Proc. Indian Acad. Sci. (Math. Sci.) Vol. 115, No. 4, November 2005, pp. 437–444.

A note on generalized characters

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MS received 23 November 2004; revised 9 July 2005

Abstract. For a compactly generated LCA group G, it is shown that the set $H(G)$ of all generalized characters on G equipped with the compact-open topology is a LCA group and $H(G) = \widehat{G}$ (the dual group of G) if and only if G is compact. Both results fail for arbitrary LCA groups. Further, if G is second countable, then the Gel'fand space of the commutative convolution algebra $C_c(G)$ equipped with the inductive limit topology is topologically homeomorphic to $H(G)$.

Keywords. Compactly generated LCA group; character; generalized character; Gel'fand space; commutative topological algebra.

1. Introduction

Throughout, let G be a LCA group with Haar measure λ and let $\widehat{G}$ denote the dual group of G, i.e., the set of all characters on G. Then it is well-known that $\widehat{G}$ is a LCA group in compact-open topology. A *generalized character* on G is a continuous function $\alpha: G \longrightarrow \mathbb{C}^{\bullet}$, where $\mathbb{C}^{\bullet} = \mathbb{C}\backslash\{0\}$ such that $\alpha(s+t) = \alpha(s)\alpha(t), s, t \in G$. Let $H(G)$ denote the set of all generalized characters on G equipped with the compact-open topology. For $\alpha, \beta \in H(G)$, define $(\alpha + \beta)(s) = \alpha(s)\beta(s), s \in G$. Then $(H(G), +)$ is an abelian topological group (23.34(b) of [4]). It is straightforward to verify that $H(\mathbb{Z}) \cong (\mathbb{C}^{\bullet}, \times)$ and $H(\mathbb{T}) \cong (\mathbb{Z}, +)$, where $\mathbb{T}$ is the unit circle in $\mathbb{C}$.

Let $C_c(G)$ denote the set of all complex-valued continuous functions on G with compact support. Then $C_c(G)$ is a commutative algebra with respect to the usual convolution product. Let τ denote the inductive limit topology on $C_c(G)$. Then, by Lemma 2.1, p. 114 of [6], $(C_c(G), \tau)$ is a commutative topological algebra.

In this paper our main goal is to show that if G is compactly generated, then $H(G)$ is a LCA group and that $H(G) = \widehat{G}$ if and only if G is compact. Both results fail for LCA groups. The results appear to be a mathematical folklore; however we failed to find a proof in the literature. In fact, the present note arises out of our investigations of uniform norms in Beurling algebras and weighted measure algebras [1, 2]. As an application we show that if, further, G is second countable, then the Gel'fand space $\Delta(C_c(G))$ of $C_c(G)$ is homeomorphic to $H(G)$; in particular, $\Delta(C_c(G))$ is a locally compact space.

2. Generalized characters

Lemma 2.1. *Let $m > 1$ be an integer and let $0 < \varepsilon < 1/m$. Then there exists a natural number N such that, for each complex number z satisfying $\varepsilon \leq |z-1| \leq 1/m$, there exists $1 \leq k \leq N$ such that $|z^k - 1| > 1/m$.*

Proof. For $r > 0$ and for $z \in \mathbb{C}$, let $\Gamma(z, r)$ denote the circle with radius r and center z. For $\delta > 0$, let $L_\delta := \{re^{i\delta}: r > 0\}$, the open ray with angle δ. Choose $0 < \delta < \pi/2$ such that L_δ cuts the circle $\Gamma(1, \varepsilon)$ in two points $z_0 = r_0 e^{i\delta}$ and $z_1 = r_1 e^{i\delta}$, where $r_0 < 1 < r_1$.

Now fix $z = re^{i\theta}$ such that $\varepsilon \le |z-1| \le 1/m$. Then $|\theta| < \pi/2$. Without loss of generality, we may assume that $\theta \ge 0$. Then we have the following three possibilities:

Case (i). $\theta \ge \delta$. Choose $n_1 \in \mathbb{N}$ such that $n_1\delta \le \pi/2$ and $L_{n_1\delta}$ does not intersect the circle $\Gamma(1, 1/m)$. Then there exists $1 \le k \le n_1$ such that $|z^k - 1| > 1/m$.

