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Limit algebras of differential forms in non-commutative geometry

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Abstract. Given a C^*-normed algebra A which is either a Banach $*$-algebra or a Frechet $*$-algebra, we study the algebras $\Omega_\infty A$ and $\Omega_\epsilon A$ obtained by taking respectively the projective limit and the inductive limit of Banach $*$-algebras obtained by completing the universal graded differential algebra $\Omega^* A$ of abstract non-commutative differential forms over A. Various quantized integrals on $\Omega_\infty A$ induced by a K-cycle on A are considered. The GNS-representation of $\Omega_\infty A$ defined by a d-dimensional non-commutative volume integral on a d^+-summable K-cycle on A is realized as the representation induced by the left action of A on $\Omega^* A$. This supplements the representation A on the space of forms discussed by Connes (Ch. VI.1, Prop. 5, p. 550 of [C]).

Keywords. Frechet $*$-algebra; graded differential algebra; non-commutative differential forms; quantized integrals; K-cycle; GNS representation.

1. Introduction

Let $(A, \|\cdot\|)$ be a C^*-normed algebra with identity 1. Let $\tilde{A}$ be the C^*-algebra completion of A. We recall the construction of the universal graded differential algebra $\Omega^* A$ over A (Ch. III.1, p. 185 of [C], §8.1, p. 320 of [GVF]) whose elements are the abstract non-commutative differential forms over A. Consider the A-bimodule $A \otimes A$. Let $d\colon A \to A \otimes A$, $da := 1 \otimes a - a \otimes 1$ which defines a derivation. Let $\Omega^1 A$ be the submodule generated by $\{adb = a \otimes b - ab \otimes 1\colon a, b \in A\}$, the module operations being $a^1(adb) = a^1 adb$, $(adb)a^1 = ad(ba^1) - abda^1$. Taking $\bar{A} := A/\mathbb{C}$ and denoting the elements of $\bar{A}$ by $\bar{a} = a + \mathbb{C}$, $a \in A$, the map $a_0 \otimes \overline{a_1} \to a_0 da_1$ establishes an A-bimodule isomorphism from $A \otimes \bar{A}$ to $\Omega^1 A$. The elements of $\Omega^1 A$ are abstract non-commutative differential forms of degree 1 over A. Defining non-commutative forms of degree k as $\Omega^k A := \Omega^1 A \otimes_A \Omega^1 A \otimes_A \cdots \otimes_A \Omega^1 A \simeq A \otimes \bar{A}^{\otimes k}$ an A-bimodule for each $k = 1, 2, 3, \ldots$, we form the graded algebra $\Omega^* A := \oplus_{n=0}^{\infty} \Omega^n A$. The multiplication in $\Omega^* A$ is given by

$$\begin{aligned}(a_0 da_1 da_2 \cdots da_n)(a_{n+1} da_{n+2} \cdots da_m) = {} & \sum_{j=1}^{n} (-1)^{n-j} a_0 da_1 da_2 \cdots d(a_j a_{j+1}) \\ & \times da_{j+2} \cdots da_n da_{n+1} \cdots da_m \\ & + (-1)^n a_0 da_1 da_2 \cdots da_m.\end{aligned}$$

The map $d\colon \Omega^n A \to \Omega^{n+1} A$, $d(a_0 da_1 da_2 \cdots da_n) = da_0 da_1 da_2 \cdots da_n$ defines a derivation on $\Omega^* A$ satisfying $d^2 = 0$, $d(\omega_1\omega_2) = (d\omega_1)\omega_2 - (-1)^{\deg \omega_1}\omega_1 d\omega_2$. Then $\Omega^* A$ is a $*$-algebra with $(dx)^* = -dx^*$, $x \in A$. We consider the limit algebras of $\Omega^* A$ in the following situations.

(i) A is a Banach $*$-algebra with a norm $|\cdot|$.
(ii) A is a Frechet $*$-algebra with a topology defined by a sequence of seminorms $\{|\cdot|_n\}$.

These are prototype situations that occur frequently.

In each of these cases, locally convex $*$-algebras $\Omega_\infty A$ and $\Omega_\epsilon A$ are obtained by taking respectively projective limits and inductive limits of a sequence of Banach $*$-algebras $\Omega_r A$, $r > 0$ which are completions of $\Omega^* A$ in suitable norms. The construction $\Omega_\infty A$ is essentially due to Arveson [A] (done in a different but related context), whereas that of $\Omega_\epsilon A$ is due to Connes (p. 373 of [C]). Basic structural properties of these algebras are discussed in §2 and §3. Connes (Ch. IV.7, Prop. 10, p. 374 of [C]) showed that $\Omega_\epsilon A$ is a quasinilpotent extension of A via the augmentation $\epsilon\colon \Omega_\epsilon A \to A$, $\epsilon\ (\omega = \sum_{k=0}^{\infty} \omega_k) = \omega_0$. This is supplemented by showing that for $\Omega_\epsilon A$, the star radical coincides with the radical which is the kernel of ϵ. In §4, concrete realizations of $\Omega_\infty A$ and $\Omega_\epsilon A$ as operator algebras are obtained by imposing a non-commutative geometric data on A via a K-cycle (spectral triple) $(\pi, H, \mathcal{D})$. The holomorphic functional calculus closure of Connes' non-commutative de Rham algebra $\Omega^*_{\mathcal{D}}$ (p. 549 of [C]) leads to a couple of operator algebras which are briefly discussed in this section. In §5, which contains the main contributions of the paper, quantized integrals are constructed on $\Omega_\infty A$ by using Dixmier trace assuming A to be a Banach $*$-algebra. This is made possible by extending to $\Omega_\infty A$ the canonical representation of $\Omega^* A$ defined by a K-cycle on A (p. 373 of [C]). This is obtained by using an automatic continuity theorem of Johnson and Sinclair [JS]. The GNS representation of $\Omega_\infty A$ defined by a d-dimensional non-commutative volume integral on a d^+-summable K-cycle is realized as the representation induced by the left action of A on $\Omega^* A$. This substantially supplements the representation of $\Omega^*_{\mathcal{D}}$ discussed in Ch. VI.1, Prop. 5, p. 550 of [C]. For topological algebras, we refer to [M] and [F3].

2. When A is a Banach $*$-algebra $(A, |\cdot|)$

The complete norm $|\cdot|$ on A is necessarily finer than the C^*-norm $\|\cdot\|$. Following Arveson [A], (p. 373 of [C]), the following system of norms is defined on $\Omega^* A$,

$$\left| \omega := \sum_{0}^{\infty} \omega_k \right|_r = \sum_{k=0}^{\infty} r^k |\omega_k|_\pi, \quad r \in \mathbb{R}^+$$

where $\omega_k \in \Omega^k A$ is the k-th degree part of ω, and $|\cdot|_\pi$ is the projective tensor product norm on the space $\Omega^k A \simeq A \otimes \bar{A}^{\otimes k}$ of forms of degree k arising from the complete norm $|\cdot|$ on A. Let

$$\begin{aligned} \Omega_r A &= \left(\Omega^* A, |\cdot|_r\right)^{\sim} \quad \text{the completion} \\ &= \left\{ \omega = \sum_{0}^{\infty} \omega_k \colon \omega_k \in \Omega^k A\ \forall k \quad \text{and} \quad \sum_{0}^{\infty} r^k |\omega_k|_\pi < \infty \right\} \end{aligned}$$

a Banach $*$-algebra with norm $|\omega|_r := \sum_0^\infty r^k |\omega_k|_\pi$. The following two limit algebras are formed with these system of Banach $*$-algebras.

(a) Arveson:

$$\Omega_\infty A := \varprojlim_{r\to\infty} \Omega_r A \quad \text{(inverse limit).}$$

(b) Connes:

$$\Omega_\epsilon A := \varinjlim_{r\to 0} \Omega_r A \quad \text{(direct limit).}$$

Let $\epsilon_r\colon \Omega_r A \to A$, $\epsilon_r\big(\omega = \sum_0^\infty \omega_k\big) := \omega_0$. It is a surjective $*$-homomorphism.

PROPOSITION 2.1

(1) *The algebra $\Omega_\infty A$ is a Frechet $*$-algebra whose bounded part $b(\Omega_\infty A)$ is A.*
(2) *There exists continuous $*$-homomorphisms $\varphi_r\colon C^*(\Omega_r A) \to \tilde{A}$, $\varphi\colon E(\Omega_\infty A) \to \tilde{A}$ where $C^*(\Omega_r A)$ (respectively $E(\Omega_\infty A)$) is the enveloping C^*-algebra of $\Omega_r A$ (respectively the enveloping $\sigma - C^*$-algebra* [B1] *of $\Omega_\infty A$).*

Proof.

