Uniqueness of the uniform norm and adjoining identity in Banach algebras

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Abstract. Let $A_e$ be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$. Unlike the case for a $C^*$-norm on a Banach *-algebra, $A_e$ admits exactly one uniform norm (not necessarily complete) if so does $A$. This is used to show that the spectral extension property carries over from $A$ to $A_e$. Norms on $A_e$ that extend the given complete norm $\|\cdot\|$ on $A$ are investigated. The operator seminorm $\|\cdot\|_\text{op}$ on $A$, defined by $\|\cdot\|$ is a norm (resp. a complete norm) iff $A$ has trivial left annihilator (resp. $\|\cdot\|_\text{op}$ restricted to $A$ is equivalent to $\|\cdot\|$).

Keywords. Adjoining identity to a Banach algebra; unique uniform norm property; spectral extension property; regular norm; weakly regular Banach algebra.

1. Introduction

Let $A_e = A + C1$ be the algebra obtained by adjoining identity to a non-unital Banach algebra $(A, \|\cdot\|)$ [8]. There are two natural problems associated with this elementary unification construction: (1) which are (all) algebra norms $\|\cdot\|$ on $A_e$ that are closely related with (e.g. extending) $\|\cdot\|$ on $A$? (2) Which properties of the Banach algebra $(A, \|\cdot\|)$ are shared by the normed algebra $(A_e, \|\cdot\|)$? In the present paper, it is shown that $A$ has unique uniform norm (not necessarily complete) (resp. spectral extension property [9]) iff $A_e$ has the same. This is interesting in view of the fact that for a Banach *-algebra $(A, \|\cdot\|)$ with a unique $C^*$-norm, $A_e$ can admit more than one $C^*$-norm [1, Example 4.4, p. 850]. This holds in spite of apparent similarity between the defining properties $\|x^2\| = \|x\|^2$ and $\|x^*x\| = \|x\|^2$ of uniform norms and $C^*$-norms respectively. This main result, together with a couple of corollaries, is formulated and proved in §3. Their proofs require some properties of norms on $A_e$ that are regular [5]. There are two standard constructs of norms on $A_e$, viz. the $l^1$-norm $\|x + \lambda 1\| = \|x\| + |\lambda|$ and the operator norm $\|x + \lambda 1\|_{\text{op}} = \sup \{\|xy + \lambda y\| : \|y\| \leq 1, y \in A\}$. In general, $\|\cdot\|_{\text{op}}$ need neither be a norm nor be complete [6, Example 4.2]. Also, in general, $\|\cdot\|_{\text{op,}A} \neq \|\cdot\|$. It is easy to see that if $p$ is any algebra seminorm on $A_e$ such that $p_e = \|\cdot\|$, then $\|a + \lambda 1\|_{\text{op}} \leq p(a + \lambda 1) = p(1)\|a + \lambda 1\|_1$. The norm $\|\cdot\|$ on $A$ is regular (resp. weakly regular) if the restriction of $\|\cdot\|_{\text{op}}$ on $A$ is equivalent to $\|\cdot\|_{\text{op},A}$ (resp. $\|\cdot\|_{\text{op},A}$ is equivalent to $\|\cdot\|$). These are essentially non-unital phenomena, for if $A$ is unital (resp. having a bai $(e_i)$, then $\|a + \lambda 1\|_{\text{op}} \leq p(a + \lambda 1) = p(1)\|a + \lambda 1\|_1$ is regular [5]. It is shown in §2 that $\|\cdot\|_{\text{op}}$ is a norm on $A_e$ iff the left annihilator $\text{lan}(A) = \{0\}$; and in this case, $\|\cdot\|_{\text{op}}$ is complete iff $\|\cdot\|$ is weakly regular iff $\|\cdot\|_1$ is equivalent to $\|\cdot\|_{\text{op}}$ on $A_e$.

Throughout, $A$ is a non-unital algebra. By a norm on $A$, we mean an algebra norm; i.e. a norm satisfying $\|xy\| \leq \|x\| \|y\|$ for all $x, y$. A uniform norm on $A$ (resp. a $C^*$-norm on a *-algebra) is a norm satisfying the square property $\|x^2\| = \|x\|^2$ (resp. the $C^*$-property $\|x^*x\| = \|x\|^2$) for all $x$.  

