pi : A \to B(H)$ is a continuous bounded operator *-representation, then there exists a unique continuous *-representation $\sigma : E(A) \to B(H)$ such that $\pi = \sigma \circ j$. Further, π is irreducible if and only if σ is irreducible.*

(2) *Let* $(\pi, \mathcal{D}(\pi), H)$ *be a closed* **-representation of* A *continuous in the uniform topology. Let* π *be weakly unbounded. Then there exists a closed weakly unbouned* **-representation* $(\sigma, \mathcal{D}(\sigma), H)$ *of* $E(A)$ *such that* $\pi = \sigma \circ j$ *and* $\mathcal{D}(\sigma)$ *is dense in the locally convex space* $(\mathcal{D}(\pi), t_\pi)$.

(3) *Let* A *be unital and symmetric. Assume that* A *is separable or nuclear (as a locally convex space). Let* $(\pi, \mathcal{D}(\pi), H)$ *be a separably acting, countably dominated* **-representation of* A *continuous in the uniform topology. Then there exists a closed* **-representation* $(\sigma, \mathcal{D}(\sigma), H)$ *of* $E(A)$ *such that* $\pi = \sigma \circ j$.

(b) (1) *There exists a unital, locally convex, non-m-convex,* F^**-algebra* A *such that* A *admits a faithful family of unbounded operator* **-representations, but admits no non-zero bounded operator* **-representation.*

(2) *There exists a unital non-locally-convex* F^**-algebra that admits no non-zero* **-representation.*

Proof. (a) (1) follows by the definition of $E(A)$.
(2) Let $\pi = \oplus \pi_i$, where each π_i is a norm continuous bounded operator *-representation $\pi_i : A \to B(H_i)$ on a Hilbert space H_i. We take $D(\pi_i) = H_i$. Let $E_i : H \to H_i$ be the orthogonal projection. By (1), there exist continuous *-homomorphisms $\sigma_i : E(A) \to B(H_i)$, $\sigma_i \circ j = \pi_i$. Let $\sigma = \oplus \sigma_i$ on the Hilbert direct sum $\oplus H_i = H$ having the domain

$$\begin{aligned}\mathcal{D}(\sigma) &= \{\eta = \Sigma E_i \eta \in H : \Sigma ||\sigma_i(z) E_i \eta||^2 < \infty \text{ for all } z \in E(A)\} \\ &\subset \mathcal{D}(\pi) = \{\eta = \Sigma E_i \eta \in H : \Sigma ||\pi_i(x) E_i \eta||^2 < \infty \text{ for all } x \in A\}.\end{aligned}$$

On $\mathcal{D}(\sigma)$, the σ-graph topology $t_{\sigma(E(A))}$ is finer than the relativized π-graph topology $t_\pi|_{\mathcal{D}(\sigma)}$. Being closed and weakly unbounded, both σ and π are standard representations. Hence, for all $h = h^*$ in A, the operators $\sigma(j(h))$ having domain $\mathcal{D}(\sigma)$ and $\pi(h)$ with domain $\mathcal{D}(\pi)$ are essentially self-adjoint. Since self-adjoint operators are maximally symmetric, $\mathcal{D}(\sigma)$ is dense in $\mathcal{D}(\overline{\pi(h)})$ for the graph topology defined by $\xi \to ||\xi|| + ||\overline{\pi(h)}\xi||$. Thus $\mathcal{D}(\sigma)$ is dense in the locally convex space $\mathcal{D}(\pi) = \cap \{\mathcal{D}(\overline{\pi(h)}) : h = h^*$ in $A\}$.
(3) By ([22], Theorem 3.2 and remark on p. 422) and ([37], Theorem 12.3.5, p. 343), there exists a compact Hausdorff Z with a positive measure μ such that

$$\pi = \int_Z^\oplus \pi_\lambda \mathrm{d}\mu(\lambda), \quad \mathcal{D}(\pi) = \int_Z^\oplus \mathcal{D}(\pi_\lambda) \mathrm{d}\mu(\lambda), \quad H = \int_Z^\oplus H_\lambda \mathrm{d}\mu(\lambda)$$

and each π_λ is irreducible. Since A is symmetric, each $\bar{\pi}$ and $\bar{\pi}_\lambda$ are standard ([37], Corollary 9.1.4, p. 237) (the commutativity assumption in this reference is not required, as the arguments in ([2], Theorem 3.5) shows); and by [3], each π_λ is a bounded operator representaion, being irreducible. Then we can proceed as in (2).
(b) (1) Take $A = L^\omega[0,1] = \bigcap_{1 \le p < \infty} L^p[0,1]$ (the Arens algebra) with pointwise operations, complex conjugation, and the topology of L^p-convergence for each p, $1 \le p < \infty$. The algebra A is a unital, symmetric, locally convex F^*-algebra, admitting a faithful standard *-representation $(\pi, \mathcal{D}(\pi), H)$ such that $\overline{\pi(A)}$ is an extended C^*-algebra with a common dense domain [13]. However, there exists no non-zero bounded operator representation of A, as A admits no non-zero multiplicative linear functional; and hence no non-zero submultiplicative *-seminorm. Thus srad $(A) = A$ and $E(A) = (0)$. (2) Take $A = \mathcal{M}$ [0,1], the algebra of all Lebesgue measurable functions on [0,1] with the topology of

convergence in measure. It admits no non-zero positive linear functional, and hence no non-zero *-representation.

Remark. 2.7. We call a *-representation $(\pi, \mathcal{D}(\pi), H)$ of a *-algebra A *boundedly decomposable* if it can be disintegrated as $\pi = \int_Z^{\oplus} \pi_\lambda \mathrm{d}\mu(\lambda)$ with each π_λ a bounded operator *-representation. One may show that $E(A)$ is universal for all closed boundedly decomposable *-representations of a locally convex F^*-algebra A. We do not know whether in (2) and (3) of Corollary (2.4) (a), σ is continuous in the uniform topology.

The *bounded vectors* [4] for a *-representation π of a *-algebra A are $B(\pi) = \bigcap\{B(\pi(x)) : x \in A\}$, where, for an operator T, the bounded vectors for T are

$$B(T) = \{\xi \in \mathcal{D}(T) : \text{ there exists a } > 0, c > 0 \text{ such that } \\ ||T^n\xi|| \leq ac^n \text{ for all } n \in \mathbb{N}\}.$$

The following is motivated by [35]. It shows that unbounded representations of locally m-convex *-algebras cannot be wildly unbounded,

Lemma 2.8. *Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of a complete locally m-convex *-algebra A continuous in the uniform topology on $\pi(A)$. Then the following hold.*

(1) *$\mathcal{D}(\pi) = B(\pi)$; and π is a direct sum of norm-continuous cyclic bounded operator *-representations.*
(2) *π is standard. For commuting normal elements x, y of A, the normal operators $\overline{\pi(x)}$ and $\overline{\pi(y)}$ have mutually commuting spectral projections.*
(3) *The uniform topology $\tau_{\mathcal{D}}$ on $\pi(A)$ is a pro-C^*-topology, i.e., it is determined by a family of C^*-seminorms.*
(4) *If A is Frechet, then $\tau_{\mathcal{D}}$ is metrizable and π is direct sum of a countable number of cyclic bounded-operator *-representations.*

Proof. Let $\xi \in \mathcal{D}(\pi)$. Let ω_ξ on A be the positive functional $\omega_\xi(x) = \langle \pi(x)\xi, \xi\rangle$ for $x \in A$. By Lemma 2.1, ω_ξ is representable and admissible. Hence the closed GNS representation (π_{ω_ξ}, $\mathcal{D}(\pi_{\omega_\xi})$, H_{ω_ξ}) associated with ω_ξ is a cyclic, norm-continuous bounded operator *-representation with $\mathcal{D}(\pi_{\omega_\xi}) = H_{\omega_\xi}$. Let ξ_ω denote the cyclic vector for π_{ω_ξ}. Let $\mathcal{D}(\pi_\xi) = (\pi(A)\xi)^{-t_\pi}$ and $H_\xi = [\pi(A)\xi]^-$. Since π is closed, $\mathcal{D}(\pi_\xi) \subset \mathcal{D}(\pi)$. The π-invariant subspace $\mathcal{D}(\pi_\xi)$ defines a closed subrepresentation $\langle \pi_\xi, \mathcal{D}(\pi_\xi), H_\xi\rangle$ of π as $\pi_\xi(x) = \pi(x)|_{\mathcal{D}(\pi_\xi)}$. Since $\langle \pi_{\omega_\xi}(x)\xi_{\omega_\xi}, \xi_{\omega_\xi}\rangle = \omega_\xi(x) = \langle \pi_\xi(x)\xi, \xi\rangle$ for all $x \in A$, it follows that π_{ω_ξ} and π_ξ are unitarily equivalent. Thus π_ξ is a bounded operator representation, and $\mathcal{D}(\pi_\xi) = H_\xi \subset B(\pi)$. This also implies that H_ξ is reducing in the sense of ([37], § 8.3). Thus the following is established.

Assertion I. For any ξ in $\mathcal{D}(\pi)$, $[\pi(A)\xi]^{-t_\pi} = [\pi(A)\xi]^- \subset B(\pi)$.

It follows that $\pi(A)\mathcal{D}(\pi) \subset B(\pi)$, hence $B(\pi)$ is dense in $(\mathcal{D}(\pi), t_\pi)$ and norm dense in H. Since $B(\pi)$ forms a set of common analytic vectors for $\pi(A)$, the conclusion (2) follows, using ([40], Theorem 2). Also, a standard Zorn's lemma argument gives $\pi = \oplus\pi_i$, with each π_i a cyclic, continuous, bounded operator representation.

Assertion II. For each bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A)$ such that $||\pi(x)\eta|| \leq ||\eta||p(x)$ for all $x \in A$, $\eta \in M$.

By continuity, given M as above, there is $k > 0$ and $p \in K_s(A)$ such that $q_M(\pi(x)) \leq kp(x)$ for all $x \in A$. Hence, for each $\eta \in M$ and $x \in A$, $||\pi(x)\eta||^2 \leq kp(x^*x) \leq kp(x)^2$. By Corollary 2.3, $||\pi(x)\eta||^2 \leq lp(x)^2$, where $l = \limsup \omega_\eta(e_\gamma e_\gamma^*) \leq ||\eta||^2$. Hence $||\pi(x)\eta|| \leq ||\eta||p(x)$ for all $x \in A$, all $\eta \in M$.

Now let $\xi \in \mathcal{D}(\pi)$. By (II) above, there exists $p \in K_s(A)$ such that for all $n \in \mathbb{N}$,

$$||\pi(x)^n\xi||^2 = \langle \pi(x^{*^n}x^n)\xi, \xi\rangle \leq ||\xi||^2 p(x)^{2n}$$

showing that $\xi \in B(\pi(x))$. Thus $\mathcal{D}(\pi) = B(\pi)$ proving (1).