(1) By definition, the bounded part of the Frechet $*$-algebra $\Omega_\infty A$ is the Banach $*$-algebra

$$b(\Omega_\infty A) := \left\{\omega = \sum_0^\infty \omega_k \in \Omega_\infty A\colon \sup_r |\omega|_r < \infty\right\}$$

with the norm $\|\omega\| := \sup_r |\omega|_r$. Thus if $\omega = \sum_0^\infty \omega_k \in b(\Omega_\infty A)$, then $\sup_{r>0} \sum_0^\infty r^k |\omega_k|_\pi = M < \infty$. Hence for all $k \in \mathbb{N}$ and all $r > 0$, $r^k|\omega_k|_\pi \le M$. It follows that $\omega_k = 0$ for all $k \neq 0$ taking $r > 1$. Thus $\omega = \omega_0 \in A$ and $\|\omega\| = |\omega|$.
(2) For any $\omega = \sum_0^\infty \omega_k \in \Omega_r A$, $\|\omega_0\| = \|\epsilon_r(\omega)\| \le |\omega_0| = |\epsilon_r(\omega)| \le |\omega|_r$. Thus $\omega \to \|\omega_0\|$ is a continuous C^*-seminorm on $\Omega_r A$. Therefore $\|\epsilon_r(\omega)\| \le |\omega|_r^1$, where $|\cdot|_r^1$ is the Gelfand–Naimark C^*-seminorm on $\Omega_r A$ defined as $|\omega|_r^1 := \sup_\sigma \|\sigma(\omega)\|$, σ running over all $*$-representations of $\Omega_r A$ on Hilbert spaces. Recall that $C^*(\Omega_r A)$ is the Hausdorff completion of $(\Omega_r A, |\cdot|_r^1)$. Thus the start radical $\mathrm{srad}\Omega_r A := \ker|\cdot|_r^1 \subset \ker \epsilon_r$, and there exists a $*$-homomorphism $\varphi_r\colon \Omega_r A/\mathrm{srad}\Omega_r A \to \Omega_r A/\ker \epsilon_r = A$ which is continuous in the respective C^*-norms. This then extends to a continuous surjective $*$-homomorphism $\varphi_r\colon C^*(\Omega_r A) \to \tilde{A}$. Now the enveloping $\sigma - C^*$-algebra of $\Omega_\infty A$, which is the Hausdorff completion of $(\Omega_\infty A, \{|\cdot|_r^1\})$, satisfies $E(\Omega_\infty A) = \varprojlim_{r\to\infty} C^*(\Omega_r A)$ [F2]. Thus there exists a continuous surjective $*$-homomorphism $\varphi\colon E(\Omega_\infty A) \to \tilde{A}$. □

Now we consider the algebra $\Omega_\epsilon A = \varinjlim_{r\to 0} \Omega_r A = \cup_{n=1}^\infty \Omega_{1/n} A$ with the inductive limit topology which is the finest locally convex topology τ making the embeddings $\Omega_r A \to \Omega_\epsilon A$ continuous. It is a locally convex topological $*$-algebra, and $\epsilon\colon \Omega_\epsilon A \to A$, $\epsilon(\omega = \sum_0^\infty \omega_k) = \omega_0$ is a continuous surjective $*$-homomorphism. Connes (Ch. IV.7, Prop. 10, p. 374 of [C]) pointed out that the ideal $\ker \epsilon$ is quasinilpotent in the sense that for any scalar $\lambda \neq 0$, $(\lambda 1 - \omega)$ is invertible for any $\omega \in \ker \epsilon$. The following supplements this.

Theorem 2.2.

(1) *$\Omega_\epsilon A$ is a locally m-convex m-barrelled Q-algebra; and $\Omega^* A$ is sequentially dense in $\Omega_\epsilon A$.*

(2) srad $\Omega_\epsilon A = \text{rad } \Omega_\epsilon A = \ker \epsilon$.
(3) *$\Omega_\epsilon A$ is a spectral algebra, and its enveloping pro-C^*-algebra $E(\Omega_\epsilon A)$ is isomorphic to the enveloping C^*-algebra $C^*(A)$ of A.*
(4) *If A is spectrally invariant in $\tilde{A}$, then $\Omega_\epsilon A$ is a C^*-spectral algebra satisfying $E(\Omega_\epsilon A) = \tilde{A}$, and $\Omega_\epsilon A$ and $\tilde{A}$ have the same K-theory.*

Proof.

(1) By Lemma 10.2, p. 317; Coro. 10.2, p. 319 of [M], $\Omega_\epsilon A$ is a locally m-convex Q-algebra. It is m-barrelled by p. 122 of [M]. The denseness of $\Omega^* A$ in $\Omega_\epsilon A$ follows from the definition of the inductive topology on $\Omega_\epsilon A$.
(2), (3) Since $\Omega_\epsilon A$ is a Q-algebra, Lemma 2.10 of [B2] implies that the enveloping pro-C^*-algebra of $\Omega_\epsilon A$ is a C^*-algebra. By 7.5.10, p. 374 of [C], $\ker \epsilon$ is a quasinilpotent ideal. By Thm 3.3.2, p. 55 of [R], $\ker \epsilon \subset \text{rad } \Omega_\epsilon A$. Then for all ω in $\Omega_\epsilon A$,

$$\begin{aligned}
\text{sp}_{\Omega_\epsilon A}(\omega) &= \text{sp}_{\Omega_\epsilon A/\text{rad } \Omega_\epsilon A}(\omega + \text{rad } \Omega_\epsilon A) \\
&\subset \text{sp}_{\Omega_\epsilon A/\ker \epsilon}(\omega + \ker \epsilon) \subset \text{sp}_{\Omega_\epsilon A}(\omega) \\
&= \text{sp}_A(\epsilon(\omega)).
\end{aligned}$$

Thus

$$\text{sp}_{\Omega_\epsilon A}(\omega) = \text{sp}_{\Omega_\epsilon A/\ker \epsilon}(\omega + \ker \epsilon) = \text{sp}_A(\epsilon(\omega)).$$

(The referee has pointed out that this also follows as: If $\epsilon(\omega)$ is invertible, put $\eta = \epsilon(\omega)^{-1}\omega$. As $\epsilon(\eta) = 1$, η is invertible by Connes observation, and ω is also invertible.) Let $|\cdot|_\infty$ be the Gelfand–Naimark seminorm on A, which is a norm as $(A, \|\cdot\|)$ is C^*-normed. Then $p_\infty(\omega) := |\epsilon(\omega)|_\infty$ defines a continuous C^*-seminorm on $\Omega_\epsilon A$. We show that it is the greatest C^*-seminorm on $\Omega_\epsilon A$. Let q be any C^*-seminorm on $\Omega_\epsilon A$, necessarily continuous as $\Omega_\epsilon A$ is a Q-algebra [F1]. Let $\pi_q\colon \Omega_\epsilon A \to B(H_q)$ be the $*$-representation defined by q. Then for all $\omega \in \Omega_\epsilon A$,

$$\begin{aligned}
q(\omega)^2 &= q(\omega^*\omega) = \|\pi_q(\omega^*\omega)\| \\
&= r_{B(H_q)}(\pi_q(\omega^*\omega)) \\
&\leq r_{Im(\pi_q)}(\pi_q(\omega^*\omega)) \\
&\leq r_{\Omega_\epsilon A}(\omega^*\omega) = r_A(\omega_0^*\omega_0) \\
&\leq |\omega_0^*\omega_0| \leq |\omega_0|^2 = |\epsilon(\omega)|^2.
\end{aligned}$$

It follows that $\ker \epsilon \subset \ker q$. Hence given $\omega = \sum_0^\infty \omega_k$, $\omega' = \sum_0^\infty \omega'_k$ in $\Omega_\epsilon A$, $\omega_0 = \omega'_0$ implies that $q(\omega - \omega') = 0$, $q(\omega) = q(\omega')$. Thus $q_0(\omega_0) := q(\omega = \sum_0^\infty \omega_k)$ is a well-defined C^*-seminorm on A. Hence $q_0 \leq |\cdot|_\infty$. It follows that $q(\omega) \leq p_\infty(\omega)$ for all $\omega \in \Omega_\epsilon A$. Thus $p_\infty(\cdot)$ is the greatest C^*-seminorm on $\Omega_\epsilon A$. Then the enveloping pro-C^*-algebra of $\Omega_\epsilon A$ is the C^*-algebra $C^*(\Omega_\epsilon A)$ which is the Hausdorff completion of $(\Omega_\epsilon A, p_\infty(\cdot))$. But

$$\begin{aligned}
\text{srad}\Omega_\epsilon A &= \ker p_\infty \\
&= \{\omega \in \Omega_\epsilon A\colon \epsilon(\omega) = 0\} \\
&= \ker \epsilon \subset \text{rad}\Omega_\epsilon A \subset \text{srad}\Omega_\epsilon A.
\end{aligned}$$

Thus $\mathrm{rad}\Omega_\epsilon A = \mathrm{srad}\Omega_\epsilon A = \ker\epsilon$ and

$$\begin{aligned} C^*(\Omega_\epsilon A) &= (\Omega_\epsilon A/\ker\epsilon, \tilde{p}_\infty(\cdot))^\sim \text{ completion} \\ &= C^*(A). \end{aligned}$$

(4) If A is spectrally invariant in $\tilde{A}$, then $C^*(A) = \tilde{A}$, and for any $\omega \in \Omega_\epsilon A$,

$$\begin{aligned} r_{\Omega_\epsilon A}(\omega) &= r_A(\epsilon(\omega)) = r_{\tilde{A}}(\epsilon(\omega)) \\ &\le \|\omega_0\| \end{aligned}$$

showing that $\Omega_\epsilon A$ is a C^*-spectral algebra. By [BIO2], $\Omega_\epsilon A$ is local, and $K_*(\Omega_\epsilon A) = K_*(\Omega_\epsilon A/\mathrm{rad}\Omega_\epsilon A) = K_*(\Omega_\epsilon A/\ker\epsilon) = K_*(A) = K_*(\tilde{A})$ the last equality being a consequence of spectral invariance of A in $\tilde{A}$. This completes the proof. $\square$

3. When A is a Frechet $*$-algebra $(A, \{|\cdot|_n\})$

In this case, we assume that the enveloping $\sigma - C^*$-algebra $E(A)$ of A is the C^*-algebra $\tilde{A}$ and that each $|\cdot|_n$ is closable with respect to the C^*-norm $\|\cdot\|$ in the sense that for any sequence (x_k) in A, if (x_k) is $|\cdot|_n$-Cauchy and $\|x_k\| \to 0$, then $|x_k|_n \to 0$. This is a typical situation exemplified in the following.

