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Stinespring representability and Kadison's Schwarz inequality in non-unital Banach star algebras and applications

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Abstract. A completely positive operator valued linear map ϕ on a (not necessarily unital) Banach $*$-algebra with continuous involution admits minimal Stinespring dilation iff for some scalar $k > 0$, $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ for all x iff ϕ is hermitian and satisfies Kadison's Schwarz inequality $\phi(h)^2 \leq k\phi(h^2)$ for all hermitian h iff ϕ extends as a completely positive map on the unitization A_e of A. A similar result holds for positive linear maps. These provide operator state analogues of the corresponding well-known results for representable positive functionals. Further, they are used to discuss (a) automatic Stinespring representability in Banach $*$-algebras, (b) operator valued analogue of Bochner–Weil–Raikov integral representation theorem, (c) operator valued analogue of the classical Bochner theorem in locally compact abelian group G, and (d) extendability of completely positive maps from $*$-subalgebras. Evans' result on Stinespring respresentability in the presence of bounded approximate identity (BAI) is deduced. A number of examples of Banach $*$-algebras without BAI are discussed to illustrate above results.

Keywords. Stinespring representability; completely positive map; Kadison's Schwarz inequality; automatic representability; positive definite functions on a group; Bochner theorem.

1. Introduction

Let $\phi : A \to B(H)$ be a completely positive linear map from, a not-necessarily unital, Banach $*$-algebra A to bounded linear operators on a Hilbert space H. Theorem 2.1 asserts that ϕ admits a minimal Stinespring dilation iff ϕ satisfies CP-Schwarz inequality $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ $(x \in A)$ for some scalar $k > 0$ iff ϕ is hermitian and satisfies Kadison's Schwarz inequality $\phi(h)^2 \leq k\phi(h^2)$ $(h = h^* \in A)$ iff ϕ is extendable as a completely positive map on the unitization A_e. Theorem 2.2 is a positive map analogue of this. These operator state analogues of [4, Theorem 37.11; 25, Theorem 3.2] are proved in §2 using elementary properties of CP-maps, Jordan order structure in Banach $*$-algebras (which is distinct from usual order in non-C^*-situation), enveloping C^*-algebra $C^*(A)$ of A and by creating operator state analogue, within the formalism of Stinespring dilation, of the arguments in [24]. The results are applied in § 3 to a variety of situations. Corollary 3.1 contains several Cauchy–Schwarz inequalities for CP-maps, including a dilation-free proof of the CP-Schwarz inequality for a 2-positive map on a Banach $*$-algebra with BAI. A simple example shows that a positive map on a C^*-algebra need not satisfy CP-Schwarz inequality, though Kadison's Schawarz inequality does hold for such a map on a Banach $*$-algebra with BAI. This brings out an essential difference between these two inequalities in the non-commutative case. Sufficient conditions for automatic Stinespring representability of CP-maps (as well as automatic representability of positive functionals) are developed in Corollary 3.2. This operator valued version of [4, Theorem 37.15] supplements automatic continuity phenomena even in scalar case. It is shown in Corollaries 3.6, 3.7 that a CP-map on a $*$-subalgebra B of a Banach $*$-algebra A having BAI extends as a CP-map on A iff it is Stinespring representable, provided (a) A is hermitian and B is closed; or (b) B is Banachable and $C^*(B) \to C^*(A)$ injectively; or (c) B is an abstract Segal algebra over A. It follows that if B has BAI and if (a) holds, then every CP-map on B extends to A. This supplements Arveson extension theorem [1]. Corollary 3.8 implies that an operator valued positive linear map ϕ on a commutative Banach $*$-algebra A is extendable to A_e iff ϕ is an integral with respect to a semi-spectral measure on the hermitian Gelfand space. This is an operator valued analogue of Bochner–Weil–Raikov integral representation theorem [11, ch. VI, Theorem 21.2, p. 492]. It follows (Corollary 3.8) that a weakly continuous operator valued function x on a locally compact abelian group G is positive definite iff x is an integral with respect to a semi-spectral measure on the dual group. This provides an operator valued version of the classical Bochner Theorem [15, Sec. 33] as well as a linear version of Stone–Naimark–Ambrose–Godament Theorem [20, ch. XV, Theorem 3.1, p. 489] occupying its proper place midway between the two. Finally, the abstract results are illustrated in several concrete Banach $*$-algebras like convolution algebras, the algebra $C^p(H)$ of von-Neumann Schatten class operators and the Hardy space $H^p(U)$ with the Hadamard product.

2. Stinespring representability

Let $(A, \| \ \|)$ be a complex Banach $*$-algebra, not necessarily having identity, assumed throughout satisfying $\|x^*\| = \|x\| (x \in A)$. Let $A_e = A \oplus \mathbb{C}$ be the Banach $*$-algebra obtained by adjoining identity to A, $\|x + \lambda 1\| = \|x\| + |\lambda| (x + \lambda 1 \in A_e)$. For a Hilbert space H, let $P(A, H)$ be the collection of all positive linear maps ϕ from A to the C^*-algebra $B(H)$ of all bounded linear operators on H, positive in the sense that $\phi(x^*x) \geq 0$ for all $x \in A$. For $n \in \mathbb{N}$, let $M_n(A) = A \otimes M_n(\mathbb{C})$ be the full matrix algebra over A, a Banach $*$-algebra with projective cross-norm $\|z\| = \inf\{\sum \|x_i\| \|y_i\| : z = \sum x_i \otimes y_i$ in $A \otimes M_n(\mathbb{C})\}$. Let $\phi_n = \phi \otimes id : M_n(A) \to M_n(B(H)) \subset B(H_n), H_n = \sum^{\oplus} H (n \text{ times})$, be $\phi_n([x_{ij}]) = [\phi(x_{ij})]$. Then ϕ is completely positive if each ϕ_n is positive. Let $CP(A, H) = \{\phi \in P(A, H) : \phi \text{ is completely positive }\}$. If every positive functional on A is continuous, then every $\phi \in P(A, H)$ is continuous by a closed graph argument.

DEFINITION

A map $\phi \in CP(A, H)$ is Stinespring representable if there exists a Hilbert space K, a $*$-homomorphism $\pi : A \to B(K)$ and a bounded linear operator $V : H \to K$ such that (i) $\phi(x) = V^* \pi(x) V (x \in A)$, and (ii) $K = [\pi(A) VH]$, the closed linear span of $\{\pi(x) V\xi : x \in A, \xi \in H\}$.

The arguments in part (iii) of [27, ch. IV, Theorem 3.6] show that the Stinespring representation $\{\pi, K, V\}$ of ϕ is unique up to unitary equivalence. By a classic theorem of Stinespring (in the unital case, and Lance in the non-unital case (see remarks in [9, p. 89])), every completely positive map ϕ on a C^*-algebra A is Stinespring representable [27, ch. 4, Theorem 3.6]. Evans showed that this also holds when A is a Banach $*$-algebra with BAI [9, Theorem 2.13]. In the absence of the requirement (ii) in the above definition, it is shown in [10] that given a not-necessarily bounded completely positive map ϕ defined on a subspace of form N^*N of a C^*-algebra A with N a left ideal, a $*$-representation π of A can be constructed satisfying above (i).

Theorem 2.1. *Let $\phi \in CP(A,H)$. The following are equivalent.*

(1) *ϕ is Stinespring representable.*
(2) *There exists a scalar $k > 0$ such that $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ for all x in A.*
(3) *ϕ is hermitian (i.e. $\phi(x^*) = \phi(x)^*$ for all x) and there exists a scalar $k > 0$ such that $\phi(h)^2 \leq k\phi(h^2)$ for all $h = h^*$ in A.*
(4) *ϕ is extendable to $\phi^e \in CP(A_e, H)$.*
(5) *ϕ is continuous in the Gelfand–Naimark pseudo-norm p_∞.*
(6) *There exists $\tilde{\phi} \in CP(C^*(A), H)$ such that $\phi = \tilde{\phi} \circ j$ where $j : A \to C^*(A)$ is $j(x) = x +$ srad A, srad $A (= \ker p_\infty)$ being the star radical of A.*

Further, if ϕ is Stinespring representable, then ϕ is continuous and $\phi(x)^*\phi(x) \leq \|\phi^e(1)\|$ $\phi(x^*x)$ for all x in A.

A completely positive map is an operator valued analogue of a positive linear functional. Then Stinespring construction $\{\pi, K, V\}$ corresponds to the GNS construction; and the above definition provides analogue of representability of positive functionals [4, Defn. 37.10, p. 199]. Thus Theorem 1 is a *CP*-analogue of [4, Theorem 37.11, p. 199]. Note that there is a gap in the proof [4, Theorem 37.11] which has been repaired in [18] and [24]. If A is a C^*-algebra, then any $\phi \in CP(A,H)$ satisfies the *CP*-Schwarz inequality $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)(x \in A)$ [27, ch. IV, Corollary 3.8, p. 199]; whereas $\phi \in P(A,H)$ is known to satisfy Kadison's Schwarz inequality $\phi(h)^2 \leq \|\phi\|\phi(h^2)(h = h^* \in A)$ [17, Ex.10.5.9, p. 770]. Thus Theorem 1.1, part (3) iff (4) of which is a *CP*-analogue of [25, Theorem 3.2], shows that these inequalities are intimately connected with Stinespring dilation. The following positive map analogue of the above theorem further clarifies the role of Kadison's inequality in extendability.

Theorem 2.2. *Let $\phi \in P(A,H)$. The following are equivalent.*

(1) *ϕ is extendable to $\phi^e \in P(A_e, H)$.*
(2) *ϕ is hermitian and for some scalar $k > 0, \phi(h)^2 \leq k\phi(h^2)$ for all $h = h^*$ in A.*
(3) *ϕ is continuous in p_∞.*
(4) *There exists $\tilde{\phi} \in P(C^*(A), H)$ such that $\phi = \tilde{\phi} \circ j$.*

When ϕ is extendable, it is continuous and $\phi(h)^2 \leq \|\phi^e(1)\|\phi(h^2)$ hold for all $h = h^* \in A$.

Examples in [14, § 21.39, p. 332] show that, even in the scalar case, in the relevant inequalities in above theorems, $\|\phi^e(1)\|$ cannot be replaced by $\|\phi\|$ in general. When A has BAI, every positive functional on A is representable [21, Theorem 4.5.14, p. 219], [25, Theorem 3.2]; and hence continuous. The following, part (b) of which recaptures [9, Theorem 2.13], contains operator valued analogue.

COROLLARY 2.3

Let A have BAI (e_i) with $\|e_i\| \leq 1$.

(a) *Let $\phi \in P(A,H)$. Then ϕ is extendable and $\phi(h)^2 \leq \|\phi\|\phi(h^2)$ for all $h = h^*$ in A.*
(b) *Let $\phi \in CP(A,H)$. Then ϕ is Stinespring representable and $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ for all x in A.*

The proofs are based on several auxiliary results some of which appears to be of independent interest. The positive elements of A is $A^+=\{\sum x_i^* x_i : x_i \text{ in } A, \text{ finite sums}\}$. We write $x \geq 0$ for $x \in A^+$. An n-state on A is $f = [f_{ij}] \in M_n(A')(A' = \text{dual of } A)$ such that for each $x = [x_{ij}] \geq 0$ in $M_n(A)$, $[f_{ij}(x_{ij}] \geq 0$ in $M_n(\mathbb{C})$.