The proof of (3) is based on arguments in ([35], Theorem 1). Let $\mathcal{F}$ be the collection of all subspaces (linear manifolds) K of $\mathcal{D}(\pi)$ such that K is π-invariant, and $\pi|_K$ is a bounded operator *-representation. For $K \in \mathcal{F}$, let s_K be the C^*-seminorm

$$s_K(\pi(x)) = \sup\{||\pi(x)\eta|| : \eta \in K, ||\eta|| \leq 1\}.$$

Let τ_1 be the topology on $\pi(A)$ defined by $\{s_K : K \in \mathcal{F}\}$. We show that $\tau_{\mathcal{D}} = \tau_1$. Clearly $\tau_1 \leq \tau_{\mathcal{D}}$. Let M be a bounded subset of $(\mathcal{D}(\pi), t_\pi)$. Choose k and p as in assertion (II) above. By Corollary 2.3, $|\omega_\xi(x)| \leq ||\xi||p(x)$ for all $x \in A$, all $\xi \in M$. Thus

$$M \subset \mathcal{D}_p := \{\eta \in \mathcal{D}(\pi) : |\langle \pi(x)\eta, \eta\rangle| \leq ||\eta||^2 p(x) \text{ for all } x \text{ in} A\}.$$

Then $\mathcal{D}_p \in \mathcal{F}$; $||\pi(x)\eta|| \leq ||\eta||^2 p(x)^2$ for all $\eta \in \mathcal{D}_p$; and, as π is closed, ([35], Lemma 3) implies that $\mathcal{D}_p$ is $||\ \ ||$-closed. Let $S = \{\xi \in \mathcal{D}_p : ||\xi|| \leq 1\}$. As M is also $||\cdot||$ bounded, $||\eta|| \leq r$ for all $\eta \in M$; and $M \subset rS$. Then, for all $x \in A$, $q_M(\pi(x)) \leq r^2 s_{\mathcal{D}_p}(\pi(x))$. Thus $\tau_{\mathcal{D}} \leq \tau$. This gives (3). Finally (4) is consequence of the fact that the topology of a metrizable A is determined by a countable cofinal subfamily of $K_s(A)$. This completes the proof of Lemma 2.8.

Now let A be commutative. Let $\mathcal{M}(A)$ be the Gelfand space consisting of all non-zero continuous multiplicative linear functionals on A. Let $\mathcal{M}^*(A) = \{\varphi \in \mathcal{M}(A) : \varphi = \varphi^*\}$ and $\varphi^*(x) = \overline{\varphi(x^*)}$. For each $x \in A$, let $\hat{x} : \mathcal{M}^*(A) \to \mathbb{C}$ be the map $\hat{x}(\varphi) = \varphi(x)$. The following, which incorporates the spectral theorem for unbounded normal operators, describes all unbounded *-representations of A. The proof can be constructed using Lemma 2.8 and ([9], Theorem 7.3), in which all bounded *-representations of A have been realized.

COROLLARY 2.9

Let A be a commutative complete locally m-convex *-algebra. Let $(\pi, \mathcal{D}(\pi), H)$ be a closed *-representation of A continuous in the uniform topology. Then there exist a positive regular Borel measure μ on $\mathcal{M}^*(A)$ and a spectral measure E on the Borel sets in $\mathcal{M}^*(A)$ with values in $B(H)$ such that the following hold.

(1) *π is a unitarily equivalent to the representation $(\sigma, \mathcal{D}(\sigma), H_\sigma)$ by multiplication operators in $H_\sigma = L^2(\mathcal{M}^*(A), \mu)$ with domain*

$$\mathcal{D}(\sigma) = \{f \in H_\sigma : \varphi \to \hat{x}(\varphi)f(\varphi) \text{is in} H_\sigma \text{ for all } x \in A\}$$

defined as $(\sigma(x)f)(\varphi) = \hat{x}(\varphi)f(\varphi)$.

(2) *For each $x \in A$, $\pi(x) = \int_{\mathcal{M}^*(A)} \hat{x}(\varphi)\mathrm{d}E(\varphi)$.*

We say that a locally convex *-algebra A is an *algebra with a C^*-enveloping algebra* if the pro-C^*-algebra $E(A)$ is a C^*-algebra. In view of Lemma 2.5, we do not need to assume A to be complete or unital. In [5], A is further assumed to be *m*-convex. The following extends the main results in ([5], § 2) to the present more general set up, and can be proved as in [5]. A is called an *sQ-algebra* if for some $k > 0$, $p \in K(A)$, the spectral radius r satisfies $r(x^*x)^{1/2} \leq kp(x)$ for all $x \in A$; A is *-*sb* if $r(x^*x) < \infty$ for each x, equivalently, $r(h) < \infty$ for all $h = h^*$. Thus $Q => sQ => {}^*$-*sb*.

Lemma 2.10. *Let A be a complete locally convex *-algebra with jointly continuous multiplication.*

(1) *A is an algebra with a C^*-enveloping algebra if and only if A admits greatest continuous C^*-seminorm.*
(2) *If A is sQ, then A admits a greatest C^*-seminorm, which is also continuous.*
(3) *Let A be an F^*-algebra. If A is *-sb, then A has a C^*-enveloping algebra; but the converse does not hold* (see ([5], Example 2.4)).

The enveloping AO^*-algebra $O(A)$

For a locally convex *-algebra (A, t) (t denoting the topology of A), let $P_c(A, t)$ (respectively $P_{ca}(A, t)$) be the set of all continuous (respectively continuous admissible) representable positive functionals on A. For each f in $P_c(A, t)$, let $(\pi_f, \mathcal{D}(\pi_f), H_f)$ denote the strongly cyclic GNS representation defined by f as in Lemma 2.1. Let $I = \cap\{\ker \pi_f : f \in P_{ca}(A, t)\}$ and $J = \cap\{\ker \pi_f : f \in P_c(A, t)\}$. Then I and J are closed *-ideal of A, $J \subset I$, and $I = \operatorname{srad}(A)$ in view of the cyclic decomposability of any $\pi \in R(A)$. The *universal representation of* (A, t) is $\pi_u = \oplus\{\pi_f : f \in P_c(A, t)\}$. This is a slight variation of ([37], p. 228). Then $\sigma_u(x + J) = \pi_u(x)$ define a one–one *-homomorphism of A/J into the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi_u))$. Let $\sigma_u(t)$ be the topology on A/J induced by the uniform topology on $\pi_u(A)$; viz. $\sigma_u(t)$ is determined by the seminorms $\{q_M : M$ is a bounded subset of $(\mathcal{D}(\pi_u), t_{\pi_u})\}$, where $q_M(x + J) = \sup\{|\langle\pi_u(x)\xi, \eta\rangle| : \xi, \eta \text{ in } M\}$. Then $(A/J,\ \sigma_u(t))$ is an AO^*-algebra [36] in the sense that it is algebraically and topologically *-isomorphic to an O^*-algebra with uniform topology [37]. We call $(A/J,\ \sigma_u(t)$ the *enveloping AO^*-algebra* of A, denoted by $O(A)$.

Lemma 2.11. *Let A be as above.*

(1) *Every *-representation of A which is continuous in the uniform topology and which is a direct sum of strongly cyclic representations factors through $O(A)$. When A is either complete and m-convex, or is countably dominated, every *-representation of A continuous in the uniform topology factors through $O(A)$.*
(2) *Let A be barrelled. Then $\sigma_u(t)$ is coarser than the quotient topology t_q on A/J.*
(3) *There exists a continuous *-homomorphism from $O(A)$ into the pro-C^*-algebra $E(A)$.*
(4) *The following are equivalent.*
   (i) *$\sigma_u(t)$ is normable.*
   (ii) *$\sigma_u(t)$ is C^*-normable.*
   (iii) *There exists a linear norm on A/J defining a topology finer than $\sigma_u(t)$.*

When any of these conditions hold, and if A is barrelled, then A has a C^*-enveloping algebra; but the converse does not hold.

Proof. (1) follows from the construction of $O(A)$ and Lemma 2.8. (2) Let A be barrelled. Since J is closed. $(A/J, t_q)$ is barrelled ([32], ch. II, §7, Corollary 1, p. 61). Further, σ_u is

weakly continuous. Hence, σ_u is continuous in the uniform topology ([20], Theorem 4.1).
(3) Since $J \subset \operatorname{srad} A$, the map

$$\phi : A/J \to A/\operatorname{srad} A \to E(A), \phi(x+J) = x + \operatorname{srad} A$$

is a well defined *-homomorphism. Now, as $E(A)$ is a pro-C^*-algebra, $E(A)/\ker q_p$ is a C^*-algebra for any $p \in K(A)$, denoted by $E_p(A)$, with the norm $||z + \ker q_p|| = q_p(z)$ and $E(A) = \text{proj lim}\, E_p(A)$, inverse limit of C^*-algebras [26]. Let $\varphi_p : E(A) \to E_p(A)$ be $\varphi_p(z) = z + \ker q_p$. For the continuity of $\phi : (O(A), \sigma_u(t)) \to (E(A), \tau)$, it is sufficient to show the continuity of the *-homomorphism $\phi_p = \varphi_p \circ \phi : O(A) \to E_p(A)$. Now the map

$$\psi : A \to A/\operatorname{srad}(A) \to E(A) \to E_p(A), \psi(x) = (x + \operatorname{srad}(A)) + \ker q_p$$

is a continuous bounded operator *-representation; and $\psi = \phi_p \circ j_u$, $j_u(x) = x + I$. Hence ϕ_p is continuous for each $p \in K(A)$.
(4) (i) if and only if (ii) if and only if (iii) follows from ([20], Theorems 3.2, 3.3). Let A be barrelled. Let $|\ |$ be a norm on A/J determining $\sigma_u(t)$. Since $t_q \geq \sigma_u(t)$, $p_\infty(x) = |x + J|$ defines a continuous C^*-seminorm on A. Let p be any continuous C^*-seminorm on A. Let A_p be the completion of $A/\ker p$ in the C^*-norm $|x + \ker p| = p(x)$. Then $\pi_p : A \to A_p$, $\pi_p(x) = x + \ker p$ defines a continuous bounded operator *-representation. By (1), there exists a continuous *-homomorphism σ_p such that $\sigma_p \circ j_u = \pi_p$. Since the uniform topology on A_p is the $|\ |_p$-topology, and since $\sigma_u(t)$ is determined by $|\ |$, it follows that for some $k > 0$, $|\sigma_p(z)| \leq k|z|$ for all $z \in A/J$. Thus $p(x) \leq kp_\infty(x)$; and so $p(x) \leq p_\infty(x)$ for all $x \in A$, both being C^*-seminorms. Thus p_∞ is the greatest continuous C^*-seminorm on A. By Lemma 2.8, $E(A)$ is a C^*-algebra. That the converse does not hold is illustrated by Arens' algebra $A = L^\omega[0,1]$, wherein $E(A) = (o)$, $O(A) = A$ topologically as well.

3. Proofs of theorems 1.1, 1.2 and 1.4

Proof of Theorem 1.2. First we prove the following.