(a) For a compact manifold M, $A = C^\infty(M)$ and $\tilde{A} = C(M)$.
(b) For a Lie group G acting on a C^*-algebra B, $A = C^\infty(B)$ the C^∞-elements of B determined by the action [Br]. For a densely defined closable derivation δ on B, $A = C^\infty(\delta)$.
(c) For a finitely algebraically generated dense $*$-subalgebra K of a C^*-algebra B, $A = \mathcal{S}(K)$ is the smooth envelop of K in the sense of Blackadar and Cuntz [BC].

Since $E(A) = \tilde{A}$, the C^*-norm $\|\cdot\|$ on A is the greatest C^*-seminorm on A, automatically continuous [F1] in the topology t defined by the sequence $\{|\cdot|_n\}$ of seminorms assumed increasing without loss of generality. Thus there exists n_0 such that $\|x\| \le |x|_{n_0}$ for all $x \in A$. We can assume that $n_0 = 1$, $\|\cdot\| \le |\cdot|_n$ for all n, each $|\cdot|_n$ is a norm of the form $\|\cdot\| + |\cdot|_n$ and $\{|\cdot|_n\}$ is increasing.

Now let $A = \varprojlim A_n$ be the Arens–Micheal decomposition of A expressing A as an inverse limit of Banach $*$-algebra A_n [F3]. Here $A_n = (A, |\cdot|_n)^\sim$ completion of A in $|\cdot|_n$. The closability of $|\cdot|_n$ with respect to $\|\cdot\|$ implies that $A_n \subset \tilde{A}$. Indeed, $\|\cdot\| \le |\cdot|_n$ on A implies that the identity map on A extends as a continuous $*$-homomorphism $\varphi_n \colon A_n \to \tilde{A}$. Let $x \in \ker\varphi_n$. Then for some sequence (x_k) in A, $|x_k - x|_n \to 0$ and $x_k = \varphi_n(x_k) \to \varphi_n(x) = 0$ in $\|\cdot\|$. By the closability, $|x_k|_n \to 0$. It follows that $x = 0$. Thus $\ker\varphi_n = 0$, and $A_n \subset \tilde{A}$. Further, closability of each $|\cdot|_n$ with respect to $\|\cdot\|$ implies that $|\cdot|_n$ is closable with respect to $|\cdot|_m$ for any $n > m$. Thus $A_n \subset A_m$ for $n > m$. Hence $A = \varprojlim A_n = \cap_{n=1}^\infty A_n$. This makes available the techniques and results of previous section.

For each $r > 0$ and $n \in \mathbb{N}$, let $|\cdot|_{n,r}$ be the norm on $\Omega^* A$ defined by $|\omega = \sum_0^\infty \omega_k|_{n,r} := \sum_{k=0}^\infty r^k |\omega_k|_{n,\pi}$ where $|\cdot|_{n,\pi}$ is the projective cross-norm on $\Omega^k A$ arising from $|\cdot|_n$. Let $(\Omega^* A)^\sim_{n,r} = (\Omega^* A, |\cdot|_{n,r})^\sim$ completion which is a Banach $*$-algebra. Further, $A \subset A_n$ implies that $\Omega^* A \subset \Omega^* A_n$; and $|\cdot|_{n,r}$ also defines a $*$-algebra norm on $\Omega^* A_n$. Let $\Omega_r A_n := (\Omega^* A_n, |\cdot|_{n,r})^\sim$ completion. Since A is dense in A_n, $\Omega^* A$ is dense in $\Omega^* A_n$ in

$|\cdot|_{n,r}$. Hence $(\Omega^*A)^{\sim}_{n,r} = \Omega_r A_n = \{\omega = \sum_0^\infty \omega_k \colon \omega_k \in \Omega^k A_n, \sum_{k=0}^\infty r^k |\omega_k|_{n,\pi} < \infty\}$. Let the Frechet $*$-algebra $\Omega_r A$ be the completion of $\Omega^* A$ in the topology τ_r defined by the sequence of norms $\{|\cdot|_{n,r} \colon n \in \mathbb{N}\}$. Then

$$\Omega_r A = \lim_{\substack{\leftarrow \\ n\to\infty}} (\Omega^*A)^{\sim}_{n,r} = \lim_{\substack{\leftarrow \\ n\to\infty}} (\Omega_r A_n)^{\sim}_{n,r}.$$

Thus we have the following:

$$\Omega_\infty A = \lim_{\substack{\leftarrow \\ r\to\infty}} (\Omega_r A) = \lim_{\substack{\leftarrow \\ r\to\infty}} \lim_{\substack{\leftarrow \\ n\to\infty}} (\Omega_r A_n),$$

$$\Omega_\epsilon A = \lim_{\substack{\rightarrow \\ r\to 0}} (\Omega_r A) = \lim_{\substack{\rightarrow \\ r\to 0}} \lim_{\substack{\leftarrow \\ n\to\infty}} (\Omega_r A_n).$$

Further for $m \leq n$, $\Omega_r A_n \subset \Omega_r A_m$ for any $r > 0$, because $A_n \subset A_m$, $|\cdot|_m \leq |\cdot|_n$, and so $|\cdot|_{m,\pi} \leq |\cdot|_{n,\pi}$. Hence

$$\Omega_r A = \lim_{\substack{\leftarrow \\ n\to\infty}} (\Omega_r A_n) = \cap_{n=1}^\infty \Omega_r A_n;$$

and so

$$\Omega_\infty A = \cap_{r=0}^\infty \cap_{n=1}^\infty \Omega_r A_n,$$

$$\Omega_\epsilon A = \cup_{r=0}^\infty \cap_{n=1}^\infty \Omega_r A_n.$$

We may investigate these limit algebras; in particular look for the analogous of the results in the previous section. This is illustrated by the following. We omit the proof which is along the lines of §2.

Theorem 3.1. *Under the assumptions stated above, the following hold*:

(a) $\Omega_\epsilon A$ *is a locally convex* Q*-algebra, and* $\ker \epsilon$ *is a quasinilpotent ideal of* $\Omega_\epsilon A$.
(b) *The enveloping pro-*C^**-algebra of* $\Omega_\epsilon A$ *is the* C^**-algebra* $\tilde{A}$.

4. Non-commutative de Rham algebra

The $*$-algebra $\Omega^* A$, and hence the limit algebras $\Omega_\infty A$ and $\Omega_\epsilon A$, are too big and abstract. A concrete realization of $\Omega^* A$ is obtained as follows by imposing a non-commutative geometric data on A i.e. a K-cycle on A (p. 310 of [C]).

DEFINITION 4.1

Let A be a $*$-algebra. A *K-cycle* on A is a triple $(\pi, H, \mathcal{D})$ satisfying the following:

(a) H is a Hilbert space.
(b) $\pi \colon A \to B(H)$ is a $*$-representation of A into the C^*-algebra $B(H)$ of all bounded linear operators on H.
(c) $\mathcal{D}$ is a generally unbounded self-adjoint operator on H satisfying the following.

  (i) $\{x \in A \colon [\mathcal{D}, \pi(x)] \in B(H)\} = A$.
  (ii) $\mathcal{D}$ has compact resolvent so that for all $\lambda \notin \operatorname{sp} \mathcal{D}$, $(\lambda 1 - \mathcal{D})^{-1}$ is a compact operator.

Thus $\mathcal{D}$ has to be unbounded unless H is finite dimensional. Generally π is assumed faithful so that $A \simeq \pi(A)$ is a C^*-normed algebra, and $(A, H, \mathcal{D})$ is also called a *spectral triple*.