Lemma 2.4. (*Banach* $*$-*algebra analogues of* [27, *Ch.* 4, 3.1–3.4])

(a) *Let* $x \in M_n(A)$. *Then* $x \geq 0$ *iff* x *is a finite sum of elements of form* $[x_i^* x_j]$ *with* $x_1, \ldots, x_n$ *in* A.
(b) *Let* $x = [x_{ij}] \geq 0$ *in* $M_n(A)$. *Then for all* $y_1, \ldots, y_n$ *in* A, $\sum_{ij} y_i^* x_{ij} y_j \geq 0$.
(c) *Let* B *be a* C^*-*algebra. Then* $\phi : A \to B$ *is completely positive iff for each n, for all* $x_1, \ldots, x_n$ *in* A *and for all* $y_1, \ldots, y_n$ *in* B, $\sum_{ij} y_i^* \phi(x_i^* x_j) y_j \geq 0$.
(d) *Let* B *be an abelian* C^*-*algebra. Then every positive linear map* $\phi : A \to B$ *is completely positive.*
(e) (*Analogue of* [17, *Ex.* 11.5.21, p. 884]) *Let* $\phi \in P(A,H)$ *be continuous. Then* $\phi \in CP(A,H)$ *iff for each n and for each n-state* $[f_{ij}]$ *on* $B(H)$, $[f_{ij} \circ \phi]$ *is an n-state on A.*

Lemma 2.5. *Let* $\phi \in P(A,H)$.

(a) $\phi(y^*x) = \phi(x^*y)^*$ *holds for all* x, y *in* A.
(b) *Assume at least one of the following.*

(i) *There exists* $k > 0$ *such that* $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ *for all* x.
(ii) A *has BAI.*

Then ϕ *is continuous and extendable as a positive linear map on* A_e.

Proof. Applying [4, Lemma 37.6, p. 147] to the functionals $f_\xi(x) = \langle \phi(x)\xi, \xi \rangle$, $\xi \in H$, (a) follows. Assume (b(i)). Then $|f_\xi(x)|^2 \leq \|\phi(x)\xi\|^2 \|\xi\|^2 = \|\xi\|^2 \langle \phi(x)^*\phi(x)\xi, \xi \rangle \leq \|\xi\|^2 k \langle \phi(x^*x)\xi, \xi \rangle = k\|\xi\|^2 f_\xi(x^*x)$. Hence by [24,18], f_ξ is representable; hence is hermitian; which in turn implies that $f_\xi^e(x + \lambda 1) = f_\xi(x) + \lambda k \|\xi\|^2$ gives a positive linear extension to A_e (using, e.g., the arguments in [4,Theorem 37.11, p. 199] wherein hermiticity is implicitly used). Thus $\phi^e(x + \lambda 1) = \phi(x) + \lambda k 1$ gives the desired extension of ϕ. The continuity of ϕ follows from continuity of positive functionals on unital Banach $*$-algebras. Assume (b(ii)) with (e_i) a BAI for A. Then f_ξ and ϕ are continuous by [4, Theorem 37.15, p. 201]; and by Cauchy–Schwarz inequality, $|f_\xi(x)|^2 \leq (\overline{\lim}\, f_\xi(e_i^* e_i)) f_\xi(x^*x) (x \in A)$; which gives the representability of f_ξ, and hence continuity and extendability of ϕ.

Lemma 2.6. *Let* $\phi \in CP(A,H)$. *Let* x, y *in* A. *Then* $\phi(y^*x)^*\phi(y^*x) \leq \|\phi(y^*y)\| \phi(x^*x)$.

Proof. Let $X = A \otimes H$. For x, y in $X, x = \sum x_i \otimes \xi_i, y = \sum y_j \otimes \eta_j$, let $\beta(x,y) = \sum \langle \phi(y_i^* x_j)\xi_j, \eta_i \rangle$. Then β defines a sesquilinear form on X; and $\beta(x,x) \geq 0$ for all x, as ϕ is completely positive. By Cauchy–Schwarz inequality [28, Theorem 1.4, p. 4], $|\beta(x,y)|^2 \leq \beta(x,x)\beta(y,y)$. Taking $x \otimes \xi$ and $y \otimes \eta$ in X, we get $\langle (\phi(y^*x)^*\phi(y^*x)\,\xi, \xi \rangle = \|\phi(y^*x)\xi\|^2 = \sup\{|\langle \phi(y^*x)\xi, \eta \rangle|^2 : \|\eta\| \leq 1 \text{ in } H\} = \sup\{|\beta(x \otimes \xi, y \otimes \eta)|^2 : \|\eta\| \leq 1 \text{ in } H\} \leq \beta(x \otimes \xi, x \otimes \xi) \sup\{\beta(y \otimes \eta, y \otimes \eta) : \|\eta\| \leq 1\} = \sup\{\langle \phi(y^*y)\eta, \eta \rangle| : \|\eta\| \leq 1\} \langle \phi(x^*x)\xi, \xi \rangle = \|\phi(y^*y)\| \langle \phi(x^*x)\xi, \xi \rangle$.

Lemma 2.7. *Let* $\phi \in P(A,H)$ *be such that there exists* $\tilde{\phi} \in P(C^*(A),H)$ *satisfying* $\tilde{\phi} \circ j = \phi$. *Then* $\phi \in CP(A,H)$ *iff* $\tilde{\phi} \in CP(A,H)$.

Proof. Since $\tilde{\phi}$ has to be continuous, ϕ is also continuous. Also, $\tilde{\phi} \circ j = \phi$ implies that $(\tilde{\phi})_n \circ (j \otimes id) = \phi_n$ for each n. Indeed, for any $[x_{ij}] \in M_n(A)$, $(\tilde{\phi})_n([j(x_{ij})]) = [\tilde{\phi} \circ j(x_{ij})] = \phi_n([x_{ij}])$. Since j is a $*$-homomorphism, ϕ_n is positive implies that $(\tilde{\phi})_n$ is positive on $M_n(j(A))$. As $M_n(j(A))$ is dense in $M_n(C^*(A))$, and as $(\tilde{\phi})_n$ is continuous on $M_n(C^*(A))$, it follows that $(\tilde{\phi})_n : M_n(C^*(A)) \to M_n(B(H))$ is positive if $\phi \in CP(A,H)$. The converse is similarily verified.

Lemma 2.8. *Let A have BAI* (e_i). *Let* $\|e_i\| \leq 1$ *for all i. Let* $\phi \in P(A,H)$.

(a) *There exists a unique* $\tilde{\phi} \in P(C^*(A),H)$ *such that* $\phi = \tilde{\phi} \circ j$.
(b) ϕ *is continuous, hermitian and satisfies* $\phi(h)^2 \leq \|\phi\|\phi(h^2)(h^* = h \in A)$. *Further* $\|\phi\| = \sup \|\phi(e_i)\|$.
(c) *Let A be abelian. Then* $\phi \in CP(A,H)$ *and* $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ *for all* $x \in A$.
(d) *Let A be unital,* $\|1\| = 1$. *Then* $\|\phi|_{H(A)}\| = \|\phi(1)\| = \|\phi\|$.

Above (c) extends [27, ch. IV, Prop. 3.9, p. 199] to Banach $*$-algebras. Does it hold in the absence of BAI for continuous ϕ?

Proof. (a) The functional $f_\xi(x) = \langle\phi(x)\xi, \xi\rangle, \xi \in H$, is continuous [4, Theorem 37.15, p. 201]; and by [7, Prop. 2.7.5, p. 49], there exists a positive functional $\tilde{f}_\xi$ on $C^*(A)$ such that $f_\xi = \tilde{f}_\xi \circ j$. Define $\tilde{\phi} : A/\text{srad } A \to B(H)$ by $\tilde{\phi}(j(x)) = \phi(x)$. Then $\tilde{\phi}$ is well defined. Indeed, $x \in \text{srad } A$ implies that $f(x^*x) = 0$ for all positive functionals f on A [4, Theorem 40.9, p. 223]. As A has BAI, each such f is representable [21, Theorem 4.5.14, p. 219], hence [4, Theorem 37.11, p. 199] gives $f(x) = 0$. Thus $f_\xi(x) = 0$ for all ξ in H; and so the operator numerical range [5, § 9, p. 85] $W(\phi(x)) = 0$, so that $\phi(x) = 0$. Thus $\tilde{\phi}$ defines a positive map. Further, $\tilde{\phi}$ is $|\,|$ continuous, $|\,|$ denoting the norm on the C^*-algebra $C^*(A)$. Indeed, let $\xi \in H$ be such that $\|\xi\| \leq \|\phi\|^{-1/2}$. Then for all x in A, $|f_\xi(x)| = |\langle\phi(x)\xi, \xi\rangle| \leq \|\phi\|\|x\|\|\xi\|^2 \leq \|x\|$, hence $\|f_\xi\| \leq 1$. By the Cauchy–Schwarz inequality, $|\langle\tilde{\phi}(j(x))\xi, \xi\rangle| = |\langle\phi(x)\xi, \xi\rangle| = \lim |f_\xi(e_i x)| \leq \overline{\lim} f_\xi(e_i^* e_i)^{1/2} f_\xi(x^*x)^{1/2} \leq \overline{\lim}\|f_\xi\|\|e_i\||j(x)| \leq |j(x)|$. Hence for all ξ with $\|\xi\| = 1, |\langle\tilde{\phi}(j(x))\xi, \xi\rangle| \leq \|\phi\||j(x)|$; and so the numerical radius $\nu(\tilde{\phi}(j(x))) \leq \|\phi\|\,|j(x)|$. Since numerical radius is a norm equivalent to the given norm [5, Theorem 9.8, p. 86 and Theorem 4.1, p. 34], it follows that $\tilde{\phi}$ is continuous; and by extention, we get desired positive map $\tilde{\phi} : C^*(A) \to B(H)$.
(b) The inequality follows from (a) and Kadison's Schwarz inequality in C^*-algebra.
(c) Let A be abelian. Then so is $C^*(A)$. By [27, ch. IV, Prop. 3.9, p. 199], $\tilde{\phi}$ is completely positive. But then $C^*(M_n(A)) = C^*(A \otimes M_n(\mathbb{C})) = C^*(A\hat{\otimes}_\pi M_n(\mathbb{C})) = C^*(A)\hat{\otimes}_\nu C^*(M_n(\mathbb{C})) = C^*(A) \otimes_\nu M_n(\mathbb{C}) = M_n(C^*(A))$ shows that ϕ is also completely positive. By [27, ch. IV, Corollary 3.8, p. 199], for any $x \in A, \phi(x)^*\phi(x) = \tilde{\phi}(j(x))^*\tilde{\phi}(j(x)) \leq \|\tilde{\phi}\|\tilde{\phi}(j(x)^*j(x)) \leq \|\phi\|\phi(x^*x)(x \in A)$.
(d) For any $h = h^* \in A$, above (b) and Lemma 2.6(a) imply that $\|\phi(h)\|^2 = \|\phi(h)^2\| \leq \|\phi(1)\|\,\|\phi(h^2)\| \leq \|\phi(1)\|\,\|\phi|_{H(A)}\|\,\|h\|^2$, so that $\|\phi|_{H(A)}\| \leq \|\phi(1)\|$. Hence $\|\phi|_{H(A)}\| = \|\phi(1)\|$. Further, $j(1)$ being the identity of $C^*(A)$, [17, Ex. 10.5.10, p. 770] implies that $\|\tilde{\phi}\| = \|\tilde{\phi}(j(1))\| = \|\phi(1)\|$. The conclusion follows from $\|\phi\| \leq \|\tilde{\phi}\|$.

Lemma 2.9. *Let* $\phi \in CP(A,H)$ *be Stinespring representable having the Stinespring representation* $\{\pi, K, V\}$. *Then* ϕ *is continuous and* $\|\phi\| \leq \|V\|^2$. *If A has BAI* (e_i), $\|e_i\| \leq 1$, *then* $\|\phi\| = \|V\|^2$.

Proof. Let $\phi(x)=V^*\pi(x)V$. Then $\|\phi(x)\|\leq\|V\|^2\|\pi(x)\|\leq\|V\|^2\|x\|$; hence $\|\phi\|\leq\|V\|^2$. If (e_i) is a BAI for A, then $\pi(e_i)\to 1$ strongly. Hence, for all $\xi\in H$, $\|V\xi\|^2=\langle V\xi,V\xi\rangle\leq\|\xi\|\,\|V^*V\xi\|=\|\xi\|\lim\|V^*\pi(e_i)V\xi\|=\|\xi\|\lim\|\phi(e_i)\xi\|\leq\|\xi\|^2\|\phi\|$; hence $\|V\|^2\leq\|\phi\|$.