Assertion I. Given a bounded subset M of $(\mathcal{D}(\pi), t_\pi)$, there exists $p \in K_s(A)$ and $k > 0$ such that $q_M(\pi(x)) \leq kr_p(x)$ for all $x \in A$.

By the continuity of π, given M, there exists $k > 0$ and $p \in K_s(A)$ such that $q_M(\pi(x)) \leq k\,p(x)$ for all $x \in A$. Let $\xi \in M$. Then

$$|\omega_\xi(x)| = |\langle \pi(x)\xi, \xi\rangle| \leq q_M(\pi(x)) \leq k\,p(x)$$

for all x. Since ω_ξ is representable, it is extendable to A^1. The arguments in the proof of Corollary 2.3(1) applied to the extension of ω_ξ to A^1 give

$$\omega_\xi(x^*x) \leq ||\xi||^2 p(x^*x) \leq ||\xi||^2 p(x)^2$$

for all x in A. Thus $||\pi_{\omega_\xi}(x)\xi|| \leq ||\xi|| p(x)$: and by the definition of r_p, $||\pi_{\omega_\xi}(x)\xi|| \leq ||\xi|| r_p(x)$ for all x in A. Since M is $||\ ||$-bounded, there exists $l > 0$ such that for all ξ in M, all x in A,

$$|\omega_\xi(x^*x)| = ||\pi_{\omega_\xi}(x)\xi||^2 \leq l^2 r_p(x)^2.$$

It follows that for all x in A, and all ξ, η in M,

$$|\langle \pi(x)\xi, \eta\rangle| \leq ||\eta|| |\omega_\xi(x^*x)^{1/2} \leq l^2 r_p(x).$$

Thus $q_M(\pi(x)) \leq l^2 r_p(x)$ for all x in A.

Now, by Lemma 2.8, $\pi = \oplus\pi_i$, with each $\pi_i : A \to B(H_i)$ norm continuous. By Lemma 2.6, there exists a closed representation $(\sigma', \mathcal{D}(\sigma'), H)$ $\sigma' = \oplus\sigma_i$ of $E(A)$, with each $\sigma_i : E(A) \to B(H_i)$ norm continuous, $\sigma_i \circ j = \pi_i$ for all i. We shall eventually show $\mathcal{D}(\pi) = \mathcal{D}(\sigma')$.

On the other hand, consider the *-representation $(\sigma, \mathcal{D}(\sigma), H)$ of $A/\text{srad}\,(A)$ having domain $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, and given by $\sigma(j(x)) = \pi(x)$ for all $x \in A$. By ([37], Proposition 2.2.3, p. 39), on $\mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))}$ which is the graph topology on $\mathcal{D}(\pi)$ due to the maximal O^*-algebra $\mathcal{L}^+(\mathcal{D}(\pi))$. Hence, on $\pi(A)$, the uniform topology $\tau_{\mathcal{D}}^{\pi(A)} = \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}$ $|_{\pi(A)} = \tau_1$ (say), which, by lemma 2.8, is a pro-C^*-topology. By ([37], Proposition 3.3.20, p. 85), $\sigma(A/\text{srad}\,(A))$ is contained in a $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}$-complete *-subalgebra of $\mathcal{L}^+(\mathcal{D}(\pi))$; and σ can be extended as a continuous *-homomorphism $\sigma(E(A), \tau) \to [\mathcal{L}^+(\mathcal{D}(\pi)), \tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}]$ giving a closed *-representation σ of $E(A)$ on H with domain $D(\sigma) = D(\pi)$. Next we prove the following.

Assertion II. As representations of $E(A)$, $\sigma = \sigma'$.

This, we do, in the following steps.

(a) σ is an extension of σ'.

Clearly, $\mathcal{D}(\sigma') \subset \mathcal{D}(\pi) = \mathcal{D}(\sigma)$. We show $\sigma(z)|_{\mathcal{D}(\sigma')} = \sigma'(z)$ for all $z \in E(A)$. Fix $z \in E(A)$. Let $\eta \in \mathcal{D}(\pi)$. Choose a net (x_r) in A such that for all $p \in K_s(A)$, $q_p(j(x_r) - z) \to 0$. Choose an appropriate p by (I) above. Then

$$\begin{aligned}
||\sigma(j(x_r))\eta - \sigma(j(x_{r'}))\eta||^2 &= ||\pi(x_r)\eta - \pi(x_{r'})\eta||^2 \\
&= \omega_\eta((x_r - x_{r'})^*(x_r - x_{r'})) \\
&\leq k\, r_p(x_r - x_{r'}) \\
&= k\, q_p(j(x)_r) - j(x_{r'})) \to 0.
\end{aligned}$$

Hence $\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$; and similarly, $\pi(x)\pi(x_r)\eta$ is norm Cauchy in $\mathcal{D}(\pi)$ for all $x \in A$. Thus $\pi(x_r)\eta$ is Cauchy in $(\mathcal{D}(\pi), t_\pi)$, which is complete as π is closed. Thus there exists $\xi \in \mathcal{D}(\pi)$ such that $\lim(x_r)\eta = \xi$ in t_π. This defines $\sigma(z)$ as $\sigma(z)\eta = \xi$, which gives $\sigma(z)|_{\mathcal{D}(\sigma')} = \sigma'(z)$.

(b) σ is a closed representation of $E(A)$.

Indeed, as π is closed.

$$\begin{aligned}
\mathcal{D}(\sigma) = \mathcal{D}(\pi) &= \bigcap\{\mathcal{D}(\overline{\pi_e(x)}) : x \in A^1\} \\
&= \bigcap\{\mathcal{D}(\overline{\sigma_e(j(x))} : j(x) \in j(A^1) = (j(A)^1)\} \\
&\supset \{\mathcal{D}(\overline{\sigma_e(z)}) : z \in (E(A))^1\} \\
&= \mathcal{D}(\bar{\sigma}) \supset \mathcal{D}(\sigma),
\end{aligned}$$

hence $\mathcal{D}(\sigma) = \mathcal{D}(\sigma')$. This also follows from the fact that π is closed: on $\mathcal{D}(\sigma) = \mathcal{D}(\pi)$, $t_\pi = t_{\mathcal{L}^+(\mathcal{D}(\pi))} = t_{\sigma((E(A)^1))}$; as well as $\pi(A) \subset \sigma(E(A)) \subset \mathcal{L}^+(\mathcal{D}(\pi))$. This further implies $\tau_{\mathcal{D}}^{\mathcal{L}^+(\mathcal{D}(\pi))}|_{\sigma(E(A))} = \tau_{\mathcal{D}}^{\sigma(E(A))}$; which, in turn gives the following.

(c) σ' is continuous in the uniform topology as a *-representation of $(E(A), \tau)$.

Now, by (c), Lemma 2.8 implies that the closed representation σ' is standard; hence self-adjoint, and so maximal hermitian ([31], (I), Lemma 4.2). Then (a) gives $\sigma' = \sigma$, thereby verifying (II). This completes the proof of Theorem 1.2.

If π is irreducible, then σ is irreducible, hence is a bounded operator representation by [3], ([6], Theorem 4.7). This gives Corollary 1.3.

Proof of Theorem 1.1. Let A be Frechet. Then $A = \text{proj lim} A_n$, an inverse limit of a sequence of Banach *-algebras A_n. Assume that each *-representation (and hence the universal representation π_u) of A is a bounded operator representation. Since A is Frechet, π_u is continuous. Let σ be the representation of $E(A)$ defined by Theorem 1.2 corresponding to π_u. Then σ is also a bounded operator *-representation. Further, as A is Frechet, $E(A) = \text{proj lim}\, C^*(A_n)$ is also Frechet. Thus σ is continuous and there exists a continuous C^*-seminorm $q_\circ$ on $E(A)$ such that $||\sigma(z)|| \leq q_\circ(z)$ for all $z \in E(A)$. Now the bounded part of $E(A)$

$$b(E(A)) = \{z \in E(A) : q(z) < \infty \text{ for all continuous } C^*\text{-seminorm } q\}$$

is a C^*-algebra with the norm

$$\begin{aligned} ||z||_\infty &= \sup\{q(z) : q \text{ is a continuous } C^*\text{-seminorm on } E(A)\} \\ &= \sup\{q_p(z) : p \in K_s(A)\}. \end{aligned}$$

Since σ is one–one, the restriction $\sigma' = \sigma|_{b(E(A))}$ is a *-isomorphism of the C^*-algebra $b(E(A))$ into $B(H_\sigma)$. Hence, for all $z \in b(E(A))$,

$$||\sigma'(z)|| = ||z||_\infty \geq q_\circ(z) \geq ||\sigma(z)||.$$

It follows that $b(E(A)) = E(A)$. As $E(A)$ is Frechet, the continuous inclusion map $(b(A), || \ ||_\infty) \to (E(A), \tau)$ is a homeomorphism. The converse follows from Theorem 1.2.

Proof of Theorem 1.4. By Corollary 2.3, $I = J = \text{srad}\,(A)$ in the notations of Lemma 2.9. Let $K = A/J$, a Frechet *-algebra in the quotient topology from A. By Lemma 2.8, the uniform topology $\tau_{\mathcal{D}}$ on $\pi_u(A)$ is a σ-C^*-topology; and the topology $\sigma_u(t)$ on K is determined by the (continuous) C^*-seminorms $\{s_G(\cdot) : G \in \mathcal{F}\}$, where $\mathcal{F}$ is the collection of all subspaces $\mathcal{D}$ of $\mathcal{D}(\pi_u)$ such that $\mathcal{D}$ is π_u-invariant and $\pi_u|_{\mathcal{D}}$ is a bounded operator *-representation; and $s_G(z) = ||\pi_u|_G(x)||$ for all $z = x + J$, $x \in A$. Thus $\sigma_u(t) \leq \tau$ where τ is the relative topology from $E(A)$ defined by all C^*-seminorms on $E(A)$. To show that $\tau \leq s_u(t)$, let $z_n = x_n + J \in K$, $z_n \to 0$ in $\sigma_u(t)$. Let q be any C^*-seminorm on A. There exists $\pi \in R(A)$ such that $q(x) = ||\pi(x)||$, and $\pi = \oplus\{\pi_f | f \in F_\pi\}$ for a suitable $F_\pi \subset P_c(A, t)$. Now $H_\pi = \oplus_{f \in F_\pi} H_f \subset \mathcal{D}(\pi_u)$, $H_\pi \in \mathcal{F}$, and $||\pi(x_n)|| = s_{H_\pi}(z_n) \to 0$. Hence $z_n \to 0$ in τ. Thus $\tau = \sigma_u(t)$, and $E(A) = (O(A), \sigma_u(t))$, the completion. The remaining assertion follows from Lemma 2.11.

4. Remarks

PROPOSITION 4.1

Let A be a *-sb Frechet *-algebra. If A is hermitian, then A is a Q-algebra.