Now π extends to a $*$-representation $\pi\colon \Omega^* A \to B(H)$ as follows.

$$\begin{aligned}\pi(a_0 da_1 da_2 \cdots da_n) &= \pi(a_0)[\mathcal{D}, \pi(a_1)] \cdots [\mathcal{D}, \pi(a_n)]\\ &= a_0[\mathcal{D}, a_1] \cdots [\mathcal{D}, a_n].\end{aligned}$$

Following p. 549 of [C], let $J_0 := \ker \pi$ in $\Omega^* A$; $J = J_0 + dJ_0$; $\Omega^*_{\mathcal{D}} = \Omega^* A/J$; viz. for $k = 0, 1, 2, 3, \ldots$

$$\begin{aligned}\Omega^k_{\mathcal{D}} &= \Omega^k A/J \cap \Omega^k A\\ &\simeq \pi(\Omega^k A)/\pi(d(J_0 \cap \Omega^{k-1} A))\end{aligned}$$

so that $\Omega^*_{\mathcal{D}} = \oplus_{k=0}^{\infty} \Omega^k_{\mathcal{D}}$. We may call $\Omega^*_{\mathcal{D}}$ the *Connes' non-commutative de Rham algebra*. In the present context, there are three canonical norms on $\Omega^k_{\mathcal{D}}$.

(a) Let $\|\cdot\|$ be the C^*-norm on A. Let $\|\cdot\|_{k,\pi}$ be the projective tensor product norm on $\Omega^k A = A \otimes A^{\otimes k}$. Let $\|\cdot\|_{\pi,q}$ be the quotient norm on $\Omega^k_{\mathcal{D}}$ from $(\Omega^k A, \|\cdot\|_{k,\pi})$.

(b) Consider $\Omega^k_{\mathcal{D}} \simeq \pi(\Omega^k(A))/\pi(d(J_0 \cap \Omega^{k-1} A))$. Let $\|\cdot\|$ be the operator norm on $\pi(\Omega^k A)$ from $B(H)$. Let $\|\cdot\|_q$ be the quotient norm on $\Omega^k_{\mathcal{D}}$ arising from operator norm. Notice that in general, $d(J)$ is not an ideal and therefore $\|\cdot\|_q$ is not an algebra norm.

(c) Assuming A to be a Banach $*$-algebra with a norm $|\cdot|$, let $|\cdot|_\pi := |\cdot|_{\pi,k}$ be the projective tensorial norm on $\Omega^k A \simeq A \otimes \bar{A}^{\otimes k}$. Let $|\cdot|_{\pi,q}$ be the quotient norm of $|\cdot|_\pi$ on $\Omega^k_{\mathcal{D}} = \Omega^k A/J \cap \Omega^k A$.

Accordingly the limit algebras are constructed taking different norms. To compensate for the absence of completeness of $(A, \|\cdot\|)$, we assume that A is closed under the holomorphic functional calculus of the C^*-algebra $\tilde{A}$. This is in spirit with Ch. III, Appendix C, p. 285 of [C].

(i) Considering the system of norms on $\Omega^*_{\mathcal{D}}$ as

$$\|\omega\|^{\pi}_{r} := \sum_{k=0}^{\infty} r^k \, \|\omega_k\|_{\pi,q}\,, \qquad (r \in \mathbb{R}^+),$$

let $\Omega_{r,\pi}(A, \mathcal{D}) = (\Omega^*_{\mathcal{D}}, \|\cdot\|^{\pi}_{r})^{\sim}$ the completion, which is a Banach $*$-algebra. Let

$$\begin{aligned}\Omega^h_{r,\pi}(A, \mathcal{D}) = {} & \text{the smallest } *\text{-subalgebra of } \Omega_{r,\pi}(A, \mathcal{D}) \text{ containing } \Omega^*_{\mathcal{D}}\\ & \text{and closed under the holomorphic functional calculus of } \Omega_{r,\pi}(A, \mathcal{D}).\end{aligned}$$

Then the following limit algebras are defined:

$$\begin{aligned}\Omega^h_{\infty,\pi}(A, \mathcal{D}) &:= \lim_{\substack{\longleftarrow\\ r\to\infty}} \Omega^h_{r,\pi}(A, \mathcal{D})\\ &\subset \lim_{\substack{\longleftarrow\\ r\to\infty}} \Omega_{r,\pi}(A, \mathcal{D}) = \Omega_{\infty,\pi}(A, \mathcal{D});\end{aligned}$$

and

$$\Omega^h_{\epsilon,\pi}(A,\mathcal{D}) := \varinjlim_{r\to 0} \Omega^h_{r,\pi}(A,\mathcal{D}) \subset \varinjlim_{r\to 0} \Omega_{r,\pi}(A,\mathcal{D}) = \Omega_{\epsilon,\pi}(A,\mathcal{D}).$$

(ii) Similarly considering the norms $\|\omega\|_r := \sum_{k=0}^{\infty} r^k \|\omega_k\|_q$ on $\Omega^*_{\mathcal{D}}$, the Banach $*$-algebras $\Omega_r(A,\mathcal{D})$ are obtained by completion. These then lead to the limit algebras

$$\Omega^h_{\infty}(A,\mathcal{D}) := \varprojlim_{r\to\infty} \Omega^h_{r}(A,\mathcal{D}) \subset \varprojlim_{r\to\infty} \Omega_{r}(A,\mathcal{D}) = \Omega_{\infty}(A,\mathcal{D});$$

and

$$\Omega^h_{\epsilon}(A,\mathcal{D}) := \varinjlim_{r\to 0} \Omega^h_{r}(A,\mathcal{D}) \subset \varinjlim_{r\to 0} \Omega_{r}(A,\mathcal{D}) = \Omega_{\epsilon}(A,\mathcal{D}),$$

where $\Omega^h_r(A,\mathcal{D})$ is the smallest $*$-subalgebra of $\Omega_r(A,\mathcal{D})$ containing $\Omega^*_{\mathcal{D}}$ and closed under the holomorphic functional calculus of $\Omega_r(A,\mathcal{D})$. The following illustrates the behaviour of these algebras.

Theorem 4.2. *Let A be closed under the holomorphic functional calculus of $\tilde{A}$. Then the following hold*:

(1) $\Omega^h_{\epsilon}(A,\mathcal{D})$ *is a locally convex Q-algebra spectrally invariant in* $\Omega_\epsilon(A,\mathcal{D})$ *and having $\tilde{A}$ as its enveloping C^*-algebra.*
(2) $\Omega^h_{\infty}(A,\mathcal{D})$ (*respectively,* $\Omega^h_{\infty,\pi}(A,\mathcal{D})$) *is closed under the holomorphic functional calculus of* $\Omega_{\infty}(A,\mathcal{D})$ (*resp.* $\Omega_{\infty,\pi}(A,\mathcal{D})$).

Proof.

(1) Since A is closed under the holomorphic functional calculus of $\tilde{A}$; A is inverse closed in $\tilde{A}$, A is Q-normed algebra, and is spectrally invariant in $\tilde{A}$. (Notice that, if A is Frechet, then the converse hold). We claim that for $0 < r' < r$, $\Omega^h_r(A,\mathcal{D}) \subset \Omega^h_{r'}(A,\mathcal{D})$. Indeed, $\Omega_r(A,\mathcal{D}) \subset \Omega_{r'}(A,\mathcal{D})$. Let $\omega \in \Omega^h_{r'}(A,\mathcal{D}) \cap \Omega_r(A,\mathcal{D})$. Let f be holomorphic on $\mathrm{sp}_{\Omega_r}(\omega) \supset \mathrm{sp}_{\Omega_{r'}}(\omega)$. Then $f(\omega) \in \Omega^h_{r'}(A,\mathcal{D})$. Also, $f(\omega) \in \Omega_r(A,\mathcal{D})$ as $\Omega_r(A,\mathcal{D})$ is a Banach algebra. Thus $\Omega^h_{r'}(A,\mathcal{D}) \cap \Omega_r(A,\mathcal{D})$ is a subalgebra of $\Omega_r(A,\mathcal{D})$ containing $\Omega^*_{\mathcal{D}}$ and closed under holomorphic functional calculus of $\Omega_r(A,\mathcal{D})$. Since $\Omega^h_r(A,\mathcal{D})$ is the smallest $*$-subalgebra with this property, it follows that $\Omega^h_r(A,\mathcal{D}) \subset \Omega^h_{r'}(A,\mathcal{D})$.

Next we show that $\ker\epsilon$ is a quasinilpotent ideal of $\Omega^h_\epsilon(A,\mathcal{D})$. Let $\omega \in \Omega^h_\epsilon(A,\mathcal{D})$, $\epsilon(\omega) = 0$, $\lambda \neq 0$ in $\mathbb{C}$. There exists $r > 0$ such that $\omega \in \Omega^h_r(A,\mathcal{D})$. Choose $r' \ll r$ such that $\|\lambda^{-1}\omega\|_{r'} < 1$. Then $\lambda^{-1}\omega$ has quasiinverse in $\Omega_{r'}(A,\mathcal{D})$. Since $\Omega^h_{r'}(A,\mathcal{D})$ is inverse closed in $\Omega_{r'}(A,\mathcal{D})$, $\lambda^{-1}\omega$ is quasiregular in $\Omega^h_{r'}(A,\mathcal{D})$. Thus $\lambda^{-1}\omega$ is quasiregular in $\Omega^h_\epsilon(A,\mathcal{D})$. Hence $\ker\epsilon$ is a quasinilpotent ideal in $\Omega^h_\epsilon(A,\mathcal{D})$.