Case (ii). $r < r_0$. Choose $n_2 \in \mathbb{N}$ such that $r_0^{n_2} < 1 - 1/m$. Then $|z^{n_2} - 1| \ge 1 - |z|^{n_2} = 1 - r^{n_2} > 1 - r_0^{n_2} \ge 1/m$.

Case (iii). $r_1 < r$. Choose $n_3 \in \mathbb{N}$ such that $r_1^{n_3} \ge 1 + 1/m$. Then $|z^{n_3} - 1| \ge |z|^{n_3} - 1 = r^{n_3} - 1 > r_1^{n_3} - 1 \ge 1/m$.

Finally take $N = \max\{n_1, n_2, n_3\}$. Then N has the required property. □

Theorem 2.2. *Let G be a compactly generated LCA group. Then*

(i) *$H(G)$ is a LCA group.*
(ii) *$H(G) = \widehat{G}$ if and only if G is compact.*

Proof.

(i) Fix an integer $m > 1$. Define $V_m := \{z \in \mathbb{C}: |z-1| < 1/m\}$. Since G is a compactly generated LCA group, there exists a neighbourhood U of 0 in G such that its closure $\overline{U}$ is compact and it generates G due to Theorem 5.13 of [4]. Take $T_m := N(\overline{U}, V_m) := \{\alpha \in H(G): \alpha(\overline{U}) \subseteq V_m\}$. Then T_m is a neighbourhood of the identity 1_G in $H(G)$. First we show that T_m is equicontinuous at 0 in G. Let $\varepsilon > 0$. If $\varepsilon \ge 1/m$, then $V := U$ is a neighbourhood of 0 in G such that

$$s \in V \text{ and } \alpha \in T_m \implies |\alpha(s) - \alpha(0)| = |\alpha(s) - 1| < 1/m \le \varepsilon.$$

So we may assume that $\varepsilon < 1/m$. Then, by Lemma 2.1, one can find an integer N such that, for each $\varepsilon \le |z-1| \le 1/m$, there exists $1 \le k \le N$ such that $|z^k - 1| > 1/m$. Choose a neighbourhood W of 0 in G such that $\sum_{k=1}^{N} W_k \subseteq U$, where $W_k = W$. Suppose, if possible, there exist $t \in W$ and $\alpha \in T_m$ such that $|\alpha(t) - 1| \ge \varepsilon$. Then, by the definition of N, there exists $1 \le k \le N$ such that $|\alpha(kt) - 1| = |\alpha(t)^k - 1| > 1/m$. On the other hand, $kt \in U$ and so $|\alpha(kt) - 1| \le 1/m$. This is a contradiction. Hence, we have

$$s \in W \text{ and } \alpha \in T_m \implies |\alpha(s) - 1| < \varepsilon.$$

This proves that T_m is equicontinuous at 0 in G. Finally, let $t \in G$ be arbitrary. Since G is generated by U, there exist $t_1, \ldots, t_p \in U$ such that $t = t_1 + \cdots + t_p$. Then, for each $\alpha \in T_m$,

$$|\alpha(t)| = |\alpha(t_1)| \ldots |\alpha(t_p)| \le (1 + 1/m)^p.$$

By the above argument, one can choose a neighbourhood W of 0 in G such that

$$s \in W \text{ and } \alpha \in T_m \implies |\alpha(s) - \alpha(0)| < \frac{\varepsilon}{(1+1/m)^p}.$$

Hence

$$|\alpha(s+t)-\alpha(t)| = |\alpha(s)-\alpha(0)||\alpha(t)| \le |\alpha(s)-1|(1+1/m)^p < \varepsilon.$$

This proves that T_m is equicontinuous. So its closure $\mathrm{Cl}_p(T_m)$ in the pointwise topology is equicontinuous (p. 17 of [5]). Let $\mathrm{Cl}_c(T_m)$ denote the closure of T_m in the compact-open topology. Then $\mathrm{Cl}_c(T_m) \subseteq \mathrm{Cl}_p(T_m)$. Hence $\mathrm{Cl}_c(T_m)$ is equicontinuous.