Proof. We can assume that A is unital. Let P be a sequence of submultiplicative *-seminorms defining the topology of A. Let $A = \text{proj lim} A_q$ be the Arens–Michael decomposition expressing A as an inverse limit of a sequence of Banach *-algebras; where, for $q \in P$, A_q is the Banach *-algebra obtained by completing $A/\ker q$ in the norm $||x + \ker q|| = q(x)$. Let $\pi_q : A \to A_q$ be $\pi_q(x) = x + \ker q$.

Case 1. Assume that A is commutative. By hermiticity, $sp_A(h) = \{\phi(h) : \phi \in \mathcal{M}(A)\} \subset \mathbb{R}$ for all $h = h^* \in A$. Note that since A is hermitian. $\mathcal{M}(A) = \mathcal{M}^*(A)$. Using ([23], Proposition 7.5), it follows that for each q, $\mathcal{M}(A_q) = \mathcal{M}^*(A_q)$; hence by ([7], Theorem 35.3, p. 188), each A_q is hermitian. Now by ([17], Lemma 41.2, p. 225), for each $z \in A_q$, the spectral radius satisfies

$$r_{A_q}(z) \leq r_{A_q}(z^*z)^{1/2} = |z_q|_q,$$

$|\ |_q$ denoting the Gelfand–Naimark pseudonorm on A_q. Then $m_q(x) = |\pi_q(x)|_q$ defines a continuous C^*-seminorm on A. By Lemma 2.10, there exists a greatest continuous C^*-seminorm $p_\infty(\quad)$ on A. By ([23], Corollary 5.3), for each $x \in A$,

$$r_A(x) = \sup\{r_{A_q}(\pi_q(x))\} \leq \sup\{m_q(x)\} \leq p_\infty(x).$$

By the continuity of p_∞, there exists a $p \in K_s(A)$ and $k > 0$ such that for all x in A, $r(x) \leq p_\infty(x) \leq kp(x)$. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

Case 2. Let A be non-commutative. Let M be a maximal commutative *-subalgebra of A containing the identity of A. Since M is spectrally invariant in A, M is also hermitian. By *-spectral boundedness and hermiticity, each positive functional on M can be extended to a positive functional on A ([17], Theorem 9.3, p. 49). It follows from ([15], Corollary 2.8) and the continuity of positive functionals on unital Frechet *-algebras, that for all $z \in M$, $p_\infty(z) = p^M_\infty(z) \leq r_M(z^*z)^{1/2}$, p^M_∞ being the greatest C^*-seminorm on M and $r_M(\cdot)$ denoting the spectral radius in M. Thus M is a commutative hermitian algebra with a C^*-enveloping algebra. By case 1, M is a Q-algebra. Further, M being hermitian, the Ptak's function $x \to r_M(x^*x)^{1/2}$ is a C^*-seminorm on M ([17], Corollary 8.3, p. 38; Theorem 8.17, p. 45).

Now let $x \in A$, and take M to be the maximal commutative *-subalgebra containing x^*x. Let $r_K(\cdot)$ denote the spectral radius in an algebra K. Then by Ptak's inequality in hermitian Frechet *-algebras ([17], Theorem 8.17, p. 45)

$$r_A(x) \leq r_A(x^*x)^{1/2} = r_M(x^*x)^{1/2} = p^M_\infty(x^*x)^{1/2} = p_\infty(x^*x)^{1/2} \leq q(x),$$

q being a *-algebra seminorm on A depending on p_∞ only. It follows from ([23], Proposition 13.5) that A is a Q-algebra.

(4.2) (i) It is claimed in ([5], Corollary 2.4) that a complete hermitian m-convex *-algebra with a C^*-enveloping algebra is a Q-algebra. Regrettably, there is a gap in the proof. The author sincerely thanks Prof. M Fragoulopoulou for pointing out this. It is implicitely used in the 'proof' therein that the completion of a hermitian normed algebra is hermitian. By Gelfand theory, this is certainly true in the commutative case, but is not true in non-commutative case (see ([17], p. 18)). Thus ([15], Corollary 2.4) remains valid in commutative case; and the above proposition partially repairs the gap in the non-commutative case. Consequently ([15], Lemma 2.15, Theorem 2.14) remains valid for Frechet algebras. Is a hermitian Frechet algebra with a C^*-enveloping algebra a Q-algebra? (ii) The algebra $C(\mathbb{R})$ of continuous functions on $\mathbb{R}$ exhibits that the condition *-*sb* can not be omitted from the above proposition. It also follows from above that a *-*sb* σ-C^*-algebra is a C^*-algebra.

(4.3) In Theorem 1.2, the assumption that π is closed can not be omitted. Let $A = C^\infty(\mathbb{R})$, the Frechet *-algebra of C^∞ functions on $\mathbb{R}$, with pointwise operations and the topology of uniform convergence on compact subsets of $\mathbb{R}$ of functions as well as their derivatives. Then $E(A) = C(\mathbb{R})$, the algebra of continuous functions on $\mathbb{R}$ with the compact open topology. On the Hilbert space $H = L^2(\mathbb{R})$, the *-representation π of A with $\mathcal{D}(\pi) = C_c^\infty(\mathbb{R})$, $\pi(a)f = af$, cannot be extended to a *-representation of $C(\mathbb{R})$ with the same domain ([10], Example 4.7).

(4.4) Theorem 1.1 means that a Fechet *-algebra has a C^*-enveloping algebra if and only if it is a BG^*-algebra [24]. In the non-metrizable case, it follows from Theorem 1.2 that if A is a complete topological m-convex *-algebra with a C^*-enveloping algebra, then every *-representation of A which is continuous in the uniform topology is a bounded operator representation. However, the converse does not hold. This is exhibited by the BG^*-algebra $C[0,1]$ of continuous functions on [0,1] with the pro-C^*-topology τ of uniform convergence on all countable compact subsets of [0,1]. Thus Theorem 1.1 is false without the assumption that A is Frechet. It would be of interest to find an example of a topological algebra with a C^*-enveloping algebra which is not a BG^*-algebra.

(4.5) Yood [42] has shown that a *-algebra A admits a greatest C^*-seminorm if and only if $\sup|f(x)| < \infty$ for each x, where the sup is taken over all admissible states S; and by Lemma 2.10, this happens for a Frechet A if and only if A has a C^*-enveloping algebra. Yood's result is an algebraic version of ([5], Corollary 2.9) that states that a complete m-convex algebra has a C^*-enveloping algebra if and only if S is equicontinuous.

(4.6) (i) Let π be a *-representation of a complete locally m-convex *-algebra A with a bounded approximate identity. Let A have a C^*-enveloping algebra. Is π continuous in the uniform topology? In particular, let π be a bounded operator *-representation. Is π norm-continuous?

(ii) Let A be a pro-C^*-algebra (more generally, a complete m-convex *-algebra with a bounded approximate identity). Let f be a representable, not necessarily continuous, positive functional on A. Is the GNS representation π_f a bounded operator representation? Is every *-representation of A weakly unbounded?

These are motivated by the point of view ([5], Remark 2.11, p. 207) that a topological *-algebras with a C^*-enveloping algebra provide a hermitian analogue of a commutative Q-algebra. It is easy to see that a *-representation π of a locally convex Q-algebra is a bounded operator representation and is norm continuous.

5. Crossed product constructions

We recall the crossed product of a C^*-dynamical system (G, A, α). Let α be a strongly continuous action of a locally compact group G by *-automorphisms of a C^*-algebra A. Let $C_c(G, A)$ be the vector space of all continuous A-valued functions with compact supports. It is a *-algebra with twisted convolution

$$x * y(g) = \int_G x(h)\alpha_h(y(h^{-1}g))\mathrm{d}h$$

and the involution $x^*(g) = \Delta(g)^{-1}\alpha_g(x(g^{-1}))^*$. The Banach *-algebra $L^1(G, A)$ is the completion of $C_c(G, A)$ in the norm $||x||_1 = \int_G ||x(h)||\mathrm{d}h$; and the crossed product C^*-algebra $C^*(G, A, \alpha)$ is the completion of $L^1(G, A)$ in its Gelfand–Naimark pseudonorm $||x|| = \sup\{||\pi(x)|| : \pi \in R(L^1(G, A))\}$, which is, in fact, a norm. Thus it is the enveloping C^*-algebra of the Banach *-algebra $L^1(G, A)$. The C^*-algebra $C^*(G, A, \alpha)$

can also be realized as the enveloping C^*-algebra of non-normed topological *-algebras smaller than $L^1(G,A)$.

Let $\mathscr{K}$ be the collection of all compact, symmetric neighbourhoods of the identity in G. For $K \in \mathscr{K}$, let $C_K(G,A) = \{f \in C_c(G,A) : \operatorname{supp} f \subseteq K\}$, a Banach space with the norm $||f|| = \sup\{||f(x)|| : x \in K\}$. The inductive limit topology τ on $C_c(G,A)$ is the finest locally convex topology on $C_c(G,A)$ making each of the embeddings $C_K(G,A) \to C_c(G,A)$, for all $K \in \mathscr{K}$, continuous. Then $C_c(G,A)$ is a locally convex, non- m-convex, topological *-algebra with jointly continuous multiplication and continuous involution. From ([18], p. 203), $E(C_c(G,A)) = C^*(G,A,\alpha)$. This immediately leads to the following.

PROPOSITION 5.1

Let (G,A,α) be a C^*-dynamical system. Let B be any topological *-algebra containing $C_c(G,A)$ as a dense *-subalgebra and satisfying $C_c(G,A) \subseteq B \subseteq C^*(G,A,\alpha)$. Then $E(B) = C^*(G,A,\alpha)$.

For $1 \leq p < \infty$, let $A^p(G,A) = L^1(G,A) \cap L^p(G,A)$, a Banach *-algebra with the norm $|x|_p = ||x||_1 + ||x||_p$. The above applies to $B = \bigcap\{A^p(G,A) : 1 \leq p < \infty\}$, a locally m-convex Q-Frechet *-algebra with the topology of $|\ |_p$-convergence for each p.

Smooth elements of a Lie group action

Let A be a unital C^*-algebra and G be a Lie group acting on A. Let Δ denote the infinitesimal generators of actions of 1-parameter subgroups of G on A, viz.,

$$\Delta = \{(\mathrm{d}/\mathrm{d}t)\alpha_{u(t)}|_{t=0} : t \to u(t) \text{ is a continuous homomorphism of } \mathbb{R} \text{ into } G\}.$$

Then Δ consists of derivations and it is a finite dimensional vector space ([11], p. 40) having basis, say $\delta_1, \delta_2, \ldots, \delta_d$. Then *$C^n$-elements* $(1 \leq n < \infty)$ and *C^∞-elements* of A for the action α are defined as follows.

$$C^n(A) = \{x \in A : x \in \operatorname{Dom}(\delta_{i_1}\delta_{i_2}\ldots\delta_{i_n}) \text{ for all } n\text{-tuples } \{\delta_{i_1},\ldots,\delta_{i_n}\} \text{ in } \Delta\}$$
$$C^\infty(A) = \bigcap\{C^n(A) : n \in \mathbb{N}\}.$$

By ([11], Proposition 2.2.1), each $C^n(A)$ and $C^\infty(A)$ are dense *-subalgebras of A; and $C^n(A)$ is a Banach *-algebra with the norm

$$||x||_n = ||x|| + \sum_{k=1}^{n} \sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\ldots\delta_{i_k}(x)||/k!.$$

Then $C^\infty(A) = \text{proj lim}\, C^n(A)$ is a Frechet *-algebra with the topology defined by the norms $\{||\ ||_n : n = 1,2,\ldots\}$.