Now as in the preceding sections, it follows that for all $\omega \in \Omega^h_\epsilon(A, \mathcal{D})$, $\mathrm{sp}_{\Omega^h_\epsilon}(\omega) = \mathrm{sp}_{\tilde{A}}(\omega_0)$, $\omega \to p_\infty(\omega) := \|\omega_0\|$ is the greatest continuous C^*-seminorm on $\Omega^h_\epsilon(A, \mathcal{D})$ which is a spectral seminorm making $\Omega^h_\epsilon(A, \mathcal{D})$ a Q-algebra. Notice that $\Omega_\epsilon(A, \mathcal{D}) = \cup_r \Omega_r(A, \mathcal{D}) = \cup_r \Omega_r(\tilde{A}, \mathcal{D}) = \Omega_\epsilon(\tilde{A}, \mathcal{D})$. By §1, $\mathrm{sp}_{\Omega_\epsilon(A,\mathcal{D})}(\omega) = \mathrm{sp}_{\Omega_\epsilon(\tilde{A},\mathcal{D})}(\omega) = \mathrm{sp}_{\tilde{A}}(\omega_0) = \mathrm{sp}_A(\omega_0) = \mathrm{sp}_{\Omega^h_\epsilon(A,\mathcal{D})}(\omega)$ for all $\omega \in \Omega^h_\epsilon(A, \mathcal{D})$.
(2) Let $x \in \Omega^h_\infty(A, \mathcal{D}) = \varprojlim_{r\to\infty} \Omega^h_r(A, \mathcal{D})$. Then $x = (x_r : r = 1, 2, \dots)$ is a coherent sequence such that $x_r \in \Omega^h_r(A, \mathcal{D}) \subset \Omega_r(A, \mathcal{D}) = \Omega_r(\tilde{A}, \mathcal{D})$. Also $\mathrm{sp}_{\Omega^h_\infty}(x) = \cup_r \mathrm{sp}_{\Omega^h_r}(x_r) = \cup_r \mathrm{sp}_{\Omega^h_r(A,\mathcal{D})}(x_r) = \mathrm{sp}_{\Omega_\infty(A,\mathcal{D})}(x) = \mathrm{sp}(x)$ say. Let f be holomorphic on $\mathrm{sp}(x)$. Then by functional calculus in Frechet locally m-convex $*$-algebra, $f(x) \in \Omega_\infty(A, \mathcal{D})$, and $f(x_r) \in \Omega^h_r(A, \mathcal{D})$, as $\Omega^h_r(A, \mathcal{D})$ is closed under holomorphic functional calculus of $\Omega_r(A, \mathcal{D})$. Also $(f(x_n))$ is a coherent sequence, and so $f(x) = (f(x_n) = \pi_n(f(x))) \in \varprojlim_{r\to 0} \Omega^h_r(A, \mathcal{D}) = \Omega^h_\infty(A, \mathcal{D})$. □

5. Quantized integrals in $\Omega_\infty A$

It would be of interest to extend the tools of non-commutative geometry to the limit algebras $\Omega_r A$, $\Omega_\infty A$ and $\Omega_\epsilon A$, when A is a dense Banach or Frechet $*$-subalgebra of a C^*-algebra. Assuming A to be Banach, we discuss below quantized integrals in $\Omega_\infty A$. Throughout this section we assume that $(A, \|\cdot\|)$ is a C^*-normed algebra which is a Banach $*$-algebra with norm $|\cdot|$. Let $\tilde{A}$ be the C^*-algebra completion of A.

Lemma 5.1. *Let $(\pi, H, \mathcal{D})$ be a K-cycle on A. Then π extends to a $*$-representation of $\Omega_\infty A$.*

Proof. The map $x \in A \to [\mathcal{D}, \pi(x)] \in B(H)$ is a derivation on the semisimple Banach algebra A. Hence by a theorem of Johnson and Sinclair [JS], it is continuous. Thus for some $M \geq 1$, $\|[\mathcal{D}, \pi(x)]\| \leq M|x|$ for all $x \in A$. Thus for any $k \in \mathbb{N}$, any $a_0, a_1, a_2, \dots, a_k$ in A.

$$\begin{aligned}\|\pi(a_0)[\mathcal{D}, \pi(a_1)][\mathcal{D}, \pi(a_2)]\cdots[\mathcal{D}, \pi(a_k)]\| &\leq M^k |a_0||a_1|\cdots|a_k| \\ &= M^k |a_0 \otimes a_1 \otimes \cdots \otimes a_k|_\pi .\end{aligned}$$

Hence for any $\omega = \sum_0^\infty \omega_k$, $\omega_k \in \Omega^k A \simeq A \otimes A^{\otimes k}$,

$$\|\pi(\omega)\| \leq \sum \|\pi(\omega_k)\| \leq \sum M^k |\omega_k|_\pi$$

showing that π is continuous in the norms $|\cdot|_r$ for $r \geq M$. It follows that π extends as a $*$-representation π of $\Omega_r A$, and hence of $\Omega_\infty A$, into $B(H)$. □

Connes (Ch. IV of [C]) (see also Ch. 7 of [GVF]) has discussed various versions of quantized integrals on $\Omega^* A$ depending on the nature of the K-cycle under consideration like d^+-summability, θ-summability etc. They turn out to be tracial positive linear functionals. As shown in [BIO1], they can be regarded as weights or quasiweights on $\Omega^* A$. Lemma 5.1 enables us to extend these integrals on $\Omega_r A$ for large enough r, and hence on $\Omega_\infty A$.

DEFINITION 5.2 [BIO1]

Let A be a $*$-algebra. For a subspace N of A, let $P(N) := \left\{ \sum_{k=1}^{n} x_k^* x_k \colon x_k \in N \text{ for all } k = 1, 2, \ldots, n; n \in \mathbb{N} \right\}$

(a) A map $\varphi\colon P(A) \to \mathbb{R}^+ \cup \{\infty\}$ is called a *weight* on A if

 (i) $\varphi(x + y) = \varphi(x) + \varphi(y)\ \forall\ x, y \in P(A)$,
 (ii) $\varphi(\lambda x) = \lambda\varphi(x)\ \forall x \in P(A), \lambda \geq 0$.

(b) Let N be a left ideal of A. A map $\varphi\colon P(N) \to \mathbb{R}^+$ is a *quasiweight* on A if it satisfies (i) and (ii) above for $P(N)$. In this case, N is denoted by N_φ.

Weights have been introduced as abstract non-commutative analogue of infinite measures in von Neumann algebras; and quasiweights are tailored for the same purpose in non-normed $*$-algebras [BIO1].

We briefly recall Dixmier trace (Ch. IV.2 of [C], Ch. 7 of [GVF]). Let H be a separable Hilbert space. Let $K(H)$ be the ideal of compact operators on H. Let (ξ_n) be an orthonormal basis for H. For a $T \in K(H)$, let $\mu_n(T)$ denote eigenvalues of $|T|$ arranged in decreasing order, counted according to multiplicities. Let $\sigma_N(T) = \sum_{n=0}^{N-1} \mu_n(T)$. Then $\mu_n(T) \to 0$ which motivates calling compact operators the *non-commutative infinitesimals*. Then the *infinitesimals of order* α constitutes the two-sided ideal

$$K_\alpha(H) := \{T \in K(H)\colon \mu_n(T) = O(n^{-\alpha}) \text{ as } n \to \infty\}.$$

Then

$$\begin{aligned} C^{1+}(H) &:= \{T \in K(H)\colon \sigma_N(T) = O(\log N) \text{ as } N \to \infty\} \\ &\supset K_1(H) \\ &\supset K_{1+}(H) := \text{ infinitesimals of order } > 1 \\ &= \left\{ T \in K(H)\colon \mu_n(T) = \circ\left(\frac{1}{n}\right) \text{ as } n \to \infty \right\} \\ &\supset C^1(H) \text{ trace class operators.} \end{aligned}$$

Let λ denote a Banach limit on $l^\infty(\mathbb{N})$ which is a translation invariant and a scale invariant positive linear functional on $l^\infty(\mathbb{N})$ vanishing on $C_0(\mathbb{N})$.

For $T \geq 0$ in $C^{1+}(H)$,

$$\begin{aligned} \mathrm{tr}_\lambda(T) &:= \lambda\left(\left\{\frac{\sigma_N(T)}{\log N}\right\}\right) \\ &=: \lim_{N\to\infty}{}_\lambda \frac{\sigma_N(T)}{\log N} \end{aligned}$$

defines a tracial linear functional on $C^{1+}(H)$ vanishing on $K_{1+}(H)$. This tr_λ is called the *Dixmier trace*. Lemma 5.1 enables us to define a quasiweight $(\tau_\lambda, N_{\tau_\lambda})$ on $\Omega_\infty(A)$ by taking

$$N_{\tau_\lambda} := \{\omega \in \Omega_\infty A\colon \pi(\omega) \in C^{1+}(H)\},$$

$$\tau_\lambda(\omega^*\omega) := \mathrm{tr}_\lambda(\pi(\omega)^*\pi(\omega)), \omega \in N_{\tau_\lambda}.$$

This defines a *quantized integral on* $P(N_{\tau_\lambda})$ as $⨍\omega := \tau_\lambda(\omega)$.