Lemma 2.10. *$C^*(A_e)=(C^*(A))_e$ up to isometric $*$-isomorphism.*

It is interesting to note that this does not mean that A has identity iff $C^*(A)$ has identity. In fact, [2] contains an example of a non-unital Banach $*$-algebra A such that $C^*(A)$ has identity.

Proof. As noted in [25, p. 145], Gelfand–Naimark pseudonorm p_∞^A on $A=p_\infty^{A_e}|A=$ restriction to A of the Gelfand–Naimark pseudonorm on A_e. Since each $*$-representation of A can be extended to A_e, $(\text{srad } A_e)\cap A=\text{srad } A$. Note that $(x,\lambda)\in\text{srad } A_e$ iff $x\in\text{srad } A$ and $\lambda=0$. The map $\phi: A_e/\text{srad } A_e\to(A/\text{srad } A)_e$, $\phi((x,\lambda)+\text{srad } A_e)=(x+\text{srad } A)+\lambda 1=(x+\text{srad } A,\lambda)$ is a bijective $*$-isomorphism, hence extends to the desired isometric $*$-isomorphism between $C^*(A_e)$ and $(C^*(A))_e$.

A positive linear functional f on A is representable [4,21] if there exists a cyclic representation π of A into bounded linear operators on a Hilbert space such that $f(x)=\langle\pi(x)\xi,\xi\rangle(x\in A)$, where ξ is a cyclic vector for π.

Lemma 2.11. *Let f be a positive functional on A.*

(a) *f is representable iff for some $k>0$, $|f(x)|\leq kp_\infty(x)$ for all x.*
(b) (i) [25, *Lemma* 1.31 $|f(y^*xy)|\leq p_\infty(x)f(y^*y)(x,y$ *in* $A)$.
  (ii) *Let A be unital. Then $|f(x)|\leq f(1)p_\infty(x)(x\in A)$.*
  (iii) *Let A has BAI (e_i). Then $|f(x)|\leq(\overline{\lim}\, f(e_i^*e_i))p_\infty(x)(x\in A)$.*

A linear map $\phi: A\to B(H)$ is J-positive if $\phi(h^2)\geq 0$ for all $h=h^*\in A$. If each $\phi_n=\phi\otimes id$ on $M_n(A)$ is J-positive, then ϕ is completely J-positive. Note that if A is a C^*-algebra, then ϕ is positive iff ϕ is J-positive.

Lemma 2.12. *Let $\phi: A\to B(H)$ be linear satisfying the following.*

(i) *ϕ is hermitian.*
(ii) *ϕ is J-positive.*
(iii) *There exists a scalar $k>0$ such that $\phi(h)^2\leq k\phi(h^2)(h=h^*\in A)$. Then $\phi^e(x+\lambda 1)=\phi(x)+\lambda k1$ gives a J-positive extension of ϕ.*

Proof. Let $u=h+\lambda 1=u^*\in A_e$. Then $h=h^*$, $\lambda=\lambda^-$. For all $\xi\in H$,

$$\begin{aligned}\langle\phi^e(u^2)\xi,\xi\rangle&=\langle\phi(h^2)\xi,\xi\rangle+2\lambda\langle\phi(h)\xi,\xi\rangle+\lambda^2k\|\xi\|^2\\&\leq\langle\phi(h^2)\xi,\xi\rangle-2|\lambda|\,|\langle\phi(h)\xi,\xi\rangle|+\lambda^2k\|\xi\|^2\\&\leq\langle\phi(h)^2\xi,\xi\rangle/k-2|\lambda|\|\phi(h)\xi\|\|\xi\|+\lambda^2k\|\xi\|^2\\&=[\,\|\phi(h)\xi\|/k^{1/2}-k^{1/2}|\lambda|\|\xi\|\,]^2\geq 0.\end{aligned}$$

Lemma 2.13. *Assume that A is symmetric and $\phi: A\to B(H)$ is linear, J-positive. Then ϕ is positive.*

By [21, Corollary 4.7.8, p. 233], every J-positive functional on a symmetric Banach $*$-algebra is positive. The conclusion follows by applying this to $f_\xi(x)=\langle\phi(x)\xi,\xi\rangle$, $\xi\in H$.

Lemma 2.14. [25] *Let $\Im$ denote the collection of all continuous positive linear functionals f on A such that $\|f|_{H(A)}\| \leq 1$. Let $\lambda(x) = \sup\{f(x^*x)^{1/2} : f \in \mathcal{F}\}, \tau(x) = \max\{\lambda(x), \lambda(x^*)\}$. The following statements hold.*

(a) *λ and τ are submultiplicative seminorms on A satisfying $\lambda(xy) \leq p_\infty(x)\lambda(y), p_\infty(x) \leq \lambda(x) \leq \tau(x), \tau(x) = \tau(x^*)$ for all x, y in A.*
(b) *If A is $*$-semisimple, then τ is a norm.*
(c) *If τ is a norm, then the τ-completion $\bar{A}$ of A is symmetric and $\tau(h^2) \leq p_\infty(h)\tau(h)$.*

Lemma 2.15. *Let $\phi \in P(A, H)$ satisfy statement* (2) *of Theorem* 2.2. *Then ϕ is extendable and $\phi^e(x + \lambda 1) = \phi(x) + \lambda k1$ defines a positive linear extension of ϕ to A_e.*

Proof. The positive functionals $f_\xi(x) = \langle\phi(x)\xi, \xi\rangle$, $\xi \in H$, are hermitian and satisfy $|f_\xi(h)|^2 \leq k\|\xi\|^2 f_\xi(h^2)(h = h^* \in A)$. By [25, Lemma 3.2], they are representable. By Lemma 2.11, $f_\xi(x) = 0$ for all ξ, all $x \in$ srad A. This implies that ϕ vanishes on the star radical srad A. Indeed, for any $h = h^*$ in srad A, $h^2 \in$ srad A and $f_\xi(h^2) = 0$, hence $\langle\phi(h)^2\xi, \xi\rangle = 0$ for all ξ, showing that $\phi(h) = 0, \phi(\text{srad } A) = \{0\}$. It follows that ϕ factors through the $*$-semisimple Banach $*$-algebra $A/\text{srad } A$ with the quotient norm; and continues to be positive and satisfying statement (2) of Theorem 2.2. Thus we can assume that A is $*$-semisimple.

Next we show that ϕ is τ-continuous on A. Let $\xi \in H$, $\mu = k\|\xi\|^2$. For all $h = h^* \in A$, one has $|f_\xi(h)|^2 \leq \mu^2 r(h^2)$, r denoting the spectral radius in A. Indeed, $|f_\xi(h)|^2 = |\langle\phi(h)\xi, \xi\rangle|^2 \leq \|\phi(h)\xi\|^2 = \|\xi\|^2\langle\phi(h)^2\xi, \xi\rangle \leq k\|\xi\|^2\langle\phi(h^2)\xi, \xi\rangle = k\|\xi\|^2 f_\xi(h^2)(h = h^* \in A)$. By [25, Theorem 3.2], f_ξ is representable, hence extendable to A_e, with $f_\xi(e) = k\|\xi\|^2$ $(e = (0, 1)$, the identity in $A_e)$. Then [4, Lemma 37.6 (iii), p. 197] applied to f_ξ gives $f_\xi(h^2) = f_\xi(eh^2e) \leq f_\xi(e)r(h^2) = \mu r(h^2)$. It follows that $|f_\xi(h)|^2 \leq \mu^2 r(h^2)$. Thus $f_\xi/\mu \in \Im$ and $|f_\xi(h)| \leq \mu\tau(h)$ for all $h = h^*$. Then $\|\phi(h)\xi\|^2 = \langle\phi(h)^2\xi, \xi\rangle \leq k\langle\phi(h^2)\xi, \xi\rangle = kf_\xi(h^2) \leq k^2\|\xi\|^2\tau(h^2) \leq k^2 \|\xi\|^2\tau(h)^2$. This ϕ is τ-continuous on $H(A)$, hence also on A.

Now the τ-completion $\bar{A}$ of A is symmetric by Lemma 2.14 (c); and by continuity, ϕ extends to a hermitian positive linear map $\bar{\phi} : \bar{A} \to B(H)$ satisfying statement (2) of Theorem 2.2 on $\bar{A}$. By Lemma 2.12, $\bar{\phi}^e : (\bar{A})_e \to B(H), \bar{\phi}^e(x + \lambda 1) = \bar{\phi}(x) + \lambda k1$ provides a J-positive extension of $\bar{\phi}$ to $\bar{A}_e$. But as $\bar{A}$ is symmetric, $\bar{A}_e$ is also symmetric [21, Theorem 4.7.9]. Lemma 2.13 implies that $\bar{\phi}^e$ is positive on $\bar{A}_e$. It follows that ϕ^e is positive on A_e.

Lemma 2.16. *Let $\phi \in CP(A, H)$.*

(a) *Statement* (2) *of Theorem* 2.1 *is equivalent to:*
$(2°)\phi_n(x)^*\phi_n(x) \leq k\phi_n(x^*x)$ for all $x \in M_n(A)$, all $n \in \mathbb{N}$.
(b) *Statement* (3) *of Theorem* 2.1 *is equivalent to:*
$(3°)$ ϕ is hermitian and $\phi_n(h)^2 \leq k\phi_n(h^2)$ for all $h = h^* \in M_n(A)$, all $n \in \mathbb{N}$.

In the above (2°) and (3°), the scalar $k > 0$ is same as in the corresponding statements in Theorem 2.1.

Proof. (a) Assume $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in A)$. By Lemma 2.5 (b), ϕ is hermitian and extends to the positive linear map $\phi^e : A_e \to B(H), \phi^e(x + \lambda 1) = \phi(x) + \lambda 1$. By Lemma 2.8, there exists $(\phi^e)^\sim$ in $P(C^*(A_e), H)$ such that $(\phi^e) \circ j^e = \phi^e$ where $j^e : A_e \to A_e/\text{srad } A_e$ is the natural quotient map. In view of Lemma 2.10, $C^*(A_e) = (C^*(A))_e, j^e|_A = j$, srad $A = A \cap \text{srad } A_e, p_\infty^A = p_\infty^{A_e}|_A$. Hence $\tilde{\phi}$ given by $\tilde{\phi} = (\phi^e)^\sim|_{C^*(A)} \in P(C^*(A), H)$ such

that $\tilde{\phi} \circ J = \phi$. Now by Lemma 2.7, $\tilde{\phi} \in CP(C^*(A), H)$. Thus $\tilde{\phi}$ is Stinespring representable, and $\tilde{\phi}(x) = V^*\pi(x)V(x \in C^*(A))$ (in the notations of the definition). Let $n \in \mathbb{N}, V_n = V \otimes id : H \otimes \mathbb{C}^n \to K \otimes \mathbb{C}^n, \pi_n = \pi \otimes id : M_n(C^*(A)) \to M_n(B(H))$ a $*$-homomorphism. Then $(\tilde{\phi})_n = \phi \otimes id = (V^*\pi(\ )V) \otimes id = (V \otimes id)^*(\pi \otimes id)(V \otimes id) = V_n^*\pi_n(\ )V_n$. Hence for all x in $M_n(C^*(A))$, $\tilde{\phi}_n(x)^*\tilde{\phi}_n(x) = V_n^* \ \pi_n(x)^*V_nV_n^*\pi_n(x)V_n \leq \|V_n^*V_n\|V_n^*\pi_n(x^*x)V_n = \|V_n\|^2\tilde{\phi}_n(x^*x) = \|V\|^2\phi_n(x^*x) = \|\tilde{\phi}\|\tilde{\phi}_n(x^*x)$ using Lemma 2.9. (Alternatively, each $\tilde{\phi}_n$ is CP on $M_n(C^*(A))$, hence for all x in $M_n(C^*(A))$, $\tilde{\phi}_n(x)^*\tilde{\phi}_n(x) \leq \|\tilde{\phi}_n\|\tilde{\phi}_n(x^*x) = \|\tilde{\phi}\|\tilde{\phi}_n(x^*x)$. As $\tilde{\phi} \circ j = \phi, (\tilde{\phi})_n \circ (j \otimes id) = \phi_n$, we obtain $\phi_n(x)^*\phi_n(x) \leq \|\tilde{\phi}\|\phi_n(x^*x)$ $(x \in M_n(A))$, and $\|\tilde{\phi}\| \leq \|(\phi^e)^{\sim}\| = \|(\phi^e)^{\sim}(1)\| = \|\phi^e(1)\| = k$.)
(b) This can be proved using Lemma 2.15 instead of Lemma 2.5 (b).