Lemma 5.2. *$C^\infty(A)$ has a C^*-enveloping algebra and $E(C^\infty(A)) = A$.*

Proof. It is well known that $C^n(A)$ and $C^\infty(A)$ are spectrally invariant in A. Hence $(C^n(A), ||\ ||)$ and $(C^\infty(A), ||\ ||)$ are Q-algebras in the norm $||\ ||$ from the C^*-algebra A. Since $||\ || \leq ||\ ||_n$, $(C^\infty(A),\tau)$ is also a Q-algebra. By Lemma 2.10, $(C^\infty(A),\tau)$ is an algebra with a C^*-enveloping algebra. Let $\pi : B \to B(H)$, where $B = C^n(A)$ or $C^\infty(A)$, be a bounded operator *-representation on a Hilbert space H. Then for all $x \in B$,

$$\begin{aligned}||\pi(x)||^2 = ||\pi(x^*x)|| = r_{B(H)}(\pi(x^*x)) &\leq r_{\pi(B)}(\pi(x^*x)) \\ &\leq r_B(x^*x) \leq ||x||^2.\end{aligned}$$

Hence π is $||\ \ ||$-continuous; and by the density of $C^\infty(A)$ in A, π extends uniquely to a *-representation of A on H. It follows that $E(C^\infty(A)) = C^*(C^n(A)) = A$ for all n.

An element $x \in A$ is *analytic* if $x \in C^\infty(A)$ and there exists a scalar $t > 0$ such that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty;$$

whereas x is *entire* if $x \in C^\infty(A)$ and for all $t > 0$, it holds that

$$\sum_{k=0}^{\infty}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\delta_{i_2}\cdots\delta_{i_k}(x)||/k!\right)t^k < \infty.$$

Let $C^\omega(A)$ (respectively $C^{e\omega}(A)$) denote the set of all analytic (respectively entire) elements of A. Then each of $C^\omega(A)$ and $C^{e\omega}(A)$ is a *-subalgebra of A and $C^{e\omega}(A) \subset C^\omega(A) \subset C^\infty(A)$. For each $t > 0$ and $x \in C^n(A)$, define

$$p_n^t(x) = ||x|| + \sum_{k=1}^{n}\left(\sum_{i_1,i_2,\ldots,i_k=1}^{d} ||\delta_{i_1}\cdots\delta_{i_k}(x)||/k!\right)t^k.$$

Then $||\ \ ||_n$ and $p_n^t(\ \ )$ are equivalent norms. Hence $P^t = (p_n^t(\ \ ))$ and $p = (||\ \ ||_n)$ define the same C^∞-topology τ on $C^\infty(A)$. Let $A_t = \{x \in C^\infty(A) : p^t(x) = \sup_k p_k^t(x) < \infty\}$, a *-subalgebra of $C^\infty(A)$, which is a Banach *-algebra with norm $p^t(\ \ )$, and which consists of elements of $C^\infty(A)$ whose numerical ranges defined with respect to P^t are bounded. For $t < s$, the inclusion $A_s \to A_t$ is norm decreasing. Thus

$$C^{e\omega}(A) = \bigcap\{A_t : t > 0\} = \bigcap_{n=1}^{\infty} A_n = \text{proj lim} A_n,$$

a Frechet m-convex, *-algebra with the topology $\tau_{e\omega}$ defined by the family of norms $\{p^t(\ \ ) : t \in \mathbb{N}\}$ (setting $p^\circ(\ \ ) = ||\ \ ||$). Further,

$$C^\omega(A) = \bigcup_{t>0} A_t = \bigcup_{n=1}^{\infty} A_{1/n} = \text{ind lim} A_{1/n}$$

with the linear inductive limit topology τ_ω. By ([21], Corollary 10.2, Lemma 10.2, p. 317) and ([32], Proposition 6.6, p. 59), $(C^\omega(A), \tau_\omega)$ is a complete m-convex *-algebra which is a Q-algebra. Thus $C^\omega(A)$ is an algebra with a C^*-enveloping algebra. Further if each A_t is dense and spectrally invariant in $C^\infty(A)$, then $C^{e\omega}(A)$ is an algebra with a C^*-enveloping algebra and $E(C^{e\omega}(A)) = E(C^\omega(A)) = A$.

The smooth crossed product

We recall the smooth Frechet algebra crossed product [29]. Let B be a Frechet *-algebra. Let (p_n) be a sequence of submultiplicative *-seminorms defining the topology of B. Let β be a strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of B. Then β is called *m-tempered* (respectively *isometric*) if for each $m \in \mathbb{N}$, there exists a polynomial

$P(X)$ such that $p_m(\beta_r(x)) \leq P(r)p_m(x)$ for all $x \in B$, $r \in \mathbb{R}$ (respectively for each $m \in \mathbb{N}$, $p_m(\beta_r(x)) = p_m(x)$ for all $x \in B$, all $r \in \mathbb{R}$). Let $S(\mathbb{R})$ be the Schwartz space consisting of the rapidly decreasing C^∞-functions on $\mathbb{R}$. It is a Frechet space with the Schwartz topology. The completed projective tensor product $S(\mathbb{R}) \otimes B = S(\mathbb{R}, B)$ consists of B-valued Schwartz functions on $\mathbb{R}$. If β is m-tempered, then $S(\mathbb{R}, B)$ becomes an m-convex Frechet algebra with twisted convolution

$$(f * g)(r) = \int_{\mathbb{R}} f(s)\beta_s(g(r-s))\mathrm{d}s.$$

This Frechet algebra is called the *smooth Schwartz crossed product* of B by the action β of $\mathbb{R}$, and is denoted by $S(\mathbb{R}, B, \beta)$. In general, $S(\mathbb{R}, B, \beta)$ need not be a *-algebra ([34], § 4). If β is isometric, then the completed projective tensor product

$$\begin{aligned} L^1(\mathbb{R}) \otimes B &= L^1(\mathbb{R}, B) \\ &= \{f : \mathbb{R} \to B \text{ measurable function} : \int_{\mathbb{R}} p_m(f(r))\mathrm{d}r < \infty \text{ for all } m \in \mathbb{N}\} \end{aligned}$$

is a Frechet *-algebra with twisted convolution and the involution $f^*(r) = \beta_r(f(-r)^*)$, denoted by $L^1(\mathbb{R}, B, \beta)$. One has $S(\mathbb{R}, B, \beta) \subset L^1(\mathbb{R}, B, \beta)$.

The following is closely related with ([29], Lemma 1.1.9).

Lemma 5.3. *Let A be a dense Frechet *-subalgebra of a Frechet *-algebra B. Assume that A and B can be expressed as inverse limits of Banach *-algebras A_n and B_n respectively, where A_n is dense in B_n for all n; the inclusions $A \to A_n$, $B \to B_n$ have dense ranges for all n; and each A_n is spectrally invariant in B_n. Then A is spectrally invariant in B and $E(A) = E(B)$.*

Proof. By ([15], Theorem 4.3), $E(A) = \text{proj lim}\, E(A_n)$ and $E(B_n) = \text{proj lim}\, E(B_n)$. Since $A_n \to B_n$ is spectrally invariant with dense range, A_n is a Q-normed algebra in the norm of B_n. Hence every C^*-seminorm on A_n is continuous in the norm of B_n; and extends uniquely to B_n. Thus A_n and B_n have the same collection of C^*-seminorms. It follows that $E(A_n) = E(B_n)$ for all n; and so $E(A) = E(B)$.

PROPOSITION 5.4

Let α be an m-tempered strongly continuous action of $\mathbb{R}$ by continuous *-automorphisms of a Frechet *-algebra B contained as a dense *-subalgebra of a C^*-algebra A such that $E(B) = A$. Then $E(C^\infty(B)) = A$.

Proof. Let $\|\ \|$ denote the C^*-norm on A. Let (p_n) be an increasing sequence of submultiplicative *-seminorms defining the topology of B. In view of the continuity of the inclusion $B \to A$, the increasing sequence $q_n(\ ) = p_n(\ ) + \|\ \|$ of norms also determines the topology of B. Let $B_n = (B, q_n)$ be the completion, which is a Banach *-algebra. Then $B = \text{proj.}\lim B_n = \bigcap B_n$. Now, for any $n \in \mathbb{N}$, $r \in \mathbb{R}$, and $x \in B$,

$$\begin{aligned} q_n(\alpha_r(x)) &= \|\alpha_r(x)\| + p_n(\alpha_r(x)) \\ &= \|x\| + \text{poly}\,(r)p_n(x) = \text{poly}'\,(r)q_n(x) \end{aligned}$$

for some polynomial poly'( ). It follows that α is m-tempered for $(q_n(\ ))$ also; and it induces an action $\alpha^{(n)}$ of $\mathbb{R}$ by continuous *-automorphisms of B_n. Let $B_{n,m}$ be the Banach

*-algebra consisting of all C^m-vectors in B_n for $\alpha^{(n)}$. By ([33], Theorem 2.2), $B_{n,m} \to B_n$ are spectrally invariant embeddings with dense ranges. Also, $C^\infty(B) = \text{proj lim}_{n,m} B_{n,m} = \text{proj lim}_n B_{n,n}$. Now Lemma 5.2 implies that $C^\infty(B)$ is spectrally invariant in B and $E(C^\infty(B)) = A$.

PROPOSITION 5.5

Let α be a strongly continuous action of $\mathbb{R}$ by *-automorphisms of a C^*-algebra A. The following hold.

(a) *The Frechet algebras $S(\mathbb{R}, A, \alpha)$ and $S(\mathbb{R}, C^\infty(A), \alpha)$ are Q-algebras.*
(b) *The embeddings $S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ are continuous, spectrally invariant and have dense ranges.*
(c) *The Frechet algebra $S(\mathbb{R}, C^\infty(A), \alpha)$ is *-algebra and $E(S(\mathbb{R}, C^\infty(A), \alpha) = C^*(\mathbb{R}, A, \alpha)$.*

Proof. By ([34], Theorem A.2), α leaves $C^\infty(A)$ invariant. In ([34], Corollary 4.9), taking the scale σ to be the weight $w(r) = 1 + |r|$ on $G = \mathbb{R} = H$, it follows that $S(\mathbb{R}, C^\infty(A), \alpha)$, is a Frechet *-algebra. Now $\tilde{\alpha}_s(f)(r) = \alpha_s(f(r))$ defines an action $\tilde{\alpha}$ of $\mathbb{R}$ on the Frechet algebra $S(\mathbb{R}, A, \alpha)$ for which, by ([29], p. 189), $C^\infty(S(\mathbb{R}, A, \alpha)) = S(\mathbb{R}, C^\infty(A), \alpha)$ homeomorphically. Note that the embeddings

$$S(\mathbb{R}, C^\infty(A), \alpha) \to S(\mathbb{R}, A, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous; $S(\mathbb{R}, C^\infty(A), \alpha)$ is dense in $S(\mathbb{R}, A, \alpha)$ by ([34], Theorem A.2); and $S(\mathbb{R}, A, \alpha)$ is dense in $L^1(\mathbb{R}, A, \alpha)$; which, in turn, is dense in $C^*(\mathbb{R}, A, \alpha)$.