Now take $t \in G$. Then $t = t_1 + \cdots + t_p$ for some $t_1, \ldots, t_p \in U$. Then $|\alpha(t)| = |\alpha(t_1)| \cdots |\alpha(t_p)| \le (1+|\alpha(t_1)-1|)\cdots(1+|\alpha(t_p)-1|) \le (1+1/m)^p$ for each $\alpha \in T_m$. Similarly, $|\alpha(t)| = |\alpha(t_1)|\cdots|\alpha(t_p)| \ge (1-|\alpha(t_1)-1|)\cdots(1-|\alpha(t_p)-1|) \ge (1-1/m)^p$ for each $\alpha \in T_m$. Hence the closure of the set $T_m(t) := \{\alpha(t) \colon \alpha \in T_m\}$ is compact in $\mathbb{C}^\bullet$. So, by Ascoli's theorem, $\mathrm{Cl}_c(T_m)$ is compact. This proves that $H(G)$ is a LCA group.

(ii) Let G be compact and let $\alpha \in H(G)$. Since α is a continuous group homomorphism, $\alpha(G)$ is a compact subgroup of $(\mathbb{C}^\bullet, \times)$. Hence $\alpha(G)$ is contained in the unit circle. So $\alpha \in \widehat{G}$. For the converse, assume that $H(G) = \widehat{G}$ and G is compactly generated. Then, by Theorem 9.8 of [4], G is topologically isomorphic to $\mathbb{R}^m \times \mathbb{Z}^n \times K$ for some non-negative integers m, n and some compact group K. Then $\widehat{G} = H(G) \cong H(\mathbb{R}^m) \oplus H(\mathbb{Z}^n) \oplus H(K)$ due to 23.34(c) of [4]. This implies that we must have $m = n = 0$. So $G = K$ is compact. □

Remark 2.3. The following is an alternative proof of Theorem 2.2(i). By the structure theory, a compactly generated LCA group G is a direct product of $\mathbb{R}^n$, $\mathbb{Z}^m$, and a compact group. By 23.34(c) of [4], $H(G_1 \times G_2)$ is canonically homeomorphic to $H(G_1) \times H(G_2)$. So it is enough to show that $H(G)$ is locally compact for $G = \mathbb{Z}$ and $G = \mathbb{R}$. It is easy to see for $G = \mathbb{Z}$. Observe that every continuous homomorphism $\psi \colon \mathbb{R} \longrightarrow \mathbb{C}$ is differentiable and satisfies $\psi'(t) = \psi(0)\psi(t), t \in \mathbb{R}$, and so $\psi(t) = \exp(zt)$ for a unique complex number z. Thus the map $\Lambda \colon H(\mathbb{R}) \longrightarrow (\mathbb{C}, +)$ is a bijective map. For $0 < \varepsilon < 1$, let $W_{n,\varepsilon} = \{z \colon |\mathrm{e}^{zx} - 1| < \varepsilon, x \in [-n, n]\}$. Then it is easy to see that

$$W_{n,\varepsilon} \subseteq \{\alpha + i\beta \colon |\alpha| < (1/n)\log(1+\varepsilon), |\beta| < (\cos^{-1} u_{n,\varepsilon})/n\},$$

where $u_{n,\varepsilon} = (\mathrm{e}^{-2|\alpha|n} + 1 - \varepsilon^2)/(2\mathrm{e}^{|\alpha|n})$. Thus the mapping Λ is open. Now for $0 < \delta < 1$,

$$\{\alpha + i\beta \colon |\alpha| < \log(1+\delta/2)/n, |\beta| < \delta/2\} \subseteq W_{n,\varepsilon}.$$

So Λ is continuous. This completes the proof. □

Examples 2.4. The following two examples show that the above theorem is not true for arbitrary LCA groups.

(i) Let $G = \{\overline{n} = (n_1, \ldots, n_k, 0, 0, \ldots) \colon k \in \mathbb{N} \text{ and } n_i \in \mathbb{Z}\}$ with the co-ordinatewise addition and the discrete topology. Then $H(G) \cong \mathbb{C}^{\bullet\mathbb{N}}$ with the pointwise topology. Then $H(G)$ is not a LCA group.