Lemma 2.17. *Let $\phi : A \to B(H)$ be completely positive (resp. completely J-positive) satisfying statement* (2) (*resp. statement* (3)) *of Theorem* 2.1. *Then $\phi^e(x+\lambda 1) = \phi(x) + \lambda k1$ defines a completely positive (resp. completely J-positive) extension of ϕ.*

Proof. We detail out the case of complete J-positivity, the other one being similar. Let $(\phi^e)_n = \phi^e \otimes id : M_n(A_e) \to B(H) \otimes M_n(\mathbb{C})$. For $x = [x_{ij}] \in M_n(A_e)$, $\phi_n(x^*) = \phi_n([x_{ij}]^*) = \phi_n([x_{ji}^*]) = [\phi(x_{ji}^*)] = [\phi(x_{ji})^*] = [\phi(x_{ij})]^* = \phi_n(x)^*$. Also, $x_{ij} = h_{ij} + \lambda_{ij}$, $h_{ij} \in A$, $\lambda_{ij} \in \mathbb{C}$ so that $x = h + \lambda$ where $h = [h_{ij}] \in M_n(A)$, $\lambda = [\lambda_{ij}] \in M_n(\mathbb{C})$. Suppose $x = x^*$. Then $h_{ij} = h_{ji}^*, \lambda_{ij} = \bar{\lambda}_{ji}$ for all i,j, i.e. $h = h^*, \lambda = \lambda^*$ (hermitian adjoint). Let $\xi = \sum \xi_i \otimes e_i \in H \otimes \mathbb{C}^n, (e_i)$ being the standard orthonormal basis in $\mathbb{C}^n$. It is sufficient to show that $\langle (\phi^e)_n(x^2)\xi, \xi\rangle \geq 0$. Now $x^2 = [\sum_l x_{il}x_{lj}] = [\sum_l (h_{il} + \lambda_{il})(h_{lj} + \lambda_{lj})] = [\sum_l h_{il}h_{lj}] + [\sum_l \lambda_{il}h_{lj}] + [\sum_l h_{il}\lambda_{lj}] + [\sum_l \lambda_{il}\lambda_{lj}] = h^2 + \lambda h + h\lambda + \lambda^2$. Hence $(\phi^e)_n(x^2) = \phi_n(h^2) + \lambda\phi_n(h) + \phi_n(h)\lambda + \lambda^2$ (matrix multiplication). Using Lemma 2.16 (b), we obtain, for each ξ in $H \otimes \mathbb{C}^n$,

$$
\begin{aligned}
\langle (\phi^e)_n(x^2)\xi, \xi\rangle &= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \phi_n(h)\lambda\xi, \xi\rangle + k\langle \lambda^2\xi, \xi\rangle \\
&= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \xi, \lambda\phi_n(h)\xi\rangle + k\langle \lambda^2\xi, \xi\rangle \\
&= \langle \phi_n(h^2)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle + \langle \lambda\phi_n(h)\xi, \xi\rangle^- + k\langle \lambda^2\xi, \xi\rangle \\
&\geq \langle \phi_n(h^2)\xi, \xi\rangle - 2|\langle \lambda\phi_n(h)\xi, \xi\rangle| + k\|\lambda\xi\|^2 \\
&\geq \langle \phi_n(h^2)\xi, \xi\rangle - 2\|\phi_n(h)\xi\|\|\lambda\xi\| + k\|\lambda\xi\|^2 \\
&\geq (1/k)\langle \phi_n(h)^2\xi, \xi\rangle - 2\|\phi_n(h)\xi\|\|\xi\| + k\|\lambda\xi\|^2 \\
&= [(1/k^{1/2})\|\phi_n(h)\xi\| - k^{1/2}\|\lambda\xi\|]^2 \geq 0.
\end{aligned}
$$

Proof of Theorem 2.1. (1) *implies* (2) and (1) *implies* (3). These are clear.
(2) *implies* (1). Our proof is a CP-analogue of the arguments in [24], applied within the formalism of Stinespring dilation. The vector space $A \otimes H$ is endowed with the non-negative sesquilinear form $\beta(\xi, \eta) = \sum_{i,j}^n \langle \phi(b_j^*a_i)\xi_i, \eta_j\rangle = \langle \phi_n([b_j^*a_i])(\sum \xi_i \otimes e_i), \sum \eta_i \otimes e_i\rangle$ for $\xi = \sum a_i \otimes \xi_i, \eta = \sum b_j \otimes \eta_j$ in $A \otimes H$. Then $N = \{\xi = \sum a_i \otimes \xi_i \in A \otimes H : \beta(\xi, \xi) = \sum \langle \phi(a_j^*a_i)\xi_i, \xi_j\rangle = 0\}$ is a subspace. Let $J : A \otimes H \to A \otimes H/N$, $J\xi = \xi + N$. Then $\langle J\xi, J\eta\rangle = \beta(\xi, \eta)$ makes $A \otimes H/N$ into an inner product space whose completion is denoted by K. Define an action π_o of A on $A \otimes H$ by $\pi_o(a)(\sum x_i \otimes \xi_i) = \sum ax_i \otimes \xi_i$. The admissibility of a positive functional on a Banach $*$-algebra [21, Theorem. 4.5.2, p. 214] is used to conclude that $\beta(\pi_o(a)\xi, \pi_o(a)\xi) \leq \|a\|^2\beta(\xi, \xi)$. Indeed, taking $\tilde{a} = [\delta_{ij}a], \tilde{x} = [\delta_{1i}x_j]$ in $M_n(A)$,

$$\beta[\pi_o(a)\xi,\pi_o(a)\xi] = \sum_{i,j}\langle\phi(x_j^*a^*ax_i)\xi_i,\xi_j\rangle$$
$$= \sum\langle\phi((\tilde{x}^*\tilde{a}^*\tilde{a}\tilde{x})_{j,i})\xi_i,\xi_j\rangle$$
$$= \left\langle\phi_n(\tilde{x}^*\tilde{a}^*\tilde{a}\tilde{x})\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle$$
$$\leq \|\tilde{a}\|^2\left\langle\phi_n(\tilde{x}^*\tilde{x})\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle = \|a\|^2\beta(\xi,\xi),$$

the last inequality being a consequence of the fact that a positive linear functional f on a Banach $*$-algebra with isometric involution satisfies $f(v^*u^*uv)\leq \|u\|^2 f(v^*v)$. One can further verify $\pi_o(ab)=\pi_o(a)\pi_o(b), \beta(\pi_o(a)\xi,\eta)=\beta(\xi,\pi_o(a^*)\eta)$. It follows that for any $a\in A, \pi(a)J\xi = J\pi_o(a)\xi$ gives a well defined bounded linear operator $\pi(a)$: $K\to K$, $\|\pi(a)\|\leq\|a\|$, and $\pi : A\to B(K)$ is a $*$-homomorphism.

Now consider the linear map $F: A\otimes H/N\to H$, $FJ\xi=\sum\phi(x_i)\xi_i$ for $\xi=\sum x_i\otimes\xi_i$ in $A\otimes H$. Then

$$\|FJ\xi\|^2 = \left\langle\sum\phi(x_i)\xi_i,\sum\phi(x_j)\xi_j\right\rangle = \sum_{i,j}\langle\phi(x_j)^*\phi(x_i)\xi_i,\xi_j\rangle$$
$$= \left\langle[\phi(x_j)^*\phi(x_i)]\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle$$
$$= \left\langle\phi_n(x)^*\phi_n(x)\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle \quad (x=[\delta_{1i}x_j])$$
$$\leq k\left\langle\phi_n(x^*x)\sum\xi_i\otimes e_i,\sum\xi_i\otimes e_i\right\rangle \quad \text{(Lemma 2.16)}$$
$$= k\left\langle[\phi(x_j^*x_i)]\sum\xi_i\otimes e_i,\sum\xi_1\otimes e_i\right\rangle$$
$$= k\sum_{i,j}\langle\phi(x_j^*x_i)\xi_i,\xi_j\rangle = k\langle J\xi,J\xi\rangle = k\|J\xi\|^2.$$

This gives a bounded linear operator $F: K\to H$ satisfying $\|F\eta\|\leq k^{1/2}\|\eta\|(\eta\in K)$. Now let $V=F^*$. Then for all $h\in H$, $\xi=\sum x_i\otimes\xi_i\in A\otimes H$, $a\in A$, we have $\langle\sum\phi(x_i)\xi_i,h\rangle = \langle F(J\xi),h\rangle = \langle J\xi, Vh\rangle = \langle\sum x_i\otimes\xi_i, Vh\rangle$ and $\langle\sum\phi(ax_i)\xi_i,h\rangle=\langle\sum x_i\otimes\xi_i,\pi(a^*)Vh\rangle$. Next we claim that $J(b\otimes\eta)=\pi(b)V\eta(b\in A,\eta\in H)$. Indeed, taking ξ as above, $\langle J\xi, J(b\otimes\eta)\rangle = \langle J(\sum x_i\otimes\xi_i), J(b\otimes\eta)\rangle = \sum\langle\phi(b^*x_i)\xi_i,\eta\rangle = \langle\sum\phi(b^*x_i)\xi_i,\eta\rangle = \langle\sum x_i\otimes\xi_i,\pi(b)V\eta\rangle$. Since ξ is arbitrary in $A\otimes H$, and since $J(A\otimes H)$ is dense in K, it follows that $J(b\otimes\eta) = \pi(b)V\eta\,(b\in A,\eta\in H)$. This immediately gives span $(\pi(A)VH)=$ span $J(A\otimes H)$, $[\pi(A)VH]=K$. Finally, for any $a\in A$, and η and η' in H, we have $\langle V^*\pi(a)V\eta,\eta'\rangle = \langle\pi(a)V\eta,V\eta'\rangle = \langle\pi(a\otimes\eta),V\eta'\rangle = FJ(a\otimes\eta),\eta'\rangle = \langle\phi(a)\eta,\eta'\rangle$ showing that $\phi(a)=V^*\pi(a)V$. It follows that ϕ is Stinespring representable.

(3) *implies* (1). In the notations of the proof of Lemma 2.15, the τ-continuous map $\bar{\phi}:\bar{A}\to B(H)$ is hermitian, positive, and it satisfies statement (3) on $\bar{A}$. Further, $(\bar{\phi})_n$ (denoted by $\bar{\phi}_n$) also satisfies $\bar{\phi}_n(h)^2\leq k\bar{\phi}_n(h^2)$ for all $h=h^*$ in $M_n(A)$. This follows from Lemma 2.16, together with the denseness of $M_n(A)$ – as well as the continuity of $\bar{\phi}_n$ – in the projective cross-norm $\gamma=\tau\otimes\|\,\|$ on $\bar{A}\otimes M_n(\mathbb{C})$. By Lemma 2.15, the map $\bar{\phi}^e: (\bar{A})_e\to B(H)$, $\bar{\phi}^e(x+\lambda 1)=\bar{\phi}(x)+\lambda k1$ is positive, which by Lemma 2.17, is completely J-positive. Now as $\bar{A}$ is symmetric, $M_n(\bar{A})$ is also symmetric. It follows from Lemma 2.13 that $\bar{\phi}^e$ is completely positive on the unital Banach $*$-algebra $\bar{A}_e$. By the result of Evans [9, Theorem 2.13], $\bar{\phi}^e$ is Stinespring representable. Hence by (1) imp. (2)

shown earlier, $\bar{\phi}^e(x)^*\bar{\phi}^e(x) \leq m\bar{\phi}^e(x^*x)(x \in \bar{A}_e)$ for some $m > 0$. Thus $\phi(x)^*\phi(x) \leq m\phi(x^*x)(x \in A)$; and by (2) implies (1) shown earlier, ϕ is Stinespring representable.