Let the K-cycle $(\pi, H, \mathcal{D})$ be of dimension d; i.e. the operator $|\mathcal{D}|^{-1}$ is an infinitesimal of order $1/d$. Then $|\mathcal{D}|^{-d} \in C^{1+}(H)$; and the sequence $\{\sigma_n(|\mathcal{D}|^{-d})/\log n\}$ is bounded. The functional $\varphi(\omega) := \mathrm{tr}_\lambda(|\mathcal{D}|^{-d}\pi(\omega))$ defines a positive linear functional on $\Omega_\infty A$ called the *d-dimensional volume integral* on $\Omega_\infty A$. In general case, one may consider the quasiweight (φ, N_φ) on $\Omega_\infty A$ defined as follows:

$$N_\varphi := \{\omega \in \Omega_\infty A \colon \mathrm{tr}_\lambda \pi(\omega^* a^* a\omega|\mathcal{D}|^{-d}) < \infty \ \ \forall\, a \in \Omega_\infty A\}$$

$$\varphi(\omega) := \mathrm{tr}_\lambda(\pi(\omega)|\mathcal{D}|^{-d}), \ \ \omega \in P(N_\varphi).$$

We aim to analyze the d-dimensional volume integral on a d^+-summable K-cycle in detail; and compute the GNS representation of $\Omega_\infty A$ defined by it.

Lemma 5.3. *Let the K-cycle $(\pi, H, \mathcal{D})$ be d^+-summable.*

(a) *The functional* $\omega \in \Omega_\infty A \mapsto \varphi(\omega) := \mathrm{tr}_\lambda(\pi(\omega)|\mathcal{D}|^{-d}) = \mathrm{tr}_\lambda(|\mathcal{D}|^{-d}\pi(\omega))$ (*p.* 287 *of* [GVF]) *defines a continuous positive linear functional on $\Omega_\infty A$ satisfying*: *For some $K > 0$, $|\varphi(\omega)| \le K|\omega|_r$, $(\omega \in \Omega_\infty A)$ for sufficiently large r.*

(b) *If (π_φ, H_φ) is the GNS representation of $\Omega_\infty A$ defined by φ, then π_φ maps $\Omega_\infty A$ into an algebra of bounded operators*; *and for any ω, η in $\Omega_\infty A$,*

$$\mathrm{tr}_\lambda(|\pi(\omega)|^2||\mathcal{D}|^{-d/2}\pi(\eta)^*|^2) \le \left\|\pi_\varphi(\omega)\right\|^2 \mathrm{tr}_\lambda(||\mathcal{D}|^{-d/2}\pi(\eta)^*|^2).$$

Proof.

(a) φ is a positive linear functional on the unital Frechet $*$-algebra $\Omega_\infty A$, hence is continuous. Therefore there exists $r > 0$ and $K > 0$ such that $|\varphi(\omega)| \le K|\omega|_r$ for all $\omega \in \Omega_\infty A$.

(b) By a result of Brooks [Bro1], every continuous positive linear functional f on a complete locally m-convex $*$-algebra B is admissible, with the result, the associated GNS representation π_f of B is bounded, and for all x, y in B, $\varphi(y^*x^*xy) \le \|\pi_\varphi(x)\|^2\varphi(y^*y)$. We apply this to the functional φ on $\Omega_\infty A$. By using the trace property of Dixmier trace tr_λ (p. 287 of [GVF]), we get, for any ω, η in $\Omega_\infty A$,

$$\begin{aligned}\mathrm{tr}_\lambda(|\pi(\omega)|^2||\mathcal{D}|^{-d/2}\pi(\eta)^*|^2) &= \mathrm{tr}_\lambda(\pi(\omega)^*\pi(\omega)\pi(\eta)|\mathcal{D}|^{-d}\pi(\eta)^*)\\ &= \mathrm{tr}_\lambda(|\mathcal{D}|^{-d}\pi(\eta)^*\pi(\omega)^*\pi(\omega)\pi(\eta))\\ &\le \|\pi_\varphi(\omega)\|^2\mathrm{tr}_\lambda(|\mathcal{D}|^{-d}\pi(\eta)^*\pi(\eta))\\ &\le \|\pi_\varphi(\omega)\|^2\mathrm{tr}_\lambda(||\mathcal{D}|^{-d/2}\pi(\eta)^*|^2).\end{aligned}$$ □

Assume that the K-cycle $(\pi, H, \mathcal{D})$ is d^+-summable. Following Connes (p. 550 of [C]), let $\mathcal{H}_k$ be the Hilbert space (Hausdorff) completion of $\pi(\Omega^k A)$ in the inner product $\langle T_1, T_2\rangle_k = \mathrm{tr}_\lambda(T_2^* T_1|\mathcal{D}|^{-d})$. Let P_k be the orthogonal projection of $\mathcal{H}_k$ onto $[\pi(d(J_0 \cap \Omega^{k-1}A))]^\perp$. Then $\langle [\omega_1], [\omega_2]\rangle = \langle P_k\omega_1, P_k\omega_2\rangle = \langle P_k\omega_1, \omega_2\rangle$ defines an inner product on $\Omega^k_{\mathcal{D}} := \pi(\Omega^k A)/\pi[d(J_0 \cap \Omega^{k-1}A)]$ where for $\omega_j \in \pi(\Omega^k A)$, $[\omega_j]$ denotes the class in $\Omega^k_{\mathcal{D}}$. Let Λ^k be the Hilbert space completion of $\Omega^k_{\mathcal{D}}$, viz $\Lambda^k = P_k\mathcal{H}_k$. Connes in Ch. VI, §1, Propo. 5 of [C] noted that the actions of $A \simeq \pi(A)$ on Λ^k by left and right multiplications define commuting unitary representations of A on Λ^k. We aim to show that this representation by left multiplication can be extended as a representation of $\Omega_\infty A$ and is unitarily equivalent to the GNS representation $(\pi_\varphi^\infty, H_\varphi^\infty)$ of $\Omega_\infty A$ defined by the volume integral φ. Let $\mathcal{H}_\infty$ be the Hausdorff completion of $\pi(\Omega_\infty A)$ in the inner product $\langle T_1, T_2\rangle = \mathrm{tr}_\lambda(T_2^* T_1 |D|^{-d})$. Let J be the conjugate linear isometry of $\mathcal{H}_\infty$ defined by $J\pi(\eta) := \pi(\eta^*)$, $\eta \in \Omega_\infty A$. The following refines Chapter VI, §1, Propo. 5(1), p. 550 of [C].

Theorem 5.4. *Let the K-cycle be d^+-summable.*

(1) *The left action π_1 and the right action π_r each of $\Omega_\infty A$ on the $\mathcal{H}_\infty$ define unitary representations of $\Omega_\infty A$ on $\mathcal{H}_\infty$ satisfying $J\pi_1(\omega)J = \pi_\mathrm{r}(\omega)$, $(\omega \in \Omega_\infty A)$.*
(2) *The GNS representation $(\pi_\varphi^\infty, H_\varphi^\infty)$ of $\Omega_\infty A$ defined by φ is unitarily equivalent to π_1, the unitary equivalence being given by $U\colon H_\varphi^\infty \to \mathcal{H}_\infty$, $U(\eta + \ker\varphi) = \pi(\eta)$, $(\eta \in \Omega_\infty A)$.*
(3) *Let $\mathcal{H} := \oplus \mathcal{H}_k$. The map $\sigma\colon A \to B(\mathcal{H})$, $\sigma(a)(\Sigma\pi(\eta_k)) = \sum \pi(a\eta_k)$, is a continuous $*$-homomorphism; and σ extends as a continuous homomorphism $\sigma\colon \Omega_\infty A \to B(\mathcal{H})$ which fails to be $*$-homomorphism.*
(4) *There exists a bounded linear map $T\colon \mathcal{H} \to H_\varphi^\infty$ such that $\pi_\varphi^\infty(\omega)T = T\sigma(\omega)$ for all $\omega \in \Omega_\infty A$ and the range of T is $\oplus H_\varphi^k$, where H_φ^k is the Hausdorff completion of $\Omega^k A$ in the d-dimensional volume integral.*