(ii) Let G be an infinite abelian group having all elements of finite order and the topology being the discrete topology. Let $\alpha \in H(G)$ and let $s \in G$. Then there exists a natural number n such that $ns = 0$ and so $\alpha(s)^n = \alpha(ns) = \alpha(0) = 1$, i.e., $|\alpha(s)| = 1$. Hence $\alpha \in \widehat{G}$. Thus $H(G) = \widehat{G}$ and G is not compact.

3. Gel'fand space of $C_c(G)$

For $f \in C_c(G)$ and $t \in G$, let $(\tau_t f)(s) = f(s-t)$, $t \in G$. We know that, for $f \in C_c(G)$, the map $\Lambda_f \colon G \longrightarrow (C_c(G), \|\cdot\|_1)$; $s \longmapsto \tau_s f$ is continuous, where $\|\cdot\|_1$ is the L^1-norm. We prove the following:

Lemma 3.1. *Let G be second countable, and let $f \in C_c(G)$. Then the map $\Lambda_f \colon G \longrightarrow (C_c(G), \tau)$; $s \longmapsto \tau_s f$ is continuous.*

Proof. Since G is a second countable, LCA group, G is metrizable. Let d be an invariant metric on G inducing the topology on G. So it is enough to show that whenever $s_n \longrightarrow s$ in G, we have $\Lambda_f(s_n) \longrightarrow \Lambda_f(s)$ in $C_c(G)$. First, assume that $s = 0$. Let U be a symmetric neighbourhood of 0 in G such that $s_n \in U$ $(n \in \mathbb{N})$ and $\overline{U}$ is compact. Let $K = \overline{U} + \operatorname{supp} f$. Then K is compact, and the supports of $\tau_{s_n} f$ and f are contained in K.

Let $\varepsilon > 0$. Since $f|_K$ is continuous and since K is a compact metric space, $f \colon K \longrightarrow \mathbb{C}$ is uniformly continuous. Let $\delta > 0$ such that

$$s, t \in K \text{ and } d(s,t) < \delta \implies |f(s) - f(t)| < \varepsilon.$$

Choose $n_0 \in \mathbb{N}$ such that $d(s_n, 0) < \delta$ $(n \geq n_0)$. Finally, let $t \in K$ and let $n \geq n_0$.

Case (*i*). $t - s_n \in K$: This implies $d(t - s_n, t) = d(-s_n, 0) = d(s_n, 0) < \delta$; and so $|f(t-s_n) - f(t)| < \varepsilon$.

Case (*ii*). $t - s_n \notin K$: This implies $t \notin \operatorname{supp} f$; because if $t \in \operatorname{supp} f$, then $t - s_n \in \operatorname{supp} f + \overline{U} = K$ which is not the case. Hence $f(t-s_n) = f(t) = 0$; and so $|f(t-s_n) - f(t)| < \varepsilon$.

Hence $|\Lambda_f(s_n)(t) - \Lambda_f(0)(t)| = |f(t-s_n) - f(t)| < \varepsilon, t \in K, n \geq n_0$. Thus $\|\Lambda_f(s_n) - \Lambda_f(0)\|_K < \varepsilon$ $(n \geq n_0)$. Thus $\Lambda_f(s_n) \longrightarrow \Lambda_f(0)$.

Now let $s_n \longrightarrow s$ in G. Then $s_n - s \longrightarrow 0$ in G. But $\|\Lambda_f(s_n) - \Lambda_f(s)\|_K = \|\Lambda_f(s_n - s) - \Lambda_f(0)\|_K$. Hence $\Lambda_f(s_n) \longrightarrow \Lambda_f(s)$. □

Let $\Delta(C_c(G))$ denote the Gel'fand space of $C_c(G)$. For $\alpha \in H(G)$, define $\varphi_\alpha(f) = \int_G f(s)\alpha(s)\mathrm{d}\lambda(s)$, $f \in C_c(G)$. Then $\varphi_\alpha \in \Delta(C_c(G))$.