(1) *implies* (4). This is clear.

(4) *implies* (1). The extension $\phi^e : A_e \to B(H)$ has to be of the form $\phi(x + \lambda 1) = \phi(x) + \lambda\phi^e(1)$. Then ϕ^e is Stinespring representable, so that, in self-explanatory notations, $V^*\pi(x)V = \phi^e(x)(x \in A_e), [\pi(A_e)VH] = K$. Let $K_1 = [\pi(A)VH]$. Since $\pi(A)K_1 \subset K_1$, the projection $P : K \to K_1$ is in the commutant $\pi(A)'$ of $\pi(A)$. Let $\sigma : A \to B(K_1)$ be $\sigma(x) = \pi(x)|_{K_1}$. Then $\pi(x) = P\sigma(x)P = \sigma(x)P$. Let $V_1 = PV$. Then for all $x \in A, \phi(x) = V^*\pi(x)V = V^*P\sigma(x)PV = V_1^*\sigma(x)V_1$. Further, $\sigma(A)V_1H = \sigma(A)PVH = P\sigma(A)PVH = \pi(A)VH$, hence $[\sigma(A)V_1H] = K_1$.

(5) *implies* (6). This is obvious.

(1) *implies* (5). If $\phi(x) = V^*\pi(x)V$, then $\|\phi(x)\| \leq \|V\|^2\|\pi(x)\| \leq \|V\|^2 p_\infty(x)$.

(5) *implies* (1). The statement (5) implies that ϕ factors through $C^*(A)$ giving completely positive map $\tilde{\phi} : C^*(A) \to B(H)$ satisfying $\tilde{\phi} \circ j = \phi$. The conclusion follows easily.

The remaining assertions follow from Lemma 2.8 (d) and Lemma 2.9.

(2.19) For the proof of Theorem 2.2, note that (2) implies (1) follows from Lemma 2.15. That (3) iff (4) is obvious; whereas (4) implies (2) follows from Kadison's inequality for $\tilde{\phi}$. That (1) implies (4) follows as in first paragraph in the proof of Lemma 2.16.

(2.20) For the proof of Corollary 2.3, note that (a) follows from Lemma 2.8 and Theorem 2.2. For (b), Lemma 2.6 implies that $\phi(x)^*\phi(x) \leq (\lim \phi(e_i^*e_i))\phi(x^*x) = \|\phi\|\phi(x^*x)(x \in A)$; and Theorem 2.1 applies.

Theorem 2.21. (*The following is a slightly modified version of* 2.1.) *Let A be a complex $*$-algebra. Let $\phi : A \to B(H)$ be a completely positive map. The following are equivalent.*

(a) *ϕ is Stinespring representable.*

(b) (i) *There exists a scalar $k > 0$ such that $\phi(x^*)\phi(x) \leq k\phi(x^*x)(x \in A)$.*

(ii) *There exists a scalar $m > 0$ and a submultiplicative seminorm p on A such that $\|\phi(x)\| \leq mp(x)(x \in A)$.*

3. Applications and related results

(I) *Cauchy–Schwarz inequalities*: A positive functional f on A satisfies $|f(y^*x|^2 \leq f(y^*y) f(x^*x)$ (x, y in A). Let $\phi \in P(A, H)$. If A is unital (or having a BAI), Kadison's inequality $\phi(h)^2 \leq \|\phi\|\phi(h^2)(h = h^* \in A)$ is an operator valued version. If $\phi \in CP(A, H)$, then Lemma 2.6 provides a CP-version, which, in the presence of BAI, reduces to Corollary 2.8 (c). The following contain some other versions of this inequality.

COROLLARY 3.1

(a) *Let $\phi \in CP(A, H)$. Let x, y, z be in A and $t > 0$ be a scalar.*

(1) *Let ϕ be Stinespring representable. Then the following hold.*

(i) $\phi(y^*x)^*(t + \phi(y^*y))^{-1}\phi(y^*x) \leq \phi(x^*x)$.

(ii) $\phi(x^*)\phi(x) \leq \|\phi^e(1)\|p_\infty(x)^2\phi^e(1)$.

(iii) $\phi(x^*x) \leq \|\phi^e(1)\|p_\infty(x)^2 1$.

(2) *The following hold.*

(i) $\phi(y^*xy)^*\phi(y^*xy) \leq p_\infty(x)^2\|\phi(y^*y)\|\phi(y^*y)$.
(ii) $\phi(y^*xy)^*\phi(y^*xy) \leq \|\phi(y^*y)\|\phi(y^*x^*xy) \leq \|\phi(y^*y)\|p_\infty(x)^2\phi(y^*y)$.
(iii) $\phi(z^*y)^*xz^*[t+\phi(z^*y^*yz)]^{-1}\phi(z^*y^*xz) \leq \phi(z^*x^*xz)$.

(b) *If* $\phi \in P(A,H)$, *then* $\phi(y^*hy)^2 \leq \|\phi(y^*y)\|\phi(y^*h^2y)(y\in A, h=h^*$ *in* $A)$; *and* $\phi(y^*x^*xy) \leq p_\infty(x)\phi(y^*y)$. *Further, if* $\phi \in CP(A,H)$, *then* $\phi(y^*xy)^*\phi(y^*xy) \leq \|\phi(y^*y)\|s(x)^2\phi(y^*y)$. *where* $s(x)=r(x^*x)^{1/2}$, $r()$ *being the spectral radius.*
(c) *If* $\phi \in P(A,H)$ *and has abelian range, then*

$$\phi(y^*x)^*\phi(y^*x) \leq \phi(y^*y)\phi(x^*x) \quad \textit{for all } x,y \textit{ in } A.$$

Does above (1) (i) hold if ϕ is not Stinespring representable ? Above (b) is an operator valued version of the familiar [4, Lemma 37.6 (iii)].

Proof. (a) (1). Inequality (i) can be proved as in [9, Theorem 1.14, p. 15] for positive definite kernels. Representability of $f_\xi, f_\xi(x)=\langle\phi(x)\xi,\xi\rangle$, Theorem 2.1 and Lemma 2.11 (b) (ii) give $\langle\phi(x)^*\phi(x)\xi,\xi\rangle \leq \|\phi(1)\|\langle\phi(x^*x)\xi,\xi\rangle \leq \|\phi(1)\|p_\infty(x)^2\langle\phi(1)\xi,\xi\rangle$ which gives (ii). This in turn implies $\|\phi(x)\| \leq \|\phi(1)\|p_\infty(x)$, hence $\phi(x^*x) \leq \|\phi(x^*x)\|1 \leq \|\phi^e\|p_\infty(x)^2 1$.
(2) For $y\in A$, let $\phi_y(x)=\phi(y^*xy)$. Then ϕ_y is completely positive, since $(\phi_y)_n=(\phi_n)Y$ where $Y=[\delta_{ij}y]$. Thus $\phi_y(x+\lambda 1)=\phi_y(x)+\lambda\phi(y^*y)$ gives a completely positive extension to A_e. Thus ϕ_y is Stinespring representable. Hence (2) follows from (1). Also (b) follows as above from Lemma 2.11 (b) (i). (c) can be easily proved.

One may look for analogue of above inequalities for J-positive and completely J-positive maps.

A dilation-free approach to the CP-Schwarz inequality

Let A be a C^*-algebra. Let $\phi \in CP(A,H)$. The only proof of the CP-Schwarz inequality $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ $(x\in A)$ that the author knows is as in [27, ch. IV, Corollary 3.8, p. 199], which is based on Stinespring dilation. The following provides a dilation-free proof in a more general context. It uses Kadison's Schwarz inequality in a crucial way via Corollary 2.3; and exhibits the essential difference between these two inequalities.

COROLLARY 3.2

(a) *Let A be a Banach* $*$*-algebra with BAI. Let* $\phi: A\to B(H)$ *be a 2-positive map. Then* $\phi(x)^*\phi(x) \leq \|\phi\|\phi(x^*x)$ *for all x in A.*
(b) *There exists a* C^*-*algebra A and a positive, non-2-positive map* $\phi: A\to B(H)$ *for an appropriate H such that for no scalar* $k>0$, $\phi(x)^*\phi(x) \leq k\phi(x^*x)$ *hold for all x in A.*

Proof. (a) The algebra $M_2(A)$ has BAI. By Corollary 2.3, ϕ is extendable, hence hermitian; and the positive map $\phi_2: M_2(A)\to M_2(B(H)) \subset B(H\otimes\mathbb{C}^2)$, $\phi_2=\phi\otimes id$, satisfies $\phi_2(h)^2 \leq \|\phi\|\phi_2(h^2)$ for all $h=h^*$ in $M_2(A)$. Let $x\in A, \xi\in H$. Take $h=\begin{bmatrix}0 & x^*\\ x & 0\end{bmatrix}=h^*$.
Then

$$\begin{bmatrix}\phi(x)^*\phi(x) & 0\\ 0 & \phi(x)\phi(x)^*\end{bmatrix} \leq \|\phi\|\begin{bmatrix}\phi(x^*x) & 0\\ 0 & \phi(xx^*)\end{bmatrix}.$$

Taking $\xi=\xi\otimes e_1+0\otimes e_2 \in H\otimes\mathbb{C}^2$ ($\{e_1,e_2\}$ =standard basis in $\mathbb{C}^2$),

$$\left\langle \begin{bmatrix} \phi(x)^*\phi(x) & 0 \\ 0 & \phi(x)\phi(x)^* \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle$$
$$\leq \|\phi\| \left\langle \begin{bmatrix} \phi(x^*x) & 0 \\ 0 & \phi(xx^*) \end{bmatrix} \begin{bmatrix} \xi \\ 0 \end{bmatrix}, \begin{bmatrix} \xi \\ 0 \end{bmatrix} \right\rangle.$$

Hence $\langle \phi(x)^*\phi(x)\xi, \xi\rangle \leq \|\phi\| \langle \phi(x^*x)\xi, \xi\rangle$.
(b) Let $A = M_2(\mathbb{C}), \phi : M_2(\mathbb{C}) \to M_2(\mathbb{C}) = B(\mathbb{C}^2)$ be $\phi(x) = \mathrm{tr}(x)1 - x$. Then ϕ is known to be positive, but not 2-positive. Let $k > 0$. Suppose $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in A)$. Then $\mathrm{tr}(x)^*\mathrm{tr}(x)1 - \mathrm{tr}(x)^*x - \mathrm{tr}(x)x^* + x^*x \leq k(\mathrm{tr}(x^*x)1 - x^*x)$. Taking $x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathrm{tr}(x) = \mathrm{tr}(x^*) = 0, x^*x = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\mathrm{tr}(x^*x) = 1$. Thus $(k+1)x^*x \leq k1$. Hence for all $z \in \mathbb{C}$,

$$\left\langle \begin{bmatrix} 0 & 0 \\ 0 & k+1 \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix} \right\rangle \leq \left\langle \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix}, \begin{bmatrix} 0 \\ z \end{bmatrix} \right\rangle,$$

giving $(k+1)|z|^2 \leq k|z|^2$, a contradiction.

(II) *Automatic Stinespring representability*: There are three closely related aspects of positive functionals viz. admissibility [21 p. 213], continuity and representability. Admissibility is automatic in Banach $*$-algebras [4, Lemma 37.6, p. 197; 21. Theorem. 4.5.2] (but not in topological $*$-algebras); automatic continuity has been considerably discussed in the literature encompassing more general topological $*$-algebra case; whereas automatic representability seems to have received least attention even in Banach $*$-algebras. Note that representability is stronger than continuity.

COROLLARY 3.3

Assume that $A = A^2$, i.e. $A =$ span $(yx : x, y$ in $A)$.

