Proc. Indian Acad. Sci. (Math. Sci.) Vol. 113, No. 2, May 2003, pp. 179–182.


# Beurling algebra analogues of the classical theorems of Wiener and Lévy on absolutely convergent Fourier series

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MS received 20 February 2002; revised 26 September 2002

**Abstract.** Let $f$ be a continuous function on the unit circle $\Gamma$, whose Fourier series is $\omega$-absolutely convergent for some weight $\omega$ on the set of integers $\mathcal{Z}$. If $f$ is nowhere vanishing on $\Gamma$, then there exists a weight $\nu$ on $\mathcal{Z}$ such that $1/f$ had $\nu$-absolutely convergent Fourier series. This includes Wiener's classical theorem. As a corollary, it follows that if $\varphi$ is holomorphic on a neighbourhood of the range of $f$, then there exists a weight $\chi$ on $\mathcal{Z}$ such that $\varphi \circ f$ has $\chi$-absolutely convergent Fourier series. This is a weighted analogue of Lévy's generalization of Wiener's theorem. In the theorems, $\nu$ and $\chi$ are non-constant if and only if $\omega$ is non-constant. In general, the results fail if $\nu$ or $\chi$ is required to be the same weight $\omega$.

**Keywords.** Fourier series; Wiener's theorem; Lévy's theorem; Beurling algebra; commutative Banach algebra.

Let $C(\Gamma)$ be the set of all continuous functions on the unit circle $\Gamma$ in the complex plane $\mathcal{C}$. Let $f \in C(\Gamma)$ such that the Fourier series

$$f \sim \sum_{n\in\mathcal{Z}} \widehat{f}(n)\, \mathrm{e}^{int}, \text{ where } \widehat{f}(n) = \frac{1}{2\pi i}\int_{-\pi}^{\pi} f(\mathrm{e}^{it})\mathrm{e}^{-int}\mathrm{d}t \quad (n \in \mathcal{Z}),$$

is absolutely convergent. If $f(z) \neq 0$ for all $z \in \Gamma$, then the Fourier series of $1/f$ is also absolutely convergent. This is a classic Wiener's theorem ([1], §11.4.17, p. 33), a transparent proof of which by Gelfand (e.g. [2], p. 33) is often cited as the first success of the theory of Banach algebras. Lévy's generalization of Wiener's theorem states that if $\varphi$ is holomorphic on a neighbourhood of the range of $f$, then $\varphi \circ f$ also has absolutely convergent Fourier series ([1], §11.4.17, p. 33). We aim to discuss Beurling algebra analogues of these.

A *weight* on $\mathcal{Z}$ is a map $\omega : \mathcal{Z} \longrightarrow [1, \infty)$ satisfying $\omega(m+n) \leq \omega(m)\omega(n)$ for all $m, n \in \mathcal{Z}$. Let $\rho(1,\omega) = \inf\{\omega(n)^{1/n} : n \geq 1\}$ and $\rho(2,\omega) = \sup\{\omega(n)^{1/n} : n \leq -1\}$. Then by ([2], p. 118), $0 < \rho(2,\omega) \leq 1 \leq \rho(1,\omega) < \infty$. A series $\sum_{n\in\mathcal{Z}} \lambda_n$ is $\omega$-absolutely convergent if $\sum_{n\in\mathcal{Z}} \left|\lambda_n\right|\omega(n) < \infty$. A function $f \in C(\Gamma)$ has $\omega$-absolutely convergent Fourier series ($\omega$-ACFS) if its Fourier series is $\omega$-absolutely convergent.

**Theorem.** *Let $\omega$ be a weight on $\mathcal{Z}$. Let $f \in C(\Gamma)$, which has $\omega$-ACFS.*

(I) *If $f(z) \neq 0$ for all $z \in \Gamma$, then there exists a weight $\nu$ on $\mathcal{Z}$ such that*:

(a) $1/f$ *has $\nu$-ACFS*;

(b) $\nu$ *is non-constant if and only if* $\omega$ *is non-constant*;
(c) $\nu(n) \leq \omega(n)$ *for all* $n \in \mathcal{Z}$.

(II) *Let* $\varphi$ *be a function holomorphic on a neighbourhood of the range of* $f$. *Then there exists a weight* $\chi$ *on* $\mathcal{Z}$ *such that*:
(a) $\varphi \circ f$ *has* $\chi$*-ACFS*;
(b) $\chi$ *is non-constant if and only if* $\omega$ *is non-constant*;
(c) $\chi(n) \leq \omega(n)$ *for all* $n \in \mathcal{Z}$.