Now let $\{| \;|_n\}$ be an increasing sequence of submultiplicative seminorms defining the topology of $S(\mathbb{R}, A, \alpha)$. Let $(B_n, | \;|_n)$ be the Hausdorff completion of $S(\mathbb{R}, A, \alpha)$ in $| \;|_n$. Then B_n is a Banach algebra and $S(\mathbb{R}, A, \alpha) = \text{proj.}\lim B_n$. Since $||\alpha_r(x)|| = ||x||$, the action $\tilde{\alpha}$ of $\mathbb{R}$ on $S(\mathbb{R}, A, \alpha)$ extends to a strongly continuous action $\tilde{\alpha}^{(n)}$ of $\mathbb{R}$ by automorphisms of B_n. Let $C^m(B_n)$ be the Banach algebra of all C^m-vectors in B_n for the action of $\tilde{\alpha}^{(n)}$. As noted in ([29], p. 189), $C^n(B_n)$ is dense and spectrally invariant in B_n; and $S(\mathbb{R}, C^\infty(A), \alpha) = \text{proj lim}\, C^n(B_n)$. Let $x \in S(\mathbb{R}, C^\infty(A), \alpha)$, $x = (x_n)$ being a coherent sequence with $x_n \in C^n(B_n)$ for all $n \in \mathbb{N}$. Now

$$sp_{S(\mathbb{R}, C^\infty(A), \alpha)}(x) = \bigcup_n sp_{C^n(B_n)}(x_n) = \bigcup_n sp_{B_n}(x_n) = sp_{S(\mathbb{R}, A, \alpha)}(x).$$

Thus $S(\mathbb{R}, C^\infty(A), \alpha)$ is spectrally invariant in $S(\mathbb{R}, A, \alpha)$; which in turn is spectrally invariant in $C^*(\mathbb{R}, A, \alpha)$ by ([33], Corollary 7.16). Thus each of $S(\mathbb{R}, C^\infty(A), \alpha)$ and $S(\mathbb{R}, A, \alpha)$ are Q-normed algebras in the C^*-norm of $C^*(\mathbb{R}, A, \alpha)$; and hence are Q-algebras in their respective Frechet topologies. Using Lemma 2.10, $E(S(\mathbb{R}, C^\infty(A), \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Proof of Theorem 1.5. Since $C^\infty(B) = B$, the Frechet m-convex algebra $S(\mathbb{R}, B, \alpha)$ is a *-algebra by ([34], Corollary 4.9). Since B is Frechet and sits in the C^*-algebra A, B is *-semisimple. Similarly, since the inclusion $S(\mathbb{R}, B, \alpha) \to C^*(\mathbb{R}, A, \alpha)$ is continuous and one–one, $S(\mathbb{R}, B, \alpha)$ is also *-semisimple. To prove that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$, it is sufficient to prove that any *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ extends to a *-representation $(\tilde{\sigma}) : C^*(\mathbb{R}, A, \alpha) \to B(H_\sigma)$. This would imply that the C^*-norm on $S(\mathbb{R}, B, \alpha)$ induced by the C^*-algebra norm on $C^*(\mathbb{R}, A, \alpha)$ is the greatest (automatically continuous) C^*-seminorm on $S(\mathbb{R}, B, \alpha)$. This is shown below by arguments analogous to those in ([25], Proposition 7.6.4, p. 255).

Let (x_λ) be a bounded approximate identity for A contained in B and which is also a bounded approximate identity for B. For each $n \in \mathbb{N}$, let $f_n \in C_c^\infty(\mathbb{R})$ be such that $0 \leq f_n \leq 1$, $f_n(x) = 1$ for all $x \in [-n, n]$, and supp $f_n \subset [-n-1, n+1]$. Then (f_n) is a bounded approximate identity for $S(\mathbb{R})$ (pointwise multiplication) contained in $C_c^\infty(\mathbb{R})$. The inverse Fourier transforms g_n of f_n constitute a bounded approximate identity for $S(\mathbb{R})$ with convolution. Thus $y_{n,\lambda} = g_n \otimes x_\lambda$ constitute a bounded approximate identity for $S(\mathbb{R}, B, \alpha)$. Given a *-representation $\sigma : S(\mathbb{R}, B, \alpha) \to B(H_\sigma)$ automatically continuous, let $\mathcal{U}(H_\sigma)$ be the group of all unitary operators on H_σ. Define $\pi : B \to B(H_\sigma)$ and $U : \mathbb{R} \to \mathcal{U}(H_\sigma)$ by

$$\pi(x) = \lim_{(n,\lambda)} \sigma(xy_{(n,\lambda)}(\quad)),$$
$$U_t = \lim_{(n,\lambda)} \sigma(\alpha_t(y(\cdot - t))).$$

The limits are taken in the weak sense; and they exist. As in ([25], § 7.6, p. 256), it is verified that π is a *-representation of B; U is a unitary representation of $\mathbb{R}$; $U_t\pi(x)U_t^* = \pi(\alpha_t(x))$ for all $t \in \mathbb{R}$, all $x \in B$; and for all $y \in S(\mathbb{R}, B, \alpha)$, $\sigma(y) = \int \pi(y(t)) U_t \mathrm{d}t$. Now, since $E(B) = A, \pi$ extends to a *-representation $\tilde{\pi} : A \to B(H_\sigma)$ so that $(\tilde{\pi}, U, H_\sigma)$ is a covariant representation of the C^*-dynamical system $(\mathbb{R}, A, \alpha)$. Then $\tilde{\sigma}(y) = \int \tilde{\pi}(y(t))U_t \,\mathrm{d}t$ defines a non-degenerate *-representation of the Banach *-algebra $L^1(\mathbb{R}, B, \alpha)$; and hence extends uniquely to a *-representation $\tilde{\sigma}$ of $C^*(\mathbb{R}, B, \alpha)$. This $\tilde{\sigma}$ is the desired extension of σ. This shows that $E(S(\mathbb{R}, B, \alpha)) = C^*(\mathbb{R}, A, \alpha)$.

Further, suppose that the action α of $\mathbb{R}$ on B is isometric. Then by [29], $L^1(\mathbb{R}, B, \alpha)$ is a *-algebra, which is a Frechet m-convex *-algebra; and

$$S(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, B, \alpha) \to L^1(\mathbb{R}, A, \alpha) \to C^*(\mathbb{R}, A, \alpha)$$

are continuous embeddings with dense ranges. It follows that $E(L^1(\mathbb{R}, B, \alpha) = C^*(\mathbb{R}, A, \alpha)$. This completes the proof of the theorem.

Actions on topological spaces

(a) Let M be a locally compact Hausdorff space. Let $\sigma : M \to [0, \infty)$ be a Borel function, $\sigma(m) \geq 1$ for all $m \in M$. Assume that σ is bounded on compact subsets of M. Following ([34], § 5), let

$$C^\sigma(M) = \{f \in C_0(M) : ||\sigma^d f|| < \infty \text{ for all } d \in \mathbb{N}\},$$

called the algebra of continuous functions on M vanishing at infinity σ-rapidly. It is shown in [34] that $C^\sigma(M)$ is a Frechet m-convex *-algebra with the topology defined by seminorms

$$||\sigma^d f|| = \sup\{|(\sigma(x))^d f(x)| : x \in M\}, \quad \mathrm{d} \in \mathbb{N};$$

and that $C_c(M) \to C^\sigma(M) \to C_0(M)$ are continuous embeddings with dense ranges. Thus $E(C^\sigma(M)) = C_0(M)$. In fact, $C^\sigma(M)$ is an ideal in $C_0(M)$; hence inverse closed in $C_0(M)$; and so is a Q-algebra.

(b) Let G be a Lie group acting on M. If $f \in C^\sigma(M)$, define $\alpha_g(f)(m) = f(g^{-1}m)$. By ([34], § 5), if σ is uniformly G-translationally equivalent (in the sense that for every

compact $K \subset G$, there exists $l \in \mathbb{N}$ and $C > 0$ such that $\sigma(gm) \leq C\sigma(m)^l$ for all $g \in G$, $m \in M$), then $g \to \alpha_g$ defines a strongly continuous action of G by continuous *-automorphisms of $C^\sigma(M)$. Then the space $C^\infty(C^\sigma(M))$ consisting of C^∞-vectors for the action α of G on $C^\sigma(M)$ is an m-convex Frechet *-algebra with a C^*-enveloping algebra and $E(C^\infty(C^\sigma(M)) = C_0(M)$.
(c) In particular, let $G = \mathbb{R}$, M be a compact C^∞-manifold, and let the action of $\mathbb{R}$ on M be smooth. Then the induced action α on $C(M)$ is smooth, so that $\alpha_r(C^\infty(M)) \subseteq C^\infty(M)$ for all $r \in \mathbb{R}$. It follows from Theorem 5.1 that $E(S(\mathbb{R}, C^\infty(M), \alpha) = C^*(\mathbb{R}, C(M), \alpha)$ the covariance C^*-algebra.

6. The Pedersen ideal of a C^*-algebra

Let A be a non-unital C^*-algebra. Let K_A be its Pedersen ideal. It is a hereditary, minimal dense *-ideal of A. For $a \in A$, let $L_a = (Aa)^-$, $R_a = (aA)^-$, I_a be the closed *-ideal of A generated by aa^*. Since $a \in L_a \bigcap R_a$, $aa^* \in I_a$. Let $K_A^+ = K_A \bigcap A^+$ be the positive part of K_A endowed with the order relation induced from that of A^+. Let $K_A^{nc} = \bigcup\{I_a : a \in K_A^+\}$.