(a) *Assume the following:*

- (a1) *Every non-zero member of $CP(A,H)$ dominates a non-zero Stinespring representable member of $CP(A,H)$.*
- (a2) *Given $\phi \in CP(A,H)$ and letting $S(\phi) = (\psi \in CP(A,H) : \psi$ is Stinespring representable and $\phi \geq \psi)$, there exists a scalar $k = k_\phi > 0$ such that $\psi(x)^*\psi(x) \leq k\psi(x^*x)$ holds for all x and for all $\psi \in S(\phi)$.*

Then every ϕ in $CP(A,H)$ is Stinespring representable.

(b) *Assume the following:*

- (b1) *Every non-zero member of $P(A,H)$ dominates a non-zero extendable member of $P(A,H)$.*
- (b2) *Given $\phi \in P(A,H)$ and letting $P(\phi) = (\psi \in P(A,H) : \psi$ is extendable and $\phi \geq \psi)$, there exists $k > 0$ such that $\psi(h)^2 \leq k\psi(h^2)$ for all $h = h^*$, all ψ in $P(\phi)$.*

Then every ϕ in $P(A,H)$ is extendable.

(c) *Assume the following:*

- (c1) *Every non-zero positive functional on A dominates a non-zero representable positive functional.*
- (c2) *Above (b2) holds with $H = \mathbb{C}$.*

Then every positive functional on A is representable.

Even in the scalar case, above (c) gives representability analogue of the automatic continuity theorem [4, Theorem 37.13]. In the following (V), we discuss (i) examples showing that in above, assumptions (a2) (and similarily (b2) and (c2) can not be omitted; and (ii) examples in which every positive functional is continuous, but not each such functional is representable.

Lemma 3.4. *Let ϕ_1 and ϕ_2 in $CP(A,H)$ be Stinespring representable. Then $\phi_1+\phi_2$ is also Stinespring representable.*

Proof. Clearly $\phi_1+\phi_2 \in CP(A,H)$. By Theorem 2.1, there exists $k_1>0, k_2>0$ such that for all $x \in A$, $\phi_1(x)^*\phi_1(x) \le k_1\phi_1(x^*x)$, $\phi_2(x)^*\phi_2(x) \le k_2\phi_2(x^*x)$. Let $\phi = \phi_1+\phi_2$. Then

$$\begin{aligned}\phi(x)^*\phi(x) &= \{(\phi_1(x)^* + \phi_2(x)^*\}\{(\phi_1(x)+\phi_2(x))\}\\ &= \phi_1(x)^*\phi_1(x) + \phi_1(x)^*\phi_2(x) + \phi_2(x)^*\phi_1(x) + \phi_2(x)^*\phi_2(x).\end{aligned}$$

Hence, for any $\xi \in H$,

$$\begin{aligned}\langle\phi(x)^*\phi(x)\xi,\xi\rangle &= \|\phi_1(x)\xi\|^2 + \|\phi_2(x)\xi\|^2 + \langle\phi_2(x)\xi,\phi_1(x)\xi\rangle + \langle\phi_1(x),\phi_2(x)\xi\rangle\\ &\le \|\phi_1(x)\xi\|^2 + \|\phi_2(x)\xi\|^2 + 2|\langle\phi_1(x)\xi,\phi_2(x)\xi\rangle|\\ &\le \|\phi_1(x)\xi\|^2 + \|\phi_2(x)\xi\|^2 + 2\|\phi_1(x)\xi\|\|\phi_2(x)\xi\|\\ &= (\|\phi_1(x)\xi\| + \|\phi_2(x)\xi\|)^2\\ &= \{\langle\phi_1(x)^*\phi_1(x)\xi,\xi\rangle^{1/2} + \langle\phi_2(x)^*\phi_2(x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \{k_1^{1/2}\langle\phi_1(x^*x)\xi,\xi\rangle^{1/2} + k_2^{1/2}\langle\phi_2(x^*x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \max(k_1,k_2)\{\langle\phi_1(x^*x)\xi,\xi\rangle^{1/2} + \langle\phi_2(x^*x)\xi,\xi\rangle^{1/2}\}^2\\ &\le \max(k_1,k_2)\{2\langle\phi(x^*x)\xi,\xi\rangle^{1/2}\}^2 \le 4\max(k_1,k_2)\langle\phi(x^*x)\xi,\xi\rangle.\end{aligned}$$

The conclusion now follows from Theorem 2.1.

Proof of Corollary 3.3. We give details for part (a). Since $A^2 = A$, each $a \in A$ is of form $a = \sum u_jv_j$ with u_j, v_j in A. Also, for any u, v in A, $4uv = (v+u^*)^*\,(v+u^*) - (v-u^*)^*\,(v-u^*) + i(v+iu^*)^*(v+iu^*) - i(v-iu^*)^*(v-iu^*)$. It follows that

$$A = \operatorname{span} A^+. \tag{1}$$

Now consider the order relation $\psi \le \phi$ in $CP(A,H)$, where $\psi \le \phi$ means $\phi - \psi \in CP(A,H)$. Clearly, this is reflexive and transitive. Since $A = \operatorname{span} A^+$, it follows that $\le$ is antisymmetric also, hence is a partial order. Let $\phi \in CP(A,H)$. Let $S(\phi) = \{\psi \in CP(A,H) : \psi$ is Stinespring representable and $\psi \le \phi\}$. We show that the partially ordered set $(S(\phi), \le)$ has maximal element.

Let C be any chain in $(S(\phi), \le)$. Let $a \in A$, say $a = \sum \propto_j b_j^* b_j$, finite sum, $\propto_j$ scalars. Then for all $b \in A$, $\psi \in C$, $\xi \in H$, we have

$$\langle\psi(b^*b)\xi,\xi\rangle \le \langle\phi(b^*b)\xi,\xi\rangle. \tag{2}$$

We first show that

$$\phi'(a) = \lim_{\psi \in C}\psi(a) \quad (a \in A) \quad \text{is defined.} \tag{3}$$

By (2), $\lim_{\psi\in C}\langle\psi(b^*b)\xi,\xi\rangle$ exists in $\mathbb{R}$ for all $b\in A, \xi\in H$. Then

$$\lim_{\psi\in C}\langle\psi(a)\xi,\xi\rangle = \lim_{\psi\in C}\left\langle\psi\left(\sum \propto_j b_j^*b_j\right)\xi,\xi\right\rangle = \lim_{\psi\in C}\sum \propto_j \langle\psi(b_j^*b_j)\xi,\xi\rangle$$
$$\leq \sum \propto_j \langle\phi(b_j^*b_j)\xi,\xi\rangle = \langle\phi(a)\xi,\xi\rangle. \tag{4}$$

This defines $\omega_\xi(a) = \lim_{\psi\in C}\langle\psi(a)\xi,\xi\rangle$. Clearly ω_ξ is positive linear functional on A. Now, the polarization identity, for any $T\in B(H)$,

$$4\langle T\xi,\eta\rangle = \langle T(\xi+\eta),\xi+\eta\rangle - \langle T(\xi-\eta),\xi-\eta\rangle + i\langle T(\xi+i\eta),\xi+i\eta\rangle$$
$$- i\langle T(\xi-i\eta),\xi-i\eta\rangle$$

gives, using (4), that for any ξ,η in $H, a\in A$,

$$B_a(\xi,\eta) = \lim_{\psi\in C}\langle\psi(a)\xi,\eta\rangle \text{ exists.} \tag{5}$$

It is easily seen that $(\xi,\eta)\to B_a(\xi,\eta)$ defines a sesquilinear form on H. Further, it is a bounded sesquilinear form. Indeed, for any $\psi\in C, a=\sum\propto_j b_j^*b_j$ in A,

$$\|\psi(a)\| \leq \sum \propto_j \|\psi(b_j^*b_j)\| \leq \sum \propto_j \|\phi(b_j^*b_j)\| = M(a,\phi) = M \text{ (say)}.$$

Hence, by the uniform boundedness principle, there is $k>0$ such that $\|\psi\|\leq k$ for all $\psi\in C$. Hence, in (5), we get

$$|B_a(\xi,\eta)| \leq k\|a\|\|\xi\|\|\eta\|. \tag{6}$$

showing that $B_a(.,.)$ is a bounded sesquilinear form on H. This defines $\phi'(a)\in B(H)$ such that $\langle\phi'(a)\xi,\eta\rangle = B_a(\xi,\eta)$, ($\xi,\eta$ in H). This proves (3).

The mapping $\phi': A\to B(H)$ defined above is linear. Since each $\psi\in C$ is Stinespring representable, hence hermitian, it follows that $\phi'(a^*) = \phi'(a)^*(a\in A)$. Further, (6) implies that ϕ' is continuous. Lemma 2.4 (e) implies that $\phi'\in CP(A,H)$ satisfying $\phi'\leq\phi$. Further, for all $a\in A, \xi$ in H,

$$\langle\phi'(x)^*\phi'(x)\xi,\xi\rangle = \|\phi'(x)\xi\|^2 = \lim_{\psi\in C}\|\psi(x)\xi\|^2 = \lim_{\psi\in C}\langle\psi(x)^*\psi(x)\xi,\xi\rangle$$
$$\leq k_\phi \lim_{\psi\in C}\langle\psi(x^*x)\xi,\xi\rangle \leq k_\phi\langle\phi'(x^*x)\xi,\xi\rangle.$$

Hence ϕ' is Stinespring representable by Theorem 2.1. Thus $\phi'\in C$ and is an upper bound for C.

By Zorn's lemma, $S(\phi)$ admits a maximal element, say ψ_o. We show that $\phi=\psi_0$. If $\phi-\psi_o\neq 0$, then by assumption (a1), there exists a Stinespring representable $\psi_1\in CP(A,H)$ such that $\phi-\psi_o\geq\psi_1$. Thus $\phi\geq\psi_o+\psi_1$; and $\psi_o+\psi_1$ is Stinespring representable completely positive map by Lemma 3.4. Thus $\psi=\psi_o+\psi_1\in S(\phi), \psi\neq\psi_o$, contradicting the maximality of ψ_o. Hence $\phi=\psi_o$ is Stinespring representable.

(III) *Extension problem:* By a celebrated extension theorem due to Arveson [1, Theorem 1.2.3], if B is a closed self-adjoint subspace of a unital C^*-algebra A, and if $1\in B$, then any $\phi\in CP(B,H)$ extends to a $\phi\in CP(A,H)$. The following has a bearing with this.

COROLLARY 2.6

Let A be a Banach $*$-algebra with BAI. Let B be a $*$-subalgebra of A. Assume that at least one of the following holds.

(a) *A is hermitian (in particular, A is spectrally invariant in $C^*(A)$) and B is closed in A.*
(b) *B is a Banach $*$-algebra with some norm such that $C^*(B) \to C^*(A)$ injectively. In particular, the Banach $*$-algebra B is dense and spectrally invariant in A so that $C^*(B) = C^*(A)$.*

Let $\phi \in CP(B,H)$. Then ϕ extends to a $\tilde{\phi} \in CP(A,H)$ iff ϕ is Stinespring representable. If A is unital and B contains identity of A, then under above assumption, any $\phi \in CP(B,H)$ extends to a $\phi \in CP(A,H)$.

Proof. Let $\phi \in CP(B,H)$. Assume (a). Let $\tilde{\phi} \in CP(A,H)$ be such that $\tilde{\phi}|_B = \phi$. By Corollary 2.3, $\tilde{\phi}$ is Stinespring representable; hence by Theorem 2.1, $\phi(x)^*\phi(x) \leq k\phi(x^*x)(x \in B)$. Conversely, let ϕ be Stinespring representable, hence extends to a $\phi^e \in CP(B_e,H)$; and there exists a Hilbert space K, a $*$-representation $\pi : B_e \to B(K)$ and a bounded linear $V : H \to K$ such that $\phi^e(x) = V^*\pi(x)V(x \in B_e)$. Now A_e is hermitian [21, Theorem 4.7.9, p. 223]; and B_e is a $*$-subalgebra of A_e containing the identity. By Lemma [21, Theorem 4.7.20], there exists a Hilbert space W containing K as a closed subspace and a $*$-representation $\sigma : A_e \to B(W)$ such that $\sigma(x)|_K = \pi(x)(x \in B_e)$. Let $id : K \to W$ be the inclusion. Then $P = (id)^* : W \to K$ is the orthogonal projection. Let $V' = id \circ V : H \to W$. Define $\phi'(x) = (V')^*\sigma(x)V'(x \in A_e)$. Then $\phi' \in CP(A_e,H)$ and for all $x \in B, \phi'(x) = V'^*\sigma(x)V' = V'^*P\sigma(x)id\,V' = V'^*\pi(x)V' = \phi(x)$.