The present note contributes to a programme suggested some thirty years ago by Edward ([1], Ex. 11.15, p. 41). In the efforts made so far in this programme, conditions on a given weight $\omega$ (e.g., the Beurling–Domar condition; $\sum \frac{\log \omega(n)}{1+n^2} < \infty$ ([3], p. 185)) are sought, which ensure that $g$ (which is either $1/f$ or $\varphi \circ f$ whatever the case may be) has $\omega$-ACFS. Contrary to this, given an arbitrary weight $\omega$, we search for another weight $\eta$ that ensure that $g$ has $\eta$-ACFS. We shall derive (II) as a corollary of (I).

*Proof.* Let $\ell^1(\mathcal{Z}, \omega) := \left\{\lambda = (\lambda_n) : \left|\lambda\right|_\omega := \sum_{n\in\mathcal{Z}} \left|\lambda_n\right|\omega(n) < \infty\right\}$, the Beurling algebra. It is a convolution Banach algebra with norm $|\cdot|_\omega$. Let $A(\omega) = \{g \in C(\Gamma) : \widehat{g} \in \ell^1(\mathcal{Z}, \omega)\}$, the weighted Wiener algebra. It is a unital Banach algebra with the pointwise operations and the norm being $\left\|g\right\|_\omega = \left|\widehat{g}\right|_\omega$. Then $g \in C(\Gamma)$ has $\omega$-ACFS if and only if $g \in A(\omega)$ and if and only if $\widehat{g} \in \ell^1(\mathcal{Z}, \omega)$. Hence the Gelfand space $\Delta(A(\omega))$ of $A(\omega)$ is identified with the closed annulus $\Gamma(\omega) = \{z \in \mathcal{C} : \rho(2, \omega) \leq \left|z\right| \leq \rho(1, \omega)\}$ via the map $z \in \Gamma(\omega) \longmapsto \varphi_z \in \Delta(A(\omega))$, where $\varphi_z(g) = \sum_{n\in\mathcal{Z}} \widehat{g}(n)z^n (g \in A(\omega))$. Thus each function $g$ in $A(\omega)$ extends uniquely as an element (denoted by $g$ itself) in $B(\omega)$ consisting of all continuous functions on $\Gamma(\omega)$ which are analytic in its interior.

(I) Let $f \in C(\Gamma)$ have $\omega$-ACFS. Notice that $\Gamma \subseteq \Gamma(\omega)$. Let $z \in \Gamma$. Since $f(z) \neq 0$, there exists a neighbourhood $N(z)$ of $z$ in $\Gamma(\omega)$ such that $\varphi_w(f) = f(w) \neq 0$ for all $w \in N(z)$. We can assume that $N(z) = \{w \in \mathcal{C} : \left|w - z\right| < r_z\} \cap \Gamma(\omega)$ for some $r_z > 0$. By the compactness, there exist $z_1, \ldots, z_m$ in $\Gamma$, arrange in such a way that $\arg z_i < \arg z_{i+1} \big(1 \leq i \leq m-1\big)$, such that $\Gamma \subseteq U_1^m N(z_i) \subseteq \Gamma(\omega)$. Now we define positive numbers $r_1$ and $r_2$ as follows:

(i) If $\rho(2, \omega) = 1 = \rho(1, \omega)$, then take $r_2 = 1 = r_1$.
(ii) If $\rho(2, \omega) = 1 < \rho(1, \omega)$, take $r_2 = 1$; and for $0 < \varepsilon < 1 - (1/\min\{s_1, \ldots, s_m\})$, take $r_1 = (1-\varepsilon)\min\{s_1, \ldots, s_m\} > 1$, where $s_i = \max\left\{\left|z\right| : z \in N(z_i) \cap N(z_{i+1})\right\}(1 \leq i \leq m)$ and $z_{m+1} = z_1$.
(iii) If $\rho(2, \omega) < 1 = \rho(1, \omega)$, take $r_1 = 1$; and for $0 < \varepsilon < (1/\max\{s_1, \ldots, s_m\}) - 1$, take $r_2 = (1+\varepsilon)\max\{s_1, \ldots, s_m\} < 1$, where $s_i = \min\left\{\left|z\right| : z \in N(z_i) \cap N(z_{i+1})\right\}(1 \leq i \leq m)$ and $z_{m+1} = z_1$.
(iv) If $\rho(2, \omega) < 1 < \rho(1, \omega)$, then take $r_1$ and $r_2$ as in (ii) and (iii) respectively.