Theorem 3.2. *Let G be second countable. Let $T \colon H(G) \longrightarrow \Delta(C_c(G))$ be defined as $T(\alpha) = \varphi_\alpha$. Then T is a bijective continuous map.*

Proof. The mapping T is clearly one-to-one. To show that T is onto, let $\varphi \in \Delta(C_c(G))$. Then, for all $s \in G$ and for all $f \in C_c(G)$,

$$\varphi(f)^2 = \varphi(f^2) = \varphi(\tau_s f * \tau_{-s} f) = \varphi(\tau_s f)\varphi(\tau_{-s} f).$$

This implies that if $\varphi(f) \neq 0$, then $\varphi(\tau_s f) \neq 0$ for all $s \in G$. Let $f \in C_c(G)$ such that $\varphi(f) \neq 0$. Define $\alpha \colon G \longrightarrow \mathbb{C}^\bullet$ as

$$\alpha(s) = \frac{\varphi(\tau_s f)}{\varphi(f)}.$$

Note that α does not depend on f; because if $g \in C_c(G)$ is another function such that $\varphi(g) \neq 0$, then

$$\varphi(\tau_s f)\varphi(g) = \varphi(\tau_s f * g) = \varphi(f * \tau_s g) = \varphi(f)\varphi(\tau_s g), \quad s \in G.$$

Now, for $s, t \in G$,

$$\alpha(s+t) = \frac{\varphi(\tau_{s+t} f)}{\varphi(f)} = \frac{\varphi(\tau_s(\tau_t f))}{\varphi(f)} = \frac{\varphi(\tau_s(\tau_t f))}{\varphi(\tau_t f)} \frac{\varphi(\tau_t f)}{\varphi(f)} = \alpha(s)\alpha(t).$$

Since G is second countable, the mapping $G \longrightarrow C_c(G);\ s \longmapsto \tau_s f$ is continuous due to Lemma 3.1. Hence α is continuous. Thus $\alpha \in H(G)$. Let $\mu \in M_{\text{loc}}(G)$ be the Radon measure corresponding to φ (p. 838 of [3]). Then, for $g \in C_c(G)$,

$$\begin{aligned}
\varphi_\alpha(g) &= \int_G g(s)\alpha(s)\mathrm{d}\lambda(s) \\
&= \frac{1}{\varphi(f)} \int_G g(s)\varphi(\tau_s f)\mathrm{d}\lambda(s) \\
&= \frac{1}{\varphi(f)} \int_G g(s) \int_G f(t-s)\mathrm{d}\mu(t)\mathrm{d}\lambda(s) \\
&= \frac{1}{\varphi(f)} \int_G (f * g)(t)\mathrm{d}\mu(t) \\
&= \frac{1}{\varphi(f)} \varphi(f * g) = \varphi(g).
\end{aligned}$$

Thus $\varphi = \varphi_\alpha$. Hence T is bijective. Now it is easy to show that T is continuous. □

DEFINITION 3.3

For $\alpha \in H(G)$, $\varepsilon > 0$, and $\{f_1, \ldots, f_n\} \subseteq C_c(G)$, define

$$B(\alpha; \varepsilon; f_1, \ldots, f_n) = \{\beta \in H(G) \colon |\widehat{f_i}(\beta) - \widehat{f_i}(\alpha)| < \varepsilon \ (1 \le i \le n)\},$$

where $\widehat{f}(\beta) = \varphi_\beta(f) = \int_G f(s)\beta(s)\mathrm{d}\lambda(s)$. Then the collection

$$\mathcal{B} = \{B(\alpha; \varepsilon; f_1, \ldots, f_n) \colon \alpha \in H(G), \varepsilon > 0,\ n \in \mathbb{N},\ \{f_1, \ldots, f_n\} \subseteq C_c(G)\}$$

forms a basis for some topology on $H(G)$. Let τ_g denote the topology on $H(G)$ generated by this basis. Then $\tau_g \subseteq \tau_{co}$ on $H(G)$. Let $\widetilde{H}(G)$ denote the $H(G)$ equipped with the topology τ_g. We say that $\widetilde{H}(G) = H(G)$ if $\tau_{co} = \tau_g$.