Now assume (b). One way conclusion is obvious as in the above case. Let $\phi \in CP(B,H)$ be Stinespring representable, hence extends to $\phi^e \in CP(B_e,H)$. By Lemmas 2.7, 2.8, ϕ^e factors through $C^*(B_e)$ as $\phi^e = \tilde{\phi}^e \circ j_B$ with $\tilde{\phi}^e \in CP(C^*(B_e),H)$. By Lemma 2.10, $C^*(B_e) \to C^*(A_e)$ injectively, and $C^*(B_e)$ is a C^*-subalgebra of $C^*(A_e)$ containing the identity of $C^*(A_e)$. By Arveson extension Theorem, [1, Theorem 1.2.3] $\tilde{\phi}^e$ extends as a $\psi \in CP(C^*(A_e),H)$. Then $\lambda = \psi \circ j_{A_e}|A \in CP(A,H)$ and λ is an extension of ϕ. This completes the proof.

Can the additional assumptions (a) or (b) in above Corollary 2.6 be weakened? Corollary 2.6 (a) applies to closed $*$-subalgebras B of the group algebra $A = L^1(G)$ of a locally compact group G which is symmetric (i.e., $L^1(G)$ is symmetric). A Segal $*$-algebra in a Banach $*$-algebra $(A, \|\cdot\|)$ is a dense $*$-ideal B of A that is a Banach $*$-algebra with some norm $|\cdot|$ such that the inclusion $(B, |\cdot|) \to (A, \|\cdot\|)$ is continuous. By [8, Theorem 31.5], B can not have BAI.

COROLLARY 3.7

Let A have BAI. Let B be a Segal $*$-algebra in A. Let $\phi \in CP(B,H)$. The following are equivalent.

(1) *ϕ extends to a $\tilde{\phi} \in CP(A,H)$.*
(2) *ϕ is Stinespring representable.*
(3) *ϕ is continuous in the norm of A.*

Further, if $A = L^1(G)$ for a locally compact group G, then each of above is equivalent to

(4) *There exists a strongly continuous completely positive definite function $x : g \in G \to x(g) \in B(H)$ such that $\phi(f) = \int_G f(g)x(g)dg(f \in B)$.*

Let G be a locally compact group. Let $x : G \to B(H)$ be a weakly continuous function. Recall that x is positive definite (resp. completely positive definite) if for every $n \in \mathbb{N}$, each $s_1, \ldots, s_n$ in G and each scalars $\lambda_1, \ldots, \lambda_n$ (resp. each $T_1, \ldots, T_n$ in $B(H)$), it holds that $\sum_{ij} \lambda_i\bar{\lambda}_j x(s_j^{-1}s_i) \geq 0$ (resp. $\sum_{ij} T_j^* x(s_j^{-1}s_i)T_i \geq 0$) in $B(H)$.

Proof. Since B is an ideal in $A, B_{-1} = A_{-1} \cap B$, where K_{-1} denote the set of all quasi-regular elements of K. It follows that B is a Q-normed algebra in the norm $\|\cdot\|_A$ of A, i.e. B_{-1} is open in $\|\cdot\|_A$. By [12, Theorem 3.1], each $*$-representation π on B is $\|\cdot\|_A$ continuous; and, by the density of B in A, π extends uniquely to a $*$-representation of A. It follows that A_e and B_e have the same collection of $*$-representations, identified via restriction. Thus srad $B_e =$(srad $A_e \cap B_e$) [4, Theorem 40.9, p. 223], and $p_{\infty|B_e}^{A_e} = p_{\infty}^{B_e}$ (for Gelfand–Naimark pseudonorms). It follows that $C^*(B_e)$ is canonically embedded as a $*$-subalgebra of $C^*(A_e)$. Then (2)$\Leftrightarrow$(1) by Corollary 3.5. That (1)$\Leftrightarrow$(3) is due to continuity, as A has BAI. That (3) iff (4) follows from Stinespring representability on $L^1(G)$ (as it has BAI), the correspondence between unitary representations of G and $*$-representation of $L^1(G)$ and Naimark–Sz. Nagy characterization of completely positive definite functions [9, Corollary 2.6, p. 17], [27, Ex. 2, p. 203]. This completes the proof.

Let A be a $*$-subalgebra of $L^1(G)$. Suppose either G is symmetric and A is closed; or A is a Banach $*$-algebra, dense and spectrally invariant in $L^1(G)$. It follows from Corollaries 3.6 and 3.7 that every completely positive map on A (in particular, on $L^1(G)$) can be extended to a completely positive map on the measure algebra $M(G)$ and is given by a completely positive definite function on G.

(IV) *Integral representations and operator valued Bochner Theorem*: The classical Bochner Theorem states that if ϕ is a positive definite function on $\mathbb{R}^n$, then there exists a positive Radon measure μ on $\mathbb{R}^n$ having mass $\mu(\mathbb{R}^n) = \phi(0)$ such that $\phi(g) = \int_{\mathbb{R}^n} \exp\{i\langle g, \xi\rangle\}\mathrm{d}\mu(\xi)(g \in \mathbb{R}^n)$. More generally, a positive definite function ϕ on a locally compact abelian group G is determined as a positive Randon measure on the dual group $\hat{G}$ by the formula $\phi(g) = \int_{\hat{G}} x(g)\mathrm{d}\mu(x)$. Via $L^1(G)$, this determines positive linear functionals on the Banach $*$-algebra $L^1(G)$; and becomes a special case of the abstract Bochner–Weil–Raikov integral representation [11, ch. IV, Theorem 21.2; 15, Theorem 33.2] stating that a continuous linear functional f on a commutative Banach $*$-algebra A is positive and representable iff there exists a positive Borel measure μ on the hermitian Gelfand space $\mathfrak{M}^*(A) = \{\varphi \in A' : \varphi$ is multiplicative, $\varphi(x^*) = \varphi(x)^-$ for all $x\}$ such that $f(x) = \int_{\mathfrak{M}^*(A)} \hat{x}(\varphi)\mathrm{d}\mu(\varphi)(x \in A), \hat{x}$ denoting the Gelfand transform. The following is an operator valued version of this. It also provides a commutative Banach $*$-algebra analogue of the Naimark dilation theorem [26, Theorem 7.5, p. 153] which is a forerunner of Stinespring theorem. Recall that a semispectral measure on a topological space X is a mapping F from Borel σ-algebra $\mathcal{B}(X)$ into $B(H), \omega \in \mathcal{B}(X) \to F(\omega) \in B(H)$ such that for each ξ in $H, \omega \to \langle F(\omega)\xi, \xi\rangle$ is a bounded positive Borel measure.

COROLLARY 3.8

Let A be a commutative Banach $*$-algebra. Let $\phi : A \to B(H)$ be a linear map.

(A) *Let ϕ be positive. The following are equivalent.*

(1) *ϕ is hermitian and for some scalar $k > 0$, $\phi(h)^2 \leq k\,\phi(h^2)$ for all $h = h^*$ in A.*
(2) *There exists a scalar $k > 0$ such that $\phi(x)^*\phi(x) \leq k\,\phi(x^*x)$ for all $x \in A$.*
(3) *There exists a semispectral measure F on $\mathfrak{M}^*(A)$ such that $\phi(x) = \int_{\mathfrak{M}^*(A)} \hat{x}(f)\,\mathrm{d}F(f)$, $\hat{x}$ denoting the Gelfand transform of x.*
(4) *ϕ is completely positive and Stinespring representable.*

(B) *Let A have BAI. Then* (3) *above is equivalent to*
(5) ϕ is positive.

On the one hand, the following is an operator valued analogue of Bochner's theorem; on the other hand, it is a positive linear map analogue of Stone–Naimark–Ambrose–Godement theorem [20, ch. XV, Theorem 3, p. 489] occupying its proper place midway between the two.

COROLLARY 3.9

Let G be a locally compact abelian group. Let $x : G \to B(H)$ be a weakly continuous function. The following are equivalent.

(1) *x is positive definite.*
(2) *x is completely positive definite.*
(3) *There exists a semispectral measure F on Borel subsets of the dual group $\hat{G}$ such that $x(s) = \int_{\hat{G}} \overline{f(s)} \mathrm{d}F(f)$.*

Proof of Corollary 3.8. (A) That (1) iff (2) iff (4) follow from the results in § 2. Now assume (4). Take extension $\phi^e \in CP(A_e, H)$. Let $\tilde{\phi}^e \in CP(C^*(A_e), H)$ such that $\tilde{\phi}^e \circ j_e = \phi^e$, $j_e(z) = z +$ srad A_e. Then $\tilde{\phi} = \tilde{\phi}^e|_{C^*(A)} \in CP(C^*(A), H)$, $\tilde{\phi} \circ j = \phi$. By Gelfand theory, $C^*(A) = C_o(\mathfrak{M}^*(A))$. $C^*(A_e) = C(X)$, $X = \mathfrak{M}^*(A) \cup \{\infty\}$ being one point compactification of $\mathfrak{M}^*(A)$. By Naimark dilation theorem [26, Theorem 7.5, p. 153], there exists a semispectral measure G on Borel sets of X such that $\tilde{\phi}^e(f) = \int_X f(t) \mathrm{d}G(t)$, $f \in C(X)$. By restriction, G defines a semispectral measure F on $\mathfrak{M}^*(A)$ such that, for all $x \in A$, $\phi(x) = \tilde{\phi}(j(x)) = \int_{\mathfrak{M}^*(A)} \tilde{f}(t) \mathrm{d}F(t)$. Thus (4) $\Leftrightarrow$ (3). If (3) holds, then the positive $\tilde{\phi}(f) = \int_{\mathfrak{M}^*(A)} f(t) \mathrm{d}f(t)$ is completely positive and Stinespring representable, and (2) follows.

Proof of Corollary 3.9. Assume (1). Then $\phi : L^1(G) \to B(H)$, $\phi(f) = \int_G f(s) x(s) \mathrm{d}s$ defines a positive linear map on $L^1(G)$ by $\langle \phi(f)\xi, \eta \rangle = \int_G f(s) \langle x(s)\xi, \eta \rangle \mathrm{d}s (\xi, \eta$ in $H)$. As G is abelian, the Banach $*$-algebra $L^1(G)$ is semisimple and symmetric. Hence $\mathfrak{M}^*(L^1(G)) = \mathfrak{M}(L^1(G)) = \hat{G}$ (dual group) by usual identification. Also, $L^1(G)$ has BAI. It follows from Corollary 3.7 that for some semispectral measure F on $\hat{G}$, $\phi(f) = \int_{\hat{G}} \hat{f}(t) \mathrm{d}F(t)$. Then, for ξ, η in H, for all $f \in L^1(G)$,

$$
\begin{aligned}
\int_G f(s) \langle x(s)\xi, \eta \rangle \mathrm{d}s &= \langle \phi(f)\xi, \eta \rangle \\
&= \int_{\hat{G}} \hat{f}(x) \mathrm{d}\langle F(t)\xi, \eta \rangle \\
&= \int_{\hat{G}} \left(\int_G f(s) \overline{t(s)} \mathrm{d}s \right) \mathrm{d}\langle F(t)\xi, \eta \rangle \\
&= \int_G f(s) \left(\int_{\hat{G}} \overline{t(s)} \mathrm{d}\langle F(t)\xi, \eta \rangle \right) \mathrm{d}s.
\end{aligned}
$$

It follows that $x(s) = \int_{\hat{G}} \overline{t(s)} \mathrm{d}F(t)$ and (3) holds. Now assume (3) viz. $x(s) = \int_{\hat{G}} \overline{t(s)} \mathrm{d}F(t)$ for some semispectral measure F on $\hat{G}$. Then $\phi(f) = \int_{\hat{G}} \hat{f}(t) \mathrm{d}F(t)$ defines a positive linear map on $L^1(G)$. By Corollary 3.7, ϕ is completely positive and Stinespring representable. Let $\{\pi, K, V\}$ be a Stinespring representation of ϕ. By reverting the steps in previous proof, $\phi(f) = \int_G f(s) x(s) \mathrm{d}s (f \in L^1(g))$. Further, there exists a weakly continuous unitary representation $s \to U(s)$ of G on K such that $\pi(f) = \int_G f(s) U(s) \mathrm{d}s$ $(f \in L^1(G))$. Then, for all such f, and ξ, η in K,

$$\int_G f(s)\langle x(s)\xi,\eta\rangle \mathrm{d}s = \langle \phi(f)\xi,\eta\rangle = \langle \pi(f)V\xi, V\eta\rangle$$
$$= \int_G f(s)\langle U(s)V\xi, V\eta\rangle \mathrm{d}s$$
$$= \int_G f(s)\langle V^*U(s)V\xi,\eta\rangle \mathrm{d}s.$$

Hence $\langle x(s)\xi,\eta\rangle = \langle V^*U(s)V\xi,\eta\rangle$. Thus x is completely positive definite; and (2) follows.