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2. Weakly regular norms

Let $(A, \| \cdot \|)$ be a normed algebra. The following shows that if $\| \cdot \|_{op}$ is a norm on $A_e$, then $\| \cdot \|_{op}$ is also a norm on $A_e$ for all norms $\| \cdot \|$ on $A$. The left annihilator of $A$ is $\text{lan}(A) = \{ x \in A : xA = \{ 0 \} \}$.

PROPOSITION 2.1

The seminorm $\| \cdot \|_{op}$ is a norm on $A_e$ iff $\text{lan}(A) = \{ 0 \}$.

Proof. Let $\| \cdot \|_{op}$ be a norm on $A_e$. Let $a \in \text{lan}(A)$. Then $ax = 0$ ($x \in A$), hence $\| a \|_{op} = \sup \{ \| ax \| : \| x \| \leq 1, x \in A \} = 0$, so that $a = 0$. Hence $\text{lan}(A) = \{ 0 \}$. Conversely, assume that $\text{lan}(A) = \{ 0 \}$. Let $\| a + \lambda 1 \|_{op} = 0$. Then $ax + \lambda x = 0$ for all $x \in A$. Suppose $\lambda \neq 0$. Then $-\lambda^{-1} ax = x (x \in A)$. Define $L_e(x) = e(x \in A)$, where $e = -\lambda^{-1} a$. Then $L_e$ is an identity operator on $A$. Then, for $x \in A$, $L_e L_e = L_e L_e$, i.e. $xy = L_e L_e(y) = L_e L_e(y) = e(y \in A)$, i.e. $(x - e) y = 0 (y \in A)$. Hence, $xe = ex = x$. Thus $A$ has an identity which is a contradiction. Thus $\lambda = 0$. This implies $ax = 0$ for all $x \in A$, hence $a = 0$. This completes the proof.

PROPOSITION 2.2

(a) Let $\| \cdot \|$ be a uniform norm on $A$. Then $\| \cdot \|$ is regular and $\| \cdot \|_{op}$ is a uniform norm on $A_e$.

(b) Let $A$ be a $*$-algebra. Let $\| \cdot \|$ be a C* -norm on $A$. Then $\| \cdot \|$ is regular and $\| \cdot \|_{op}$ is a C* -norm on $A_e$.

Note that if a Banach algebra admits a uniform norm, then it is commutative and semisimple. In the above, the proof of (a) is similar to that of (b) in [4, Lemma 19, p. 67]. In the following, the proof of (1) implies (2) is along the lines of [7, Theorem 1]; whereas that of the remaining part is simple.

PROPOSITION 2.3

Let $(A, \| \cdot \|)$ be a Banach algebra. Then the following are equivalent.

1. $\| \cdot \|$ is weakly regular (so that $\| a + \lambda 1 \|_{op} \leq m \| a \|_{op} (a \in A_e)$, for some $m > 0$).
2. $\| a + \lambda 1 \|_{op} \leq \| a + \lambda 1 \|_1 \leq 2(1 + m(\exp 1)) \| a + \lambda 1 \|_{op} (a + \lambda 1 \in A_e)$
3. $\| \cdot \|_{op}$ is a complete norm on $A_e$.

If $\| \cdot \|$ is regular, then $m = 1$ so that $\| a + \lambda 1 \|_{op} \leq \| a + \lambda 1 \|_1 \leq 6(\exp 1) \| a + \lambda 1 \|_{op}$ for all $a + \lambda 1 \in A_e$ [7, Theorem 1].

3. Uniqueness of uniform norm and unification

A Banach algebra $(A, \| \cdot \|)$ has unique uniform norm property (UUNP) if $A$ admits exactly one (not necessarily complete) uniform norm. The uniform algebra $C(X)$ has UUNP, whereas the disc algebra does not have. In [2] and [3], Banach algebras with UUNP have been investigated. Such an $A$ is necessarily commutative, semisimple and the spectral radius $r = r_A(\cdot)$ is the unique uniform norm. We denote the Hausdorff completion of $(A, r)$ by $U(A)$. The spectral radius on $U(A)$ is the complete uniform norm on $U(A)$. A norm $\| \cdot \|$ on $A$ is functionally continuous (FC) if every multiplicative linear functional on $A$ is $\| \cdot \|$-continuous. A subset $F$ of the Gelfand space of $A$ is a set of uniqueness for $A$ if $|x|_F = \sup \{ |f(x)| : f \in F \}$ defines a norm on $A$. 
Theorem 3.1. A Banach algebra \((A, \| \cdot \|)\) has UUNP iff \(A_e\) has UUNP.