Proof. First we construct the GNS representation $(\pi_\varphi^\infty, H_\varphi^\infty)$ of $\Omega_\infty A$ defined by φ. Let $\omega \in \Omega_\infty A$, $T := \pi(\omega)|\mathcal{D}|^{-d/2}$. Then $T^*T = |\mathcal{D}|^{-d/2}\pi(\omega)^*\pi(\omega)|\mathcal{D}|^{-d/2} = |T|^2 = |\pi(\omega)|\mathcal{D}|^{-d/2}|^2$. Also, $TT^* = \pi(\omega)|\mathcal{D}|^{-d}\pi(\omega)^*$. Since $\mathrm{tr}_\lambda(\cdot)$ is a trace on $B(H)$, $\mathrm{tr}_\lambda(|\mathcal{D}|^{-d/2}\pi(\omega^*\omega)|\mathcal{D}|^{-d/2}) = \mathrm{tr}_\lambda(\pi(\omega^*\omega)|\mathcal{D}^{-d}|) = \mathrm{tr}_\lambda(T^*T) = \mathrm{tr}_\lambda(|T|^2) = \mathrm{tr}_\lambda(|\pi(\omega)||\mathcal{D}|^{-d/2})^2$. Hence

$$\begin{aligned} N_\varphi^\infty &:= \{\omega \in \Omega_\infty A\colon \varphi(\omega^*\omega) = 0\} \\ &= \{\omega \in \Omega_\infty A\colon \mathrm{tr}_\lambda(\pi(\omega)^*\pi(\omega)|\mathcal{D}|^{-d}) = 0\} \\ &= \{\omega \in \Omega_\infty A\colon \mathrm{tr}_\lambda(|\pi(\omega)|\mathcal{D}|^{-d/2}|^2) = 0\}. \end{aligned}$$

The inner product on the quotient space $\Omega_\infty A/N_\varphi^\infty$ is

$$\langle \omega + N_\varphi^\infty, \eta + N_\varphi^\infty\rangle = \varphi(\eta^*\omega) = \mathrm{tr}_\lambda(\pi(\eta)^*\pi(\omega)|\mathcal{D}|^{-d}).$$

Then the representation space H_φ^∞ is

$$\begin{aligned} H_\varphi^\infty &= (\Omega_\infty A/N_\varphi^\infty)^\sim \quad \text{completion} \\ &= (\Omega^* A/N_\varphi^\infty)^\sim \\ &= ((\oplus\Omega^k A)/(N_\varphi^\infty))^\sim. \end{aligned}$$

Let the Hilbert space H_φ be defined as $H_\varphi = \bigoplus_{k=0}^{\infty}(\Omega^k A/N_\varphi^\infty \cap \Omega^k A)^\sim = \bigoplus_{k=0}^{\infty} H_\varphi^k$ where $H_\varphi^k := (\Omega^k A/N_\varphi^\infty \cap \Omega^k A)^\sim$. The representation $\pi_\varphi^\infty\colon \Omega_\infty A \to B(H_\varphi^\infty)$ is $\pi_\varphi^\infty(\omega)(\eta + N_\varphi^\infty) = \omega\eta + N_\varphi^\infty$ for all $\omega, \eta \in \Omega_\infty A$.

Now let $\phi_k\colon \pi(\Omega^k A) \to \Omega^k A/N_\varphi^\infty \cap \Omega^k A$ be $\phi_k(\pi(\omega)) = \omega + N_\varphi^\infty \cap \Omega^k A$. Then ϕ_k is well-defined. Indeed, for $\omega, \eta \in \Omega^k A$, $\pi(\omega) = \pi(\eta)$ implies that $\pi((\omega-\eta)^*(\omega-\eta)) = 0$.

Hence $\varphi((\omega-\eta)^*(\omega-\eta)) = \mathrm{tr}_\lambda(\pi(\omega-\eta)^*\pi(\omega-\eta)) = 0$, with the result, $\omega-\eta \in N_\varphi^\infty$, $\omega + N_\varphi^\infty = \eta + N_\varphi^\infty$. Clearly ϕ_k is a linear map, and

$$\begin{aligned}\ker\phi_k &= \{\pi(\omega) \in \pi(\Omega^k A) \colon \varphi(\omega^*\omega) = 0\}\\ &= \{\pi(\omega) \colon \mathrm{tr}_\lambda(\pi(\omega^*\omega)|D|^{-d}) = 0\}.\end{aligned}$$

Thus $\pi(\Omega^k A)/\ker\phi_k \simeq \Omega^k A/N_\varphi^\infty \cap \Omega^k A$ as linear spaces under the linear map $\tilde{\phi}_k$ defined as $\tilde{\phi}_k(\pi(\omega) + \ker\varphi_k) := \omega + N_\varphi^\infty \cap \Omega^k A$. Also, the Hilbert space

$$\begin{aligned}\mathcal{H}_k &= \text{Hausdorff completion of } \pi(\Omega^k A) \text{ in the inner product } \langle\cdot,\cdot\rangle_k\\ &= \left(\frac{\pi(\Omega^k A)}{\ker\phi_k}\right)^\sim \simeq \left(\frac{\Omega^k A}{N_\varphi^\infty \cap \Omega^k A}\right)^\sim = H_\varphi^k.\end{aligned}$$

Thus the Hilbert space $\mathcal{H} := \oplus\mathcal{H}_k \simeq \oplus H_\varphi^k = H_\varphi$ under the map $\tilde{\phi} = \oplus\tilde{\phi}_k$. Notice that $\tilde{\phi}_k$ is an onto isometry from $\mathcal{H}_k$ to H_φ^k. For, given $\omega_k \in \Omega^k A$, denoting the norms in $\mathcal{H}_k$ and H_φ^k by $\|\cdot\|_k$ and $\|\cdot\|_\varphi$, we have

$$\begin{aligned}\|\pi(\omega) + \ker\phi_k\|_k^2 &= \langle\pi(\omega_k), \pi(\omega_k)\rangle_k\\ &= \mathrm{tr}_\lambda(\pi(\omega_k)^*\pi(\omega_k)|\mathcal{D}|^{-d})\\ &= \varphi(\omega_k^*\omega_k) = \|\omega_k + N_\varphi^\infty \cap \Omega^k A\|_\varphi^2\\ &= \|\tilde{\phi}_k(\pi(\omega_k) + \ker\phi_k)\|_\varphi^2.\end{aligned}$$

Let us note the following.
(i) The inner products in $\mathcal{H}$ and $\mathcal{H}_\infty$ are distinct. Indeed let $\eta_1 = \sum\eta_k^1 = (\eta_k^1)$ and $\eta_2 = \sum\eta_k^2 = (\eta_k^2)$ be in $\Omega^* A = \oplus\Omega^k A$ with η_k^1, η_k^2 in $\Omega^k A$ for all k. Then the inner product in $\mathcal{H} = \oplus\mathcal{H}_k$ is

$$\begin{aligned}\langle\pi(\eta_1), \pi(\eta_2)\rangle &= \sum_k \langle\pi(\eta_k^1), \pi(\eta_k^2)\rangle\\ &= \sum_k \mathrm{tr}_\lambda(\pi(\eta_k^2)^*\pi(\eta_k^1)|\mathcal{D}|^{-d})\\ &= \mathrm{tr}_\lambda \sum_k (\pi(\eta_k^2)^*\pi(\eta_k^1)|\mathcal{D}|^{-d}).\end{aligned}$$

On the other hand, the inner product in $\mathcal{H}_\infty = (\pi(\Omega_\infty A))^\sim$ is

$$\begin{aligned}\langle\pi(\eta_1), \pi(\eta_2)\rangle_\varphi &= \mathrm{tr}_\lambda(\pi(\eta_2)^*\pi(\eta_1)\mathcal{D}|^{-d})\\ &= \mathrm{tr}_\lambda\left(\left(\sum\pi(\eta_i^2)^*\right)\left(\sum\pi(\eta_j^1)\right)|\mathcal{D}^{-d}|\right)\\ &= \sum_k \sum_{i+j=k} \mathrm{tr}_\lambda(\pi(\eta_i^2)^*\pi(\eta_j^1)|\mathcal{D}^{-d}|).\end{aligned}$$

(ii) A repetition of our earlier arguments involving ϕ_k, $\mathcal{H}_k$ and H_φ^k show that $\mathcal{H}_\infty \simeq H_\varphi^\infty$. Indeed, the linear map $\phi\colon \pi(\Omega_\infty A) \to \Omega_\infty A/N_\varphi^\infty$, $\phi(\pi(\omega)) = \omega + N_\varphi^\infty$ is well-defined,

$\frac{\pi(\Omega_\infty A)}{\ker\phi} \simeq \frac{\Omega_\infty A}{N_\varphi^\infty}$, $\|\pi(\omega)+\ker\phi\|^2 = \|\omega + N_\varphi^\infty\|^2$ for all $\omega \in \Omega_\infty A$, with the result we get the isomorphic Hilbert spaces

$$\begin{aligned}\mathcal{H}_\infty &= (\pi(\Omega_\infty A))^\sim \quad \text{Hausdorff completion}\\ &\simeq \left(\frac{\Omega_\infty A}{N_\varphi^\infty}\right)^\sim \quad \text{completion}\\ &= H_\varphi^\infty.\end{aligned}$$

Now the left action $\pi_{\rm l}$ and the right action $\pi_{\rm r}$ of $\Omega_\infty A$ on $\mathcal{H}_\infty$ are given by

$$\begin{aligned}\pi_{\rm l}(\omega)\pi(\eta) &= \pi(\omega\eta)\\ \pi_{\rm r}(\omega)\pi(\eta) &= \pi(\eta\omega)\end{aligned}$$