Thus in any case, $\rho(2, \omega) \leq r_2 \leq 1 \leq r_1 \leq \rho(1, \omega)$. Define $\nu : \mathcal{Z} \to [1, \infty)$ as follows: If $\rho(2, \omega) = \rho(1, \omega)$, then take $\nu = \omega$; otherwise define

$$\nu(n) = \begin{cases} r_1^n & \text{if } n \geq 0 \\ r_2^n & \text{if } n \leq 0 \end{cases}.$$

It is clear that $\nu$ is non-constant if and only if $\omega$ is non-constant. Then the following holds:

(1) $\nu$ is a weight on $\mathcal{Z}$, $\rho(2,\nu) = r_2$ and $\rho(1,\nu) = r_1$;
(2) $\Gamma(\nu) \subseteq \Gamma(\omega)$;
(3) $f(z) \neq 0$ for all $z \in \Gamma(\nu)$;
(4) $1 \leq \nu(n) \leq \omega(n)$ for all $n \in \mathcal{Z}$.

Then by (4) above, $A(\omega) \subseteq A(\nu)$, and so $f \in A(\nu)$. Since $f(z) \neq 0$ for all $z$ in $\Gamma(\nu) = \Delta(A(\nu))$, it follows by the Gelfand theory that $1/f \in A(\nu)$, i.e. $1/f$ has $\nu$-ACFS.

(II) Let $K$ be the range of $f$. Let $\varphi$ be a function holomorphic on a neighbourhood $U$ of $K$. Let $C$ be a closed rectifiable Jordan contour in the open set $U$ containing $K$. Let $\mu \in C$. Then $\mu \notin K$ and $\mu 1 - f \in A(\omega)$. By part (I), there exists a weight $\eta$ (which is non-constant if and only if $\omega$ is non-constant) such that $\eta \leq \omega$ and the inverse $(\mu 1 - f)^{-1}$ of $(\mu 1 - f)$ belongs to $A(\eta)$. Now take $R_\mu = (\mu 1 - f)^{-1}$. Then its norm $\|R_\mu\|_\eta$ is positive. Define $N(\mu) = \{\lambda \in C : |\lambda - \mu| < \|R_\mu\|_\eta^{-1}\}$. Then by the elementary Banach algebra argument, it follows that for every $\lambda \in N(\mu)$, $\lambda 1 - f = (\mu 1 - f)\{1 + (\lambda - \mu)R_\mu\}$ is invertible in $A(\eta)$. Thus $N(\mu)$ is a neighbourhood of $\mu$ in $C$ such that for all $\lambda \in N(\mu)$, $\lambda 1 - f$ is invertible in $A(\eta)$.

Now by the compactness of $C$, there exist finitely many $\mu_1, \ldots, \mu_n$ in $C$ and weights $\eta_1, \ldots \eta_n$ such that $C \subseteq \cup_1^n N(\mu_i)$, and for any $\lambda \in C$, the inverse of $\lambda 1 - f$ belongs to $A(\eta_i)$ for some $i$. Now define

$$r_2 = \max\{\rho(2,\eta_i) : 1 \leq i \leq n\} \text{ and } r_1 = \min\{\rho(1,\eta_i) : 1 \leq i \leq n\}$$

so that $r_2 \leq 1 \leq r_1$. If $\rho(2,\omega) = 1 = \rho(1,\omega)$, then by Part I, each $\eta_i = \omega$. If $\rho(2,\omega) = 1 < \rho(1,\omega)$, then $\rho(2,\eta_i) = 1 < \rho(1,\eta_i)$ for each $i$, and so $r_2 = 1 < r_1$. Similarly, the cases $\rho(2,\omega) < 1 = \rho(1,\omega)$ and $\rho(2,\omega) < 1 < \rho(1,\omega)$ can be discussed. Now if $\rho(2,\omega) = 1 = \rho(1,\omega)$, then take $\chi = \omega (= \eta_i)$; otherwise define $\chi : \mathcal{Z} \longrightarrow [1,\infty)$ as

$$\chi(n) = \begin{cases} r_1^n & \text{if } n \geq 0 \\ r_2^n & \text{if } n \leq 0 \end{cases}.$$

It is clear that $\chi$ is non-constant if and ony if $\omega$ is non-constant. Then the following holds.