Remark 3.4. Let $r > 1$. Define $\omega(s) = \mathrm{e}^{r|s|}$, $s \in \mathbb{R}$. Then ω is a weight on $\mathbb{R}$ such that $\Delta(L^1(\mathbb{R}, \omega)) \cong \Pi_{-r,r} := \{x + iy \in \mathbb{C} \colon -r \le x \le r\}$ due to Theorem 4.7.33, p. 533 of [3].

Theorem 3.5. *If G is* (i) *discrete*, (ii) *compact or* (iii) $G = \mathbb{R}$, *then* $\widetilde{H}(G) = H(G)$.

Proof. In the first two cases, it is enough to prove that the point evaluation map e: $G \times \widetilde{H}(G) \longrightarrow \mathbb{C}$ is continuous due to Corollary 13.1.1, p. 281 of [7].

(i) Fix (g_0, α_0) in $G \times \widetilde{H}(G)$. Let V be a neighbourhood of $e(g_0, \alpha_0) = \alpha_0(g_0)$ in $\mathbb{C}$. Then there exists $\varepsilon > 0$ such that $S(\alpha_0(g_0), \varepsilon) \subseteq V$. Choose $U = \{g_0\}$ and $f = \delta_{g_0}$. Define $B = B(\alpha_0; \varepsilon; f)$. Then $U \times B$ is a neighbourhood of (g_0, α_0) in $G \times \widetilde{H}(G)$. Then, for $(g, \alpha) \in U \times B$,

$$|\alpha(g) - \alpha_0(g_0)| = |\alpha(g_0) - \alpha_0(g_0)| = |\widehat{f}(\alpha) - \widehat{f}(\alpha_0)| < \varepsilon.$$

Hence $e(g, \alpha) = \alpha(g) \in V$. Thus the map e is continuous.

(ii) Since G is compact, $H(G) = \widehat{G}$. Suppose $\{t_\gamma\} \subset G$ and $\{\alpha_\gamma\} \subset \widetilde{H}(G)$ are nets that converge to t and α, respectively. Let $f \in C_c(G)$ such that $\widehat{f}(\alpha) \neq 0$. Choose γ_0 such that

$$|\widehat{f}(\alpha) - \widehat{f}(\alpha_\gamma)| < \frac{|\widehat{f}(\alpha)|}{2}, \quad \gamma \geq \gamma_0.$$

Hence $|\widehat{f}(\alpha)| - |\widehat{f}(\alpha_\gamma)| < \frac{|\widehat{f}(\alpha)|}{2}$; and so $\widehat{f}(\alpha_\gamma) \neq 0$, $\gamma \geq \gamma_0$. It is elementary that, for $s \in G$ and for $\beta \in \widetilde{H}(G)$, $\beta(s)\widehat{f}(\beta) = [\tau_s(f)]^\wedge(\beta)$. Hence

$$\alpha(s) = \frac{[\tau_s(f)]^\wedge(\alpha)}{\widehat{f}(\alpha)}, \quad s \in G$$

and

$$\alpha_\gamma(s) = \frac{[\tau_s(f)]^\wedge(\alpha_\gamma)}{\widehat{f}(\alpha_\gamma)}, \quad s \in G;\ \gamma \geq \gamma_0.$$

Since $\widehat{f}(\alpha_\gamma) \longrightarrow \widehat{f}(\alpha)$, it is enough to prove that $[\tau_{t_\gamma}(f)]^\wedge(\alpha_\gamma) \longrightarrow [\tau_t(f)]^\wedge(\alpha)$. But

$$\begin{aligned}
|[\tau_{t_\gamma}(f)]^\wedge(\alpha_\gamma) - [\tau_t(f)]^\wedge(\alpha)| &\leq |[\tau_{t_\gamma}(f)]^\wedge(\alpha_\gamma) - [\tau_t(f)]^\wedge(\alpha_\gamma)| \\
&\quad + |[\tau_t(f)]^\wedge(\alpha_\gamma) - [\tau_t(f)]^\wedge(\alpha)| \\
&\leq \|[\tau_{t_\gamma}(f)]^\wedge - [\tau_t(f)]^\wedge\|_1 \\
&\quad + |[\tau_t(f)]^\wedge(\alpha_\gamma) - [\tau_t(f)]^\wedge(\alpha)|.
\end{aligned}$$

The right-hand side tends to 0 as $\gamma \longrightarrow \infty$. Hence the map e is continuous.