(V) *Examples and Remarks*: (3.10) Consider the sequence space $l^p = l^p(\mathbb{N}), 1 \leq p < \infty$. It is a non-unital commutative Banach $*$-algebra with pointwise multiplication, complex conjugation and the norm $\|x\| = (\sum |x_n|^p)^{1/p}$.

(i) $l^p \cdot l^p = \{xy | x, y \text{ in } l^p\}$ is a proper dense subset of l^p [8, p. 113]. Thus l^p is not factorizable, hence it fails to admit BAI. However, $(u_n), u_n = (1, 1, \ldots, 1_n, 0, 0, \ldots)$, constitute unbounded approximate identity for l^p.

(ii) Every positive linear functional on l^p is continuous [23, ch. V, Theorem 5.5, p. 228], and is of the form $f = f_a$ for some $a = (a_n) \in l^q, 1/p + 1/q = 1$, with $a_n \geq 0$ for all n. where $f_a(x) = \sum a_n x_n$. Not every such f is representable. In fact, f_a is representable iff $a \in l^1$. Indeed, $|f_a(x)|^2 \leq k f_a(x^*x) (x \in l^p)$ gives, taking $x = u_n$, that $(a_n) \in l^1$.

(iii) Above (i) and (ii) illustrate that the boundedness of approximate identity can not be omitted from Corollary 2.3.

(iv) By [6], every positive functional on a separable commutative Banach $*$-algebra A is continuous iff $A^2 (= \text{span } A \cdot A)$ is closed and of finite co-dimension. What is an analogous theorem for automatic representability? This result implies that $(l^p)^2$ is closed. Since $(l^p \cdot l^p)^- = l^p$, $(l^p)^2$ is dense in l^p. Thus $l^p = (l^p)^2$. This illustrates that in a commutative Banach $*$-algebra A, the condition $A^2 = A$ need not imply automatic representability; though it does imply automatic continuity of positive functionals [4, Theorem 37.14, p. 201].

(v) Let f be a positive functional on l^p, $f = f_a$. Let $b = (a_1, \ldots, a_n, 0, 0, \ldots,)$. Then f_b is representable and $f \geq f_b$. This shows that Corollary 3.3 fails if the assumption (c2) (and so (b2) and (a2)) are omitted. Thus, [4, Theorem 37.13, p. 200] does not hold if 'continuity' is replaced by 'representability'.

(vi) Every continuous hermitian linear functional on l^p is a difference of two positive linear functionals; though it need not be a difference of two representable positive functionals. This illustrates the crucial role of representability in Grothendieck's well-known dual characterization of C^*-algebras.

3.11. Here are some concrete examples to which Corollary 3.7 applies. (a) For a locally compact abelian group G, take $A = L^1(G)$, let $1 < p < \infty$. Take $B_1 = L^1(g) \cap L^p(G)$ with norm $\|f\|_{B_1} = \|f\|_1 + \|f\|_p$; $B_2 = \{f \in L^1(G) : \hat{f} \in L^p(\hat{G})\}$, $\|f\|_{B_2} = \|f\|_1 + \|\hat{f}\|_p$. Then each B_i is a convolution Segal algebra in $L^1(G)$ and having involution $f^*(s) = \overline{f(-s)}$. (b) Take $A = L^1(\mathbb{R}), B_k = \{f \in L^1(\mathbb{R}) \cap C^k(\mathbb{R}) |$ the kth derivative $f^{(k)} \in L^1(\mathbb{R})\}$, $\|f\|_{B_k} = \|f\|_1 + \|f^{(k)}\|_1$. By Corollaries 3.7, 3.8 and 3.9, Stinespring representable maps on $B (=$ any of above $B_i)$ are precisely those that are determined by semispectral measures on the dual group $\hat{G}$.

3.12. Another abstract Segal $*$-algebra is this. Let $A = K(H)$, C^*-algebra of all compact operators on a separable Hilbert space H, let $B = C^p(H) = \{x \in K(H) : \|x\|_p = [\text{trace}$

$(x^*x)^{p/2}]^{1/p} < \infty\}$, $1 \le p < \infty$, the Banach $*$-algebra with norm $\|\cdot\|_p$ of von Neumann–Schatten class of operators, which is a dense $*$-ideal in A containing all finite rank operators. Let (ξ_n) be an orthonormal basis in H. Then $(C^2(H), \|\cdot\|_2)$ is a Hilbert space with norm $\|x\|_2 = \langle x, x\rangle^{1/2}$, where $\langle x, y\rangle = \sum_n \langle x\xi_n, y\xi_n\rangle$. By using [23, ch. IV, Theorem 5.5, p. 228], every positive linear functional f on $C^p(H)$ is $\|\cdot\|_p$-continuous; and by the well-known duality $\langle C^p(H), C^q(H)\rangle$, $1/p + 1/q = 1$, $\langle x, y\rangle = \text{trace } xy^*$, is of the form $f = f_a, f_a(x) = \langle x, a\rangle$ with $a \ge 0$ in $B(H)$ [22]. Further, f_a is representable iff $a \in C^1(H)$ (= trace class operators) iff f is the restriction of a normal positive functional on $B(H)$. An appeal to Grothendieck's result refered to in (3.10) implies that not every positive functional is representable. Let $\phi \in CP(C^p(H), K)$ for some Hilbert space K. Then ϕ is Stinespring representable iff ϕ extends as a completely positive map $\tilde{\phi} : K(H) \to B(K)$ iff ϕ is the restriction of a normal completely positive map on $B(H)$.

3.13. Corollary 3.6 applies to smooth subalgebras of a C^*-algebra. A smooth subalgebra B of a C^*-algebra A is a dense $*$-subalgebra of A which is a Banach algebra with some norm and is spectrally invariant in A. Then $C^*(B) = A$. If B has BAI, then every completely positive map on B extends to a completely positive map on A. This, in particular, applies to the Banach $*$-algebra $C^m(A)$ of C^m-elements (and to the Frechet algebra $C^\infty(A)$ of C^∞-elements) of an action of a Lie group on a C^*-algebra A.

3.14. Banach $*$-algebras to which Theorems 2.1, 2.2 apply specifically are those that do not admit BAI. This includes non-factorizable Banach $*$-algebras [8], the algebra $R(G)$ which is linear span of positive definite functions on a compact group G, and the Fourier algebra of a locally compact, non-compact group [19]. Let G be a compact Lie group. The convolution Banach algebras $C^m(G)$(C^m-functions on G, $0 \le m < \infty$) and $L^p(G)$, $1 < p < \infty$ are non-unital, and not admitting BAI [15, (34.40) (b), p. 357]. They are Banach $*$-algebras with involution $f^*(g) = \overline{f(g^{-1})}$ [14, Theorem 15.14, p. 197]. A map $\phi \in CP(C^m(G), H)$ (or $\phi \in CP(L^p(G), H)$) is Stinespring representable iff it extends as a completely positive map on $L^1(G)$ iff it extends as a completely positive map on the group C^*-algebra $C^*(G)$ of G iff it is determined by a completely positive definite function on G [9, p. 20]. Note that continuous positive functionals on $C^m(G)$ are given by distributions of the positive type of order m on G; whereas representable functionals are given by continuous positive definite functions on G. A similar assertion for $L^p(G)$ explains [15, (34.42 (b)), p. 358].

In particular, consider $G = T = \{z \in \mathbb{C} | \, |z| = 1\}$. Recall [16] that an orthogonal basis in a Banach algebra A is a basis $(e_n)_0^\infty$ such that $e_n e_m = \delta_{nm} e_n$. A Banach algebra with an orthogonal basis is commutative, non-unital (if infinite dimensional) and $\mathfrak{M}(A) = \{e_n^*\} \approx \mathbb{N}$, e_n^* is the coefficient functional $e_n^*(x) = \alpha_n$, where $x = \sum_0^\infty \alpha_n e_n$ is the expansion of x in (e_n). The convolution algebra $L^p(T)$, $1 < p < \infty$, admits the sequence of trigonometric polynomials $e_n(t) = t^n, n \ge 1$, as orthogonal basis, the Fourier series $f(e^{i\theta}) \sim \sum a_n e^{in\theta}$, $a_n = \hat{f}(n)$, provides expansion $f(e^{i\theta}) = \sum \hat{f}(n) e^{in\theta}$ in $\|\cdot\|_p$. It is a Banach $*$-algebra with involution $f^*(e^{i\theta}) = \overline{f(e^{-i\theta})}$. Now let $U = \{z \in \mathbb{C} | \, |z| < 1\}$. The Hardy space $H^p(U)$ is also a Banach $*$-algebra with Hadamard product $(f * g)(x) = (1/2\pi i) \int_{|z|=r} f(z) \; g(xz^{-1}) z^{-1} dz$, $|x| < r < 1$, having involution $f^*(z) = \overline{f(\bar{z})}$. The sequence $e_n(z) = z^n$, $n \in \mathbb{N}$, is an orthogonal basis for $H^p(U)$ [16], the Taylor series $f(z) = \sum_0^\infty (f^{(n)}(0)/n!) z^n$ being expansion of f in terms of (e_n). Now, by [13, ch. II, §3, p. 59]. via the radial limit $f(z) \to f(e^{i\theta})$, $H^p(U)$ is isometric to a closed subspace K of $L^p(T)$, where K = the space of boundary functions of $H^p(U)$ = the closure in $L^p(T)$ of analytic polynomials. The Fourier series $f(e^{i\theta}) \sim \sum a_n e^{in\theta}$ of $f \in K$ is supported on non-negative integers; and the Fourier coefficients $a_n = f^{(n)}(0)/n! =$ Taylor coefficients of the H^p-function $f(z) = \sum a_n z^n$. Thus the embedding preserves the multiplication and the envolution. The Banach $*$-algebras $H^p(U)$ and $L^p(T)$ are hermitian; and their enveloping C^*-algebras are $C^*(H^p(U)) \approx c_0(\mathbb{N})$ as in [3, Proposition 5.3], $C^*(L^p(T)) = C^*L^1(T)) = C^*(T)(=$ group C^*-algebra of $T) = C_0(\hat{T}) = C_0(\mathbb{Z})$. Thus $C^*(H^p(U)) \to C^*(L^p(T))$ injectively. Let H be a Hilbert space and $\phi : H^p(U) \to B(H)$ be positive linear. Then ϕ is extendable iff ϕ is completely positive Stinespring representable iff there exists a sequence $F(n), n \in \mathbb{N}$, of positive operators in $B(H)$ such that $\phi(f) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} F(n)$.

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