(1) $\chi$ is a weight on $\mathcal{Z}$, $\rho(2,\chi) = r_2$ and $\rho(1,\chi) = r_1$;
(2) $\rho(2,\omega) \leq \rho(2,\eta_i) \leq \rho(2,\chi) \leq 1 \leq \rho(1,\chi) \leq \rho(1,\eta_i) \leq \rho(1,\omega)$ for all $i$;
(3) $1 \leq \chi \leq \eta_i \leq \omega$ on $\mathcal{Z}$ and hence $A(\omega) \subseteq A(\eta_i) \subseteq A(\chi)$ for all $i$;
(4) For any $\lambda \in C$, the inverse of $\lambda 1 - f$ belongs to $A(\chi)$.

Now the map $\lambda \in C \longrightarrow \varphi(\lambda)R_\lambda$ is a continuous map from $C$ into the Banach algebra $(A(\chi), \|\cdot\|_\chi)$, where $R_\lambda$ is the inverse of $\lambda 1 - f$. Hence the integral $(1/2\pi i)\int_C \varphi(\lambda)R_\lambda \mathrm{d}\lambda$ is in $A(\chi)$ in the sense of $\|\cdot\|_\chi$-convergence and $\varphi(f) = (1/2\pi i)\int_C \varphi(\lambda)R_\lambda \mathrm{d}\lambda$, where $\varphi(f)$ is defined by the functional calculus in $C(\Gamma)$. Thus $\varphi(f)$ has $\chi$-ACFS. It follows that $\varphi(f)(\mathrm{e}^{i\theta}) = (\varphi \circ f)(\mathrm{e}^{i\theta})$ for all $\mathrm{e}^{i\theta} \in \Gamma$. $\square$

*Remarks.*

(1) Let $\omega$ be any weight on $\mathcal{Z}$ such that $\rho(2,\omega) \neq \rho(1,\omega)$. Then $\Gamma$ is properly contained in $\Gamma(\omega)$. Let $f \in C(\Gamma)$ have $\omega$-ACFS such that $f(z) \neq 0$ for all $z \in \Gamma$, and $f(z_0) = 0$ for some $z_0 \in \Gamma(\omega)$. Then the function $f$ is clearly not invertible in $A(\omega)$, i.e., $1/f$ cannot have $\omega$-ACFS. For example, define $\omega(n) = e^{|n|}$ $(n \in \mathcal{Z})$ and let $f(z) = z_0 - z (z \in \mathcal{C})$,

where $1 < |z_0| < e$. Then $f$ has $\omega$-ACFS, $\rho(1, \omega) = e$, $\rho(2, \omega) = 1/e$ and $1/f$ does not have $\omega$-ACFS.
(2) Let $\omega$ be a weight on $\mathcal{Z}$ such that $\rho(2, \omega) = 1 = \rho(1, \omega)$. Then it follows from the proof that for any $f \in C(\Gamma)$ having $\omega$-ACFS and satisfying $f(z) \neq 0$ for all $z \in \Gamma$, the $1/f$ has also $\omega$-ACFS. Examples of such weights include:
(i) $\omega_\alpha(n) = \left(1 + |n|\right)^\alpha$, where $0 < \alpha < \infty$;
(ii) $\omega(n) = 1 + \log\left(1 + |n|\right)$;
(iii) $\omega(n) = \left(1 + |n|\right)^{\sqrt{1+|n|}}$.
(3) Let $f \in C(\Gamma)$ such that $f$ have $\omega$-ACFS for every weight $\omega$ on $\mathcal{Z}$. Suppose $f(z) \neq 0$ for all $z \in \Gamma$. One would be tempted to know whether $1/f$ has $\omega$-ACFS for every $\omega$. The answer is 'no'. For example, take $f(z) = 2z + z^2$, a trigonometric polynomial. Then the Fourier series of $1/f$ is

$$\left(\frac{1}{f}\right)(z) = \frac{1}{2z}\sum_0^\infty (-1)^k \left(\frac{z}{2}\right)^k$$

which fails to have $\omega$-ACFS for the weight $\omega(n) = 2^{|n|+2}(n \in \mathcal{Z})$.

## Acknowledgement

The authors are thankful to the referee for a careful reading of the manuscript.

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