Then by Lemmas 5.3 and 5.1, we have

$$\begin{aligned}\|\pi_{\rm r}(\omega)\pi(\eta)\|_{{\rm tr}_\lambda} &= {\rm tr}_\lambda((\pi(\eta)^*\pi(\omega)^*\pi(\omega)\pi(\eta))|\mathcal{D}|^{-d})^{1/2}\\ &= {\rm tr}_\lambda((\pi(\omega)^*\pi(\eta)^*\pi(\eta)\pi(\omega))|\mathcal{D}|^{-d})^{1/2}\\ &\le \|\pi_{\rm r}(\omega)\pi(\eta)\|_{{\rm tr}_\lambda}\\ &= \|\pi_\varphi(\omega)\|\ \|\pi(\eta)\|_{{\rm tr}_\lambda}\end{aligned}$$

for all ω, η in $\Omega_\infty A$. Furthermore, since π_φ is continuous, it follows that $\|\pi_\varphi(\omega)\| \le M|\omega|_{\rm r}$, $(\omega \in \Omega_\infty A)$ for some $M > 0$ and $r > 0$, which implies that $\pi_{\rm l}(\omega)$ and $\pi_{\rm r}(\omega)$ are bounded linear operators on $\mathcal{H}_\infty$, and π and $\pi_{\rm r}$ are continuous in the topology of $\Omega_\infty A$. The homomorphisms $\pi_{\rm l}$ and $\pi_{\rm r}$ of $\Omega_\infty A$ into $B(\mathcal{H}_\infty)$ define $*$-representations of $\Omega_\infty A$ on $\mathcal{H}_\infty$, since, for example,

$$\begin{aligned}\langle\pi_{\rm l}(\omega)\pi(\eta_1), \pi(\eta_2)\rangle_{{\rm tr}_\lambda} &= \langle\pi(\omega\eta_1), \pi(\eta_2)\rangle\\ &= {\rm tr}_\lambda(\pi(\eta_2)^*\pi(\omega)\pi(\eta_1)|\mathcal{D}|^{-d})\\ &= {\rm tr}_\lambda((\pi(\omega)^*\pi(\eta_2))^*\pi(\eta_1)|\mathcal{D}|^{-d})\\ &= {\rm tr}_\lambda((\pi(\omega^*)\pi(\eta_2))^*\pi(\eta_1)|\mathcal{D}|^{-d})\\ &= \langle\pi(\eta_1), \pi(\omega^*)\pi(\eta_2)\rangle\\ &= \langle\pi(\eta_1), \pi_{\rm l}(\omega^*)\pi(\eta_2)\rangle_{{\rm tr}_\lambda}.\end{aligned}$$

Further $J\pi_{\rm l}(\omega)J = \pi_{\rm r}(\omega)^*$, $\omega \in \Omega_\infty A$, where J is the conjugate linear isometry on $\mathcal{H}_\infty$ defined by $J\pi(\eta) = \pi(\eta^*)$, $\eta \in \Omega_\infty A$. We define

$$U\colon \eta + N_\varphi \in H_\varphi^\infty \mapsto \pi(\eta) \in \mathcal{H}_\infty.$$

Then it is easily shown that U extends to a unitary operator of $\mathcal{H}_\varphi^\infty$ onto $\mathcal{H}_\infty$ and

$$\pi_\varphi^\infty(\omega) = U^*\pi_{\rm l}(\omega)U, \quad \omega \in \Omega_\infty A$$

shows that π_φ^∞ is unitarily equivalent to $\pi_{\rm l}$. This completes the proof of (1) and (2).

The map $\sigma\colon A \to B(\mathcal{H})$, $\sigma(a)(\Sigma\pi(\eta_k)) = (\Sigma\pi(a\eta_k))$, $(\eta_k) = \Sigma\eta_k \in \Omega^* A$, defines a $*$-representation of A on $\mathcal{H} = \oplus_k \mathcal{H}_k$ satisfying $\|\sigma(a)\xi\| \le |a|\,\|\xi\|$, $\xi \in \mathcal{H}$. We show that σ extends as a homomorphism $\sigma\colon \Omega_\infty A \to B(\mathcal{H})$. Indeed, let $\omega = \Sigma\omega_j \in \Omega^* A$ with each $\omega_j \in \Omega^j A$. Let $\xi = (\xi_k\colon \xi_k \in \mathcal{H}_k) \in \mathcal{H}$ be such that each ξ_k is of the form $\xi_k = \pi(\eta_k)$, $\eta_k \in \Omega^k A$. Then define the left action of ω on ξ by $\sigma(\omega_k)\xi := \sum_j \pi(\omega_k\eta_j)$, $\sigma(\omega)\xi := \sum_{j,k} \pi(\omega_j\eta_k)$. Clearly, σ is linear on $\Omega^* A$. Further, for $\omega = \sum \omega_i$, $\delta = \sum \delta_j$ in $\Omega^* A$,

$$
\begin{aligned}
\sigma(\omega\delta) &= \sigma\left(\sum_k \left(\sum_{i+j=k} \omega_i\delta_j\right)\right) \\
&= \sum_k \sum_{i+j=k} (\sigma(\omega_i\delta_j)) \\
&= \sum_k \sum_{i+j=k} \sigma(\omega_i)\sigma(\delta_j) \\
&= \left(\sum_i (\sigma(\omega_i))\right)\left(\sum_j \sigma(\delta_j)\right) = \sigma(\omega)\sigma(\delta)
\end{aligned}
$$

shows that σ is a homomorphism. However, σ fails to be a $*$-homomorphism. Take $\omega \in \Omega^1 A$, $\eta \in \Omega^k A$. Put $\xi = \pi(\eta) \in \mathcal{H}_k$ and $\zeta = \pi(\omega\eta) \in \mathcal{H}_{k+1}$. We have $\sigma(\omega)\xi = \zeta$, thus $\langle \sigma(\omega)\xi, \zeta\rangle = \|\zeta\|^2$, while $\sigma(\omega^*)\zeta \in \mathcal{H}_{k+2}$, thus $\langle \xi, \sigma(\omega^*)\zeta\rangle = 0$.

For the continuity of $\sigma\colon \Omega_\infty A \to B(\mathcal{H})$, we show that given $\omega = \sum \omega_k$, $\xi = \sum \xi_k$ both in $\Omega^* A$, $\|\sigma(\omega)\xi\| \le \left(\sum_k \|\pi(\omega_k)\|_{\mathrm{op}}\right) \|\xi\|$. Indeed,

$$
\begin{aligned}
\|\sigma(\omega_k)\xi\|^2 &= \sum_i \|\pi(\omega_k)\xi_j\|^2_{k+j} \\
&= \sum_j \operatorname{tr}_\lambda(\pi(\xi_j)^*\pi(\omega_k)^*\pi(\omega_k)\pi(\xi_j)|\mathcal{D}|^{-d}) \\
&\le \sum_j \|\pi(\omega_k)\|^2_{\mathrm{op}} \operatorname{tr}_\lambda(\pi(\xi_j)^*\pi(\xi_j)|\mathcal{D}|^{-d}) \\
&= \|\pi(\omega_k)\|^2_{\mathrm{op}} \sum_j \|\pi(\xi_j)\|^2_j \\
&= \|\pi(\omega_k)\|^2_{\mathrm{op}} \|\xi\|^2 .
\end{aligned}
$$

Then

$$
\begin{aligned}
\|\sigma(\omega)\xi\| &= \left\|\sum_k \sigma(\omega_k)\xi\right\| \\
&\le \sum_k \|\sigma(\omega_k)\xi\| \\
&\le \left(\sum_k \|\pi(\omega_k)\|_{\mathrm{op}}\right) \|\xi\| .
\end{aligned}
$$

Thus $\sigma(\omega)$ is a bounded operator from $\mathcal{H}$ to $\mathcal{H}$; and for any $\xi \in \mathcal{H}$,

$$\begin{aligned}\|\sigma(\omega)\xi\| &\leq \left(\sum_k \|\pi(\omega_k)\|_{\text{op}}\right)\|\xi\| \\ &\leq \left(\sum_k M^k|\omega_k|_\pi\right)\|\xi\| \quad \text{as in Lemma 5.1} \\ &\leq |\omega|_r\,\|\xi\| \quad \text{if } r \geq M.\end{aligned}$$

Thus σ is continuous in the topology of $\Omega_\infty A$ and so extends as a continuous homomorphism $\sigma\colon \Omega_\infty A \to B(\mathcal{H})$. Now let $T\colon \mathcal{H} \to H_\varphi^\infty$ be the bounded linear operator defined by

$$T((\pi(\eta_k))) := \eta + N_\varphi^\infty, \quad \eta = (\eta_k) = \sum \eta_k \in \Omega^* A.$$

Then $\pi_\varphi^\infty(\omega)T = T\sigma(\omega)$ holds for all $\omega \in \Omega_\infty A$. This completes the proof. $\square$

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