(iii) Note that $H(\mathbb{R}) \cong \mathbb{C}$ and τ_{co} is exactly the usual topology $\mathcal{U}$ on $\mathbb{C}$. So we need to prove that $\tau_g = \mathcal{U}$. Let $S(z, \varepsilon)$ be an open sphere in $\mathbb{C}$ and let $w \in S(z, \varepsilon)$. Let $r > 1$ such that $S(z, \varepsilon) \subset \Pi_{-r,r}$. By Remark 3.4, there exists a weight ω on $\mathbb{R}$ such that $\Delta(L^1(\mathbb{R}, \omega)) \cong \Pi_{-r,r}$. Since $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R}, \omega)$, $\Delta((C_c(\mathbb{R}), \|\cdot\|_\omega)) \cong \Pi_{-r,r}$. So choose $g_1, \ldots, g_n$ in $L^1(\mathbb{R}, \omega)$ and $\delta > 0$ such that $B(w; \delta; g_1, \ldots, g_n) \subseteq S(z, \varepsilon)$. Choose $f_1, \ldots, f_n$ in $C_c(G)$ such that $\|f_i - g_i\|_\omega < \frac{\delta}{3}$ $(1 \leq i \leq n)$. Now let $u \in B(w; \frac{\delta}{3}; f_1, \ldots, f_n)$. Then, for $1 \leq i \leq n$,

$$\begin{aligned}
|\widehat{g_i}(u) - \widehat{g_i}(w)| &\leq |\widehat{g_i}(u) - \widehat{f_i}(u)| + |\widehat{f_i}(u) - \widehat{f_i}(w)| + |\widehat{f_i}(w) - \widehat{g_i}(w)| \\
&\leq \|f_i - g_i\|_\omega + |\widehat{f_i}(u) - \widehat{f_i}(w)| + \|f_i - g_i\|_\omega \\
&< 2\frac{\delta}{3} + \frac{\delta}{3} = \delta.
\end{aligned}$$

Hence $u \in B(w; \delta; g_1, \ldots, g_n)$. Thus $B(w; \frac{\delta}{3}; f_1, \ldots, f_n) \subseteq S(z, \varepsilon)$. Since w is arbitrary, $S(z, \varepsilon) \in \tau_g$. Hence the two topologies are identical. □

Theorem 3.6. *If $\widetilde{H}(G_i) = H(G_i)$, $i = 1, 2$, then $\widetilde{H}(G_1 \oplus G_2) = H(G_1 \oplus G_2)$.*

Proof. Let $G = G_1 \oplus G_2$. It is enough to prove that the point evaluation map e: $G \times \widetilde{H}(G) \longrightarrow \mathbb{C}$ is continuous. Let $s = s_1 \oplus s_2 \in G$ and $\alpha \in \widetilde{H}(G)$. Since $H(G) \cong$

$H(G_1) \oplus H(G_2)$, there exist $\alpha_1 \in H(G_1)$ and $\alpha_2 \in H(G_2)$ such that $\alpha = \alpha_1 \oplus \alpha_2$. Let V be a neighbourhood of $e(s, \alpha) = \alpha_1(s_1)\alpha_2(s_2)$. Choose $\varepsilon > 0$ such that $S(\alpha_1(s_1), \varepsilon) \cdot S(\alpha_2(s_2), \varepsilon) \subseteq V$. Since $\widetilde{H}(G_i) = H(G_i), i = 1, 2$, there exist basic neighbourhoods $W_1 = U_1 \times B(\alpha_1; \delta_1; f_1, \ldots, f_m)$ of (s_1, α_1) in $G_1 \times H(G_1)$ and $W_2 = U_2 \times B(\alpha_2; \delta_2; h_1, \ldots, h_n)$ of (s_2, α_2) in $G_2 \times H(G_2)$ such that

$$(t, \beta) \in W_1 \Longrightarrow \beta(t) \in S(\alpha_1(s_1), \varepsilon);$$

and

$$(t, \beta) \in W_2 \Longrightarrow \beta(t) \in S(\alpha_2(s_2), \varepsilon).$$