Lemma 6.1. *K_A^{nc} is a dense *-ideal of A containing K_A; and $A = C^*$-ind lim $\{I_a : a \in K_A^+\}$.*

Proof. Let $a \in K_A^+$. Then $a^2 = aa^* \in I_a$; and I_a being a C^*-algebra, $a = (a^2)^{1/2} \in I_a$. Thus $K_A^+ \subseteq K_A^{nc}$. Observe that for any $x = x^* \in K_A$, $x \in I_x$. Indeed, $x^2 \in K_a^+$; hence $x^2 \in I_x$ and $|x| \in I_x$. But than taking the Jordan decomposition $x = x^+ - x^-$ in A, $(x^+)^2 = (x^+)^2 + x^+x^- = x^+|x| \in I_x$; so that $x^+ \in I_x$, $x^- \in I_x$, and $x \in I_x$. In particular, $x^2 \in I_{x^2}$ and $|x| \in I_{x^2}$. By repeating this argument, $x \in I_{x^2} \subset K_A^{nc}$ for any $x = x^* \in K_A$. It follows that $K_A \subset K_A^{nc}$. Now, by ([28], Lemma 1), $0 \leq a \leq b$ in A implies $L_a \subseteq L_b$, $R_a \subseteq R_b$ and $I_a \subseteq I_b$; and $K_A = \bigcup\{L_a : a \in K_A^+\} = \bigcup\{R_a : a \in K_A^+\}$. The family $\{I_a : a \in K_A^+\}$ forms an inductive system of C^*-algebras; and C^*-ind $\lim\{I_a : a \in K_A^+\} = (\bigcup\{I_a : a \in K_A^+\})^- = A$, $(\quad)^-$ denoting the norm closure. This proves the lemma.

Let t_1 (respectively t_2) be the finest locally convex linear topology (respectively finest locally m-convex topology) on K_A^{nc} making continuous the embeddings $I_a \to K_A^{nc}$, where $a \in K_A^+$. Then (K_A^{nc}, t_1) (respectively (K_A^{nc}, t_2)) is the *linear topological inductive limit* (respectively *topological algebraic inductive limit*) of $\{I_a : a \in K_A^+\}$ ([21], ch. IV).

Proof of Theorem 1.6. In the present set up, ([21], p. 115, 118, 125) implies that $t_1 = t_2$, equal to τ say, and (K_A^{nc}, τ) is a complete m-barrelled locally m-convex *-algebra; and the $\|\ \|$-topology on K_A^{nc} is coarser than τ. Since K_A^{nc} is an ideal, it is inverse closed in its $\|\ \|$-completion A, and hence $(K_A^{nc}, \|\ \|)$ and $(K_A, \|\ \|)$ are Q-algebras. This implies that any *-homomorphism from K_A^{nc} into $B(H)$ for a Hilbert space H is $\|\ \|$-continuous and extends uniquely to A. Thus $\|\ \|$ is the greatest C^*-seminorm on K_A^{nc}. To show that $\|\ \|$ is the greatest τ-continuous C^*-seminorm on K_A^{nc} so that $E(K_A^{nc}) = A$, it is sufficient to show that (K_A^{nc}, τ) is a Q-algebra. To that end, in view of ([23], Lemma E.2), we show that 0 is a τ-interior point of the set $(K_A^{nc})_{-1}$ of quasiregular elements of K_A^{nc}. Note that, by ([21], p. 114), basic τ-neighbourhoods of 0 in K_A^{nc} are precisely of the form $V = |c\,o|$ $\{\bigcup(U_a : a \in K_A^+)\}$, where $|c\,o|$ denotes the absolutely convex hull and U_a denotes a convex balanced neighbourhood of 0 in $(I_a, \|\ \|)$. For any $a \in K_A^+$, $(I_a, \|\ \|)$ is a Q-algebra, and being an ideal in A, $(I_a)_{-1} = A_{-1} \bigcap I_a$. Hence, for the zero neighbourhood $U_a = \{x \in I_a : \|x\| \leq 1\}$ in $(I_a, \|\ \|)$,

$$U_a \subseteq (I_a)_{-1} = (K_A^{nc})_{-1} \bigcap I_a \subset (K_A^{nc})_{-1}; \text{ and}$$

$$|c\,o|\Big\{\bigcup(U_a : a \in K_A^+)\} = \{x \in K_A^{nc} : ||x|| \leq 1\Big\} = U \text{ (say)}$$

is a zero neighbourhood in (K_A^{nc}, τ) contained in $(K_A^{nc})_{-1}$. It follows that (K_A^{nc}, τ) and (K_A, τ) are Q-algebras. Now, as in the proof of ([28], Theorem 4), $K_A^{nc} = \cup\{I_{e_\lambda}\}$, (e_λ) being a bounded approximate identity for A contained in K_A. Thus if A has countable bounded approximate identity, then K_A^{nc} is an LFQ-algebra; and τ is the finest (unique) locally convex topology on K_A^{nc} such that for each λ, $\tau|_{I_{e_\lambda}}$ is the norm topology.

7. The groupoid C^*-algebra

We follow the terminology and notations of [31]. Let G be a locally compact groupoid, i.e., a locally compact space G with a specified subset $G^2 \subseteq G \times G$ so that two continuous maps $G \to G$, $x \to x^{-1}$, and $G^2 \to G$, $(x, y) \to xy$ are defined satisfying $(xy)z = x(yz)$, $x^{-1}(xy) = y$ and $(zx)x^{-1} = z$. The unit space of G is $G^o = \{xx^{-1} : x \in G\} = \{x^{-1}x : x \in G\}$. Let $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$. Assume that there exists a left Haar system $\{\lambda^u : u \in G^o\}$ on G, i.e., a family of measures λ^u on G such that $\operatorname{supp}\lambda^u = r^{-1}(u)$; for each $f \subseteq C_c(G)$, $u \to \int f \mathrm{d}\lambda^u$ is continuous; and for all $x \in G$ and $f \in C_c(G)$, $\int f(xy)\mathrm{d}\lambda^{d(x)}(y) = \int f(y)\mathrm{d}\lambda^{r(x)}(y)$. Let σ be a continuous 2-cocycle in $Z^2(G, T)$. Let t denote the usual inductive limit topology on $C_c(G)$. Then $(C_c(G), t)$ is a topological *-algebra with jointly continuous multiplication

$$f * g(x) = \int f(xy)g(y^{-1})\sigma(xy, y^{-1})\mathrm{d}\lambda^{d(x)}(y)$$

and the involution $f^*(x) = (f(x^{-1})\sigma(x, x^{-1}))^-$ ([31], Proposition II.1.1, p. 48). The I-norm on $C_c(G, \sigma)$ is $||f||_I = \max(||f||_{I,r}, ||f||_{I,l})$, where

$$||f||_{I,r} = \sup\left\{\int |f|\mathrm{d}\lambda^u : u \in G^o\right\}, \quad ||f||_{I,l} = \sup\left\{\int |f|\mathrm{d}\lambda_u : u \in G^o\right\},$$

$\lambda_u = (\lambda^u)^{-1}$ being the image of λ^u by the inverse map $x \to x^{-1}$ ([31], p. 50). Then $||\ \ ||_I$ is a submultiplicative *-norm on $C_c(G, \sigma)$. The L^1-algebra of (G, σ) is the completion $A = (C_c(G, \sigma), ||\ \ ||_I)$, a Banach *-algebra. For f in $C_c(G, \sigma)$, define $||f|| = \sup\{||\pi(f)||\}$, π running over all weakly continuous, non-degenerate *-representations $\pi : (C_c(G, \sigma), t) \to B(H_\pi)$ satisfying $||\pi(f)|| \leq ||f||_I$ for all f. Then $||\ \ ||$ defines a C^*-norm on $C_c(G, \sigma)$; and the groupoid C^*-algebra of (G, σ) is $C^*(G, \sigma) = (C_c(G, \sigma), ||\ \ ||)^-$, the completion. The following can be proved using cyclic decomposition and ([31], Corollary II.1.22, p. 72).

PROPOSITION 7.1

Let G be second countable having sufficiently many non-singular G-Borel sets. Then $E(C_c(G, \sigma)) = C^*(G, \sigma)$.

8. The universal *-algebra on generators with relations

Let G be any set. Let $F(G)$ be the free associative *-algebra on *generators* G, viz., the *-algebra of all polynomials in non-commuting variables $G \coprod G^*$ where $G^* = \{x^* : x \in G\}$. Let R be a collection of statements about elements of G, called *relations* on G, assumed throughout to be such that they make sense for elements of a locally m-convex *-algebra. A *Banach (respectively C^*-) representation* of (G, R) is a function ρ from G to a Banach *-algebra (respectively a C^*-algebra) $\rho : G \to A$ such that $\{\rho(g) : g \in G\}$ satisfies the relations R in A. Let $\mathrm{Rep}_B(G, R)$ (respectively $\mathrm{Rep}(G, R)$) be the set of all Banach representations (respectively C^*-representations) of (G, R). Motivated by ([27], Definition 1.3.4), it is assumed that R satisfies the following.

(i) The function $\rho : G \to \{0\}$ is a Banach representation of (G, R).
(ii) Let $\rho : G \to A$ be a representation of (G, R) in a Banach *-algebra A. Let B be a closed *-subalgebra of A containing $\rho(G)$. Then ρ is a representation of (G, R) in B.
(iii) Let ρ be a representation of (G, R) in a complete locally m-convex *-algebra A. Let $\phi : A \to B$ be a continuous *-homomorphism into a Banach *-algebra B. Then $\phi \circ \rho$ is a representation of (G, R) in B.
(iv) Let A be a complete locally m-convex *-algebra expressed as an inverse limit of Banach *-algebras viz. $A = \mathrm{proj.}\lim A_p$. Let $\pi_p : A \to A_p$ be the natural maps. Let $\rho : G \to A$ be a function such that for all p, $\pi_p \circ \rho$ is a representation of (G, R). Then ρ is a representation of (G, R).

DEFINITION 8.1

(a) (Blackadar) (G, R) is *C^*-bounded* if for each g in G, there exists a scalar $M(g)$ such that $||\rho(g)|| \leq M(g)$ for all $\rho \in \mathrm{Rep}\,(G, R)$.
(b) (Blackadar) (G, R) is *C^*-admissible* if it is C^*-bounded and the following holds.
(bC^*) If (ρ_α) is a family of representations $\rho_\alpha : G \to B(H_\alpha)$ of (G, R) on Hilbert spaces H_α, then $\oplus\rho_\alpha : G \to B(\oplus H_\alpha)$ is a representation of (G, R).
(c) (G, R) is *weakly Banach admissible* if given finitely many representations $\rho_i : G \to A_i$ $(1 \leq i \leq n)$ of G into Banach *-algebras, the map $g \to \rho_1(g) \oplus \rho_2(g) \oplus \ldots \oplus \rho_n(g)$ is a representation of (G, R) in $\oplus A_i$. (G, R) is *weakly C^*-admissible* [27] if this holds with Banach algebras replaced by C^*-algebras.

The class of relations making sense for elements of a Banach *-algebra is smaller than the class of relations making sense for elements of a C^*-algebra. The usual algebraic relations involving the four elementary arithmetic operations on elements of G and G^* do make sense for Banach *-algebras; but relations like $x^+ \geq x^-$ for $x = x^*$ in G, or like $|x| \geq |y|$ for elements x, y in G, which make sense for C^*-algebras, fail to make sense for Banach *-algebras. We refer to [27] for relations satisfying (i)–(iv) except (ii). The relation (suggested by the referee). "The elements a, b and c generate A" fails to satisfy Definition 8.1(c). Our definition of weakly Banach admissible relations is very much ad hoc aimed at exploring a method of constructing non-abelian locally m-convex *-algebras.