We shall need the following. The proofs are straightforward. For details we refer to [3].

Lemma A. Let \(\| \cdot \|\) be an FC norm on any commutative algebra \(A\). Let \(B\) be the completion of \((A, \| \cdot \|)\). Then the Gelfand space \(\Delta(A)\) (resp. Silove boundary \(\partial A\)) is homeomorphic to \(\Delta(B)\) (resp. \(\partial B\)).

Lemma B. Let \(A\) be a semisimple commutative Banach algebra. Then the following are equivalent.

1. \(A\) has UUNP.
2. \(U(A)\) has UUNP; and any closed set \(F\) in \(\Delta(U(A))\) which is a set of uniqueness for \(A\), is also a set of uniqueness for \(U(A)\).
3. \(U(A)\) has UUNP; and for a non-zero closed ideal \(I\) of \(U(A)\) with \(I = k(h(I))\) (kernel of hull of \(I\), \(I \cap A\) is non-zero.

Lemma C. Let \(A\) be a Banach algebra with UUNP, and \(I\) be a closed ideal such that \(I = k(h(I))\). Then \(I\) has UUNP.

Proof of Theorem 3.1. Assume that \(A\) has UUNP.

Case 1. Let \(\| \cdot \|\) have the square property. By Proposition 2.2 (a) and Proposition 2.3, \((A_e, \| \cdot \|_e)\) is a Banach algebra, \(\| \cdot \|_e\) has square property and \(\| \cdot \|_e\) is equivalent to \(\| \cdot \|_1\).

Let \(| \cdot |\) be any uniform norm on \(A_e\), then \(|1 \cdot |A|A\). Since \(A\) has UUNP, \(|1 \cdot |A|A\). Hence \(\| \cdot \|_e\) is equivalent to \(\| \cdot \|_1\), \(\| \cdot \|_e\) is uniform norm on \(A_e\), \(\| \cdot \|_e\) is equivalent to \(\| \cdot \|_1\). Then \(A_e\) has UUNP.

Case 2. In the general case, note that \(U(A)\) is an ideal of \(U(A_e)\) and, by Lemma A, the Gelfand space \(\Delta(U(A))\) is homeomorphic to the one point compactifications of each of \(\Delta(A)\) and \(\Delta(U(A))\). Define \(K = \{x \in U(A_e): x(U(A)) = \{0\}\}\). We prove that \(K = \{0\}\). Let \(x \in K\). Then its Gelfand transform \(\hat{x}: \Delta(U(A_e)) \to \mathbb{C}\) is continuous. Since \(x \in K\), \(xy = 0\) \((y \in U(A))\). We prove that \(\hat{x}\) is zero on \(\Delta(U(A))\). Since \(\Delta(U(A))\) is dense in \(\Delta(U(A_e))\), it is enough to prove that \(\hat{x}\) is zero on \(\Delta(U(A))\). Suppose there exists \(\phi \in \Delta(U(A))\) such that \(\phi(x) \neq 0\). Since \(\phi\) is non-zero, there exists \(y \in U(A)\) such that \(\phi(y)\) is non-zero. This implies \(\phi(xy) \neq 0\), hence \(xy \neq 0\) which is a contradiction. Thus \(K = \{0\}\). By Lemma B, it is enough to prove that \(U(A_e)\) has UUNP; and for every non-zero closed ideal \(I\) of \(U(A_e)\) with \(I = k(h(I))\), \(A_e \cap I\) is non-zero. Let \(I\) be a non-zero closed ideal of \(U(A_e)\) such that \(I = k(h(I))\). We prove that \(I \cap A_e \neq \{0\}\). Let \(J = I \cap U(A)\). Then, first, we prove that \(J = k(h(J))\). Clearly \(J \subseteq k(h(J))\). Let \(x \in U(A)\) such that \(x \notin J\). Then \(x \notin I\), hence there exists \(\phi \in h(I) \subseteq \Delta(U(A))\) such that \(\phi(x) \neq 0\). Then \(\psi = \phi|U(A)\) is zero on \(J\) and \(\psi(x) \neq 0\). Thus \(x \notin h(J)\), and so \(J = k(h(J))\). From \(K = \{0\}\), \(I \neq \{0\}\) and \(U(A) \subseteq J\), it follows that \(J \neq \{0\}\). Since \(A\) has UUNP and \(J\) is a non-zero closed ideal of \(U(A)\) such that \(J = k(h(J))\), \(A \cap J \neq \{0\}\) by Lemma B. Hence \(I \cap A_e \neq \{0\}\). Finally, we show that \(U(A)\) has UUNP. Note that, by Proposition 2.2 (a) and Proposition 2.3, the operator norm on \(U(A)\) is a complete uniform norm; and is the spectral radius \(r_{U(A)}\) itself. Further, \(U(A)\) is clearly isometrically isomorphic to \(U(A_e)\) via the map \(T: U(A) \to U(A_e), T(a + \lambda e) = a + \lambda e\), where \(e\) is the identity of \(U(A)\). By Lemma C,
U(A) has UUNP, hence by the isomorphism T and by Case 1, U(A_e) has UUNP. Conversely, if A_e have UUNP, then A being a closed ideal of A_e satisfying A = k(h(A)) in A_e, A has UUNP by Lemma C. This completes the proof.