Take $W = U \times B$, where $U = (U_1 \oplus U_2)$ and $B = B(\alpha_1; \delta_1; f_1, \ldots, f_m) \oplus B(\alpha_2; \delta_2; h_1, \ldots, h_n)$. Let $(s, \beta) \in W$. Then $s = s_1 \oplus s_2$ for some $s_i \in U_i, i = 1, 2$ and $\beta = \beta_1 \oplus \beta_2$ for some $\beta_1 \in B(\alpha_1; \delta_1; f_1, \ldots, f_m)$ and $\beta_2 \in B(\alpha_2; \delta_2; h_1, \ldots, h_n)$. So $\beta(s) = \beta_1(s_1)\beta_2(s_2)$. Now, for all $1 \leq i \leq m$,

$$\begin{aligned} |\widehat{h_1}(\alpha_2)||\widehat{f_i}(\beta_1) - \widehat{f_i}(\alpha_1)| &= |(f_i \times h_1)^\wedge(\beta_1 \oplus \alpha_2) - (f_i \times h_1)^\wedge(\alpha)| \\ &< \delta \leq \delta_1|\widehat{h_1}(\alpha_2)|. \end{aligned}$$

Hence $\beta_1 \in B(\alpha_1; \delta_1; f_1, \ldots, f_m)$; and so $\beta_1(s_1) \in S(\alpha_1(g_1), \varepsilon)$. Similarly, we can show that $\beta_2(s_2) \in S(\alpha_2(g_2), \varepsilon)$. Hence $e(s, \beta) = \beta(s) = \beta_1(s_1)\beta_2(s_2) \in S(\alpha_1(g_1), \varepsilon) \cdot S(\alpha_2(g_2), \varepsilon) \subseteq V$. Thus the map e is continuous. □

COROLLARY 3.7

If G is compactly generated, then $\widetilde{H}(G) = H(G)$.

Proof. Since G is compactly generated, $G \cong \mathbb{R}^m \times \mathbb{Z}^n \times K$, where m and n are non-negative integers and K is a compact group due to Theorem 9.8 of [4]. Now the result follows from Theorems 3.5 and 3.6. □

COROLLARY 3.8

If G is second countable and compactly generated, then $H(G) \cong \Delta(C_c(G))$, and hence $\Delta(C_c(G))$ is locally compact.

Proof. The topology τ_g on $H(G)$ is nothing but the Gel'fand topology on $C_c(G)$. So the result follows from Theorem 3.2 and Corollary 3.7. □

Theorem 3.9. *If G is discrete, then $H(G) \cong \Delta(C_c(G))$.*

Proof. Define $T\colon H(G) \longrightarrow \Delta(C_c(G))$ as in Theorem 3.2. Since G is discrete, T is a bijective continuous map as in the proof of Theorem 3.2. Let $\{\varphi_\gamma\}$ be a net in $\Delta(C_c(G))$ such that $\varphi_\gamma \longrightarrow \varphi$ in $\Delta(C_c(G))$. Let $\alpha_\gamma, \alpha \in H(G)$ such that $T(\alpha_\gamma) = \varphi_\gamma$ and $T(\alpha) = \varphi$. Then, for each $s \in G$,

$$\alpha_\gamma(s) = \varphi_\gamma(\delta_s) \longrightarrow \varphi(\delta_s) = \alpha(s).$$

Since G is discrete, $\alpha_\gamma \longrightarrow \alpha$ in $H(G)$. Hence the result is proved. □

Remark 3.10

(i) Let G be as in Example 2.4(i). Then $\Delta(C_c(G)) \cong H(G)$ is not locally compact.
(ii) If the condition "second countable" in Lemma 3.1 can be dropped, then the same can be dropped from Corollary 3.8; in this case, $\Delta(C_c(G))$ is locally compact for all compactly generated LCA groups.

Acknowledgements

The authors gratefully acknowledge the financial support from the University Grants Commission as a major research project, letter No. F.8-9/2004(SR). The authors are thankful to the referee for suggesting an alternative proof of Theorem 2.2(i) and for constructive suggestions leading to the present version of the paper.

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