Lemma 8.2. (a) *Let (G, R) be weakly Banach admissible. Then there exists a complete m-convex *-algebra $A(G, R)$ and a representation $\rho : G \to A(G, R)$ such that given any representation $\sigma : G \to B$ into a complete m-convex *-algebra B, there exists a continuous *-homomorphism $\phi : A(G, R) \to B$ satisfying $\phi \circ \rho = \sigma$.*
(b) ([27], Proposition 1.3.6). *Let (G, R) be weakly C^*-admissible. Then there exists a pro-C^*-algebra $C^*(G, R)$ and a representation $\rho_\infty : G \to C^*(G, R)$ such that given any representation $\sigma : G \to B$ of G into a pro-C^*-algebra B, there exists a continuous *-homomorphism $\phi : C^*(G, R) \to B$ such that $\phi \circ \rho_\infty = \sigma$.*

Proof. (a) Let $K = K(F(G))$ be the set of all submultiplicative *-seminorms p on $F(G)$ of the form $p(x) = ||\sigma(x)||$, σ running through all Banach representations of G. For $p \in K$, let $N_p = \{x \in F(G) : p(x) = 0\}$ and $N_a = \cap\{N_p : p \in K\}$ a *-ideal of $F(G)$. Let $B = F(G)/N_a$. Take $\tilde{p}(x + N_a) = p(x)$. Let t be the Hausdorff topology defined by $\{\tilde{p} : p \in K\}$. Let $A(G,R)$ be the completion of (B,t). Let $\rho : G \to A(G,R)$ be $\rho(g) = g + N_a$.

Claim 1. ρ is a representation of G in $A(G,R)$.

Let q be any t-continuous submultiplicative *-seminorm on $A(G,R)$. Let A_q be the Banach *-algebra obtained by the Hausdorff completion of $(A(G,R),q)$. By (iv) above, it is sufficient to prove that $\pi_q \circ \rho : G \to A_q$ is a representation of (G,R). Since q is t-continuous, there exists $p_1, p_2, \ldots, p_k$ in K such that $q(x) \le c \max p_i(x)$ for all $x \in F(G)$; and each p_i is of form $p_i(x) = ||\sigma_i(x)||$, $\sigma_i : G \to A(i)$ being a representation into some Banach algebra $A(i)$. By (c) of Definition 8.1, there exists a Banach *-algebra B and a representation $\sigma : G \to B$ such that $q(x) \le ||\sigma(x)||$ for all $x \in F(G)$. In view of (ii), we assume that B is generated by $\sigma(G)$. Let $\phi : B \to A_q$ be $\phi(\sigma(x)) = (x + N_a) + \ker q = \pi_q(\rho(x))$. Then ϕ is well defined, continuous and $\phi \circ \sigma = \pi_q \circ \rho$. By the assumption (iii) above, $\phi \circ \sigma$ is a representation of G.

Claim 2. Given any representation $\sigma : G \to C$ into a complete m-convex *-algebra C, there exists a unique continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.

Let $C = \text{proj.}\lim C_\alpha$, an inverse limit of Banach *-algebras C_α, $\pi_\alpha : C \to C_\alpha$ being the projection maps. By (iii) of above, $\pi \circ \sigma$ is a Banach representation of (G,R). By the construction of $A(G,R)$, there exist continuous *-homomorphisms $\phi_\alpha : A(G,R) \to C_\alpha$ such that $\phi_\alpha \circ \rho = \pi_\alpha \circ \sigma$. Hence by the definition of an inverse limit, there exists a continuous *-homomorphism $\phi : A(G,R) \to C$ such that $\phi \circ \rho = \sigma$.
(b) We only outline the (needed) construction of $C^*(G,R)$ from [27]. Let S be the set of all C^*-seminorms on $F(G)$ of form $q(x) = ||\sigma(x)||$, σ running over all representations of G into C^*-algebras. Let $N_q = \{x \in F(G) : q(x) = 0\}$ and $N = \cap\{N_q : q \in S\}$. Let τ be the pro-C^*-topology on $F(G)/N$ defined by $\tilde{q}(x + N) = q(x)$, $q \in S$. Then $C^*(G,R)$ is the completion of $(F(G)/N, \tau)$. The map $\rho_\infty : G \to C^*(G,R)$ where $\rho_\infty(x) = x + N$ is the canonical representation.

The following brings out the essential point in arguments in claim 1 above.

Lemma 8.3. *There exists a natural one-to-one correspondence between* $\text{Rep}_B(G,R)$ (*respectively* $\text{Rep}\,(G,R)$) *and t-continuous Banach* **-representations (respectively* C^**-algebra representations) of* $A(G,R)$.

Lemma 8.4. $\text{srad}\,(A(G,R)) \bigcap (F(G)/N_a) = \text{srad}\,(F(G)/N_a) = \{x + N_a : x \in N\}$.

Proof. Let $C = F(G)/N_a$. Let $x + N_a \in C \bigcap \text{srad} A$. Then $\pi(x + N_a) = 0$ for all continuous *-homomorphisms $\pi : A \to B(H_\pi)$. By Lemma 8.3, $p(x) = 0$ for all $p \in S$. Hence $x \in N$, and $x + N_a \in \text{srad}\,(F(G)/N_a)$. Conversely, let $x \in N$. Then $q(x) = 0$ for all $q \in S$. Again by Lemma 8.3, $||\pi(x + N_a)|| = 0$ for all $\pi \in R(A)$, hence $x + N_a \in \text{srad} A$.

Proof of Theorem 1.7. (1) Let $A = A(G,R)$. Let $\phi : (F(G)/N_a, t) \to (F(G)/N_a, \tau)$ be $\phi(X + N_a) = x + N$. Then ϕ is a well defined, continuous *-homomorphism; hence extends as a continuous surjective *-homomorphism $\phi : A \to C^*(G, R)$. The universal property of $C^*(G, R)$, Lemma 8.3 and weak Banach admissibility of R imply the following whose proof we omit.

Assertion 1. Given any continuous *-homomorphism $\pi : A(G, R) \to B$ to a pro-C^*-algebra B, there exists a continuous *-homomorphism $\tilde{\pi} : C^*(G, R) \to B$ such that $\pi = \tilde{\pi} \circ \phi$.

$$\begin{tikzcd}
A(G,R) \arrow[r, "\pi"] \arrow[dr, "\phi"] & B \\
G \arrow[u, "\rho"] \arrow[r, "\rho_\infty"'] & C^*(G,R) \arrow[u, "\tilde{\pi}"']
\end{tikzcd}$$

By applying the above to the maps ϕ and $j : A \to E(A)$, $j(x) = x + \operatorname{srad}(A)$, it follows that there exist continuous *-homomorphisms $\tilde{\phi} : E(A) \to C^*(G, R)$ and $\tilde{j} : C^*(G, R) \to E(A)$ such that the following diagrams commute.

$$\begin{tikzcd}
 & A \arrow[dl, "j"'] \arrow[dr, "\phi"] & \\
E(A) \arrow[rr, shift left, "\tilde{\phi}"] & & C^*(G,R) \arrow[ll, shift left, "\tilde{j}"]
\end{tikzcd}$$

Assertion 2. The maps $\tilde{\phi}$ and $\tilde{j}$ are inverse of each other.

Indeed, $\tilde{j}$ is one–one on $F(G)/N$. For given $x \in F(G)$,

$$0 = \tilde{j}(x + N) = \tilde{j} \circ \phi(x + N_a) = j(x + N_a)$$

which implies $(x + N_a) + \operatorname{srad}(A) = 0$ and $(x + N_a) \in \operatorname{srad}(A)$. Hence $x \in N$ by Lemma 8.4, so that $x + N = 0$. Similarly $\tilde{\phi}$ is one–one on $F(G)/N$. Also,

$$\begin{aligned}
(\tilde{\phi} \circ \tilde{j})(x + N) &= \tilde{\phi} \circ \tilde{j} \circ \phi(x + N_a) \\
&= \tilde{\phi} \circ j(x + N_a) = \phi(x + N_a) = x + N,
\end{aligned}$$

which implies that $\tilde{\phi} = \tilde{j}^{-1}$ on $F(G)/N_a$; and $\tilde{j} = \tilde{\phi}^{-1}$ on $F(G)/N_a + \operatorname{srad} A$. By continuity and density, $\tilde{\phi}$ establishes a homeomorphic *-isomorphism $\tilde{\phi} : E(A) \to C^*(G, R)$ with $\tilde{\phi}^{-1} = \tilde{j}$.

(2) Let (G, R) be C^*-admissible. Then $\sup\{||\sigma(x)|| : \sigma \in \operatorname{Rep}(G, R)\} < \infty$; and $\pi = \oplus\{\sigma : \sigma \in \operatorname{Rep}(G, R)\} \in \operatorname{Rep}(G, R)$. Thus $q(x) = ||\pi(x)||$ defines the greatest member of $S(F(G))$, q is a C^*-norm, and it is the greatest t-continuous C^*-seminorm on $F(G)/N$. Thus the topology τ on $C^*(G, R)$ is determined by q. Conversely suppose that $C^*(G, R)$ is a C^*-algebra so that $||z||_\infty = \sup\{q(z) : q$ is a continuous C^*-seminorm on $C^*(G, R)\} < \infty$ for all $z \in C^*(G, R)$, and τ is determined by the C^*-norm $||\ ||_\infty$. Let $p_\infty(x) = ||x + N||_\infty = \sup\{q(x) : q \in S\}$ for all $x \in F(G)$. Then $p_\infty \in S$ and $\ker p_\infty = N$. There exists a C^*-representation $\sigma : G \to C$ such that $p_\infty(g) = ||\sigma(g)||$ for all $g \in G$; and this defines a continuous C^*-representation $\sigma : C^*(G, R) \to C$. It is clear that R is C^*-bounded. We verify (bC^*) of Definition 8.1. Let $\{\rho_\alpha\} \subseteq \operatorname{Rep}(G, R)$ with $\rho_\alpha : G \to B(H_\alpha)$

for some Hilbert space H_α. Let $H = \oplus H_\alpha$. For $x \in F(G)$, let $\lambda(x) = \oplus \rho_\alpha(x)$. By the C^*-boundedness of (G, R), $\lambda(x) \in B(H)$. This defines a *-homomorphism $\lambda : F(G) \to B(H)$ satisfying $||\lambda(x)|| = \sup ||\rho_\alpha(x)|| \le p_\infty(x)$ for all $x \in F(G)$. Since ker $p_\infty = N$, λ factors to a *-representation $\tilde{\lambda} : F(G)/N \to B(H)$ satisfying $||\tilde{\lambda}(z)|| \le ||z||_\infty$. As $||\ \ ||_\infty$ is τ-continuous, so is $\tilde{\lambda}$. By lemma 8.3, $\{\lambda(g) : g \in G\}$ satisfies the relations R in $B(H)$. Thus (G, R) is C^*-admissible.

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