Following [1], a Banach *-algebra B has unique C*-norm (i.e. B has UC*NP) if B admits exactly one C*-norm (not necessarily complete). In spite of the apparent similarity between the square property and the C*-property of norms the above result differs from the corresponding situation in B, viz. UC*NP for B need not imply UC*NP for B_e [1, Example 4.4, p. 850]. In fact, by [1, Theorem 4.1, p. 849], for a non-unital B with UC*NP, B, has UC*NP iff the enveloping C*-algebra C*(B) is non-unital. Like C*(B) for B, the uniform Banach algebra U(A) is universal for A in an appropriate sense. Unlike the case of B, it happens that A is unital iff U(A) is unital. This explains why the above result for A differs from the corresponding result for B.

A Banach algebra (A, \|\cdot\|) has the spectral extension property (SEP) [9] (i.e. A is a permanent Q-algebra [10]), if for every Banach algebra B such that A is algebraically embedded in B, r_A(x) = r_B(x) for all x \in A; equivalently, every norm \|\cdot\| on A satisfies r_A(x) \leq \|x\| for all x \in A [9, Proposition 1].

**Corollary 3.2**

Let (A, \|\cdot\|) be a semisimple commutative Banach algebra. Then A has SEP iff A_e has SEP.

**Proof.** Let A have SEP. Then, by [2, Proposition 2.1] and Theorem 3.1, A_e has UUNP. By [2, Proposition 2.6], it is enough to prove that A_e has (P)-property; i.e. every non-zero closed ideal J of A_e has an element a + \lambda 1 such that r_1(a + \lambda 1) > 0, where r_1(a + \lambda 1) = \inf \{\|a + \lambda 1\| : \lambda \text{ is a norm on } A_e\}, called the permanent radius of a + \lambda 1 in A_e, for all \lambda \in [0, \infty). Let I be a non-zero closed ideal of A_e. Then J = I \cap A is a non-zero closed ideal of A by [8, Theorem 1.1.6, p. 11]. Since A has SEP, by [2, Proposition 2.6], it has (P)-property, hence there exists a \lambda \in J such that the permanent radius, say r_2(a), of a in A is positive. Then clearly r_1(a) \geq r_2(a) > 0. Thus A_e has (P)-property. Conversely, assume that A_e has SEP. Let \|\cdot\| be any norm on A. Then, since A is semisimple, Proposition 2.1 implies the operator norm \|\cdot\|_{op} is a norm on A_e. Since A_e has SEP, r_A(a) = r_A_e(a) \leq |a|_{op} \leq |a| (a \in A). Thus r_A(a) \leq |a| for all a in A and for any norm \|\cdot\| on A. Hence, A has SEP. This completes the proof.

By [9, Corollary 2], a regular Banach algebra has SEP. In understanding the relation between UUNP and SEP, a weaker notion of regularity has been found useful in [2], viz. a semisimple commutative Banach algebra (A, \|\cdot\|) is weakly regular if for any proper closed subset F of the Gelfand space \Delta(A) of A, there exists a non-zero element a in A such that \partial A_e = \partial F = 0.

**Corollary 3.3**

Let (A, \|\cdot\|) be a semisimple commutative Banach algebra. Then A is weakly regular iff A_e is weakly regular.

**Proof.** Let A be weakly regular. Then, by [2, Corollary 2.4(II)], A has UUNP and \Delta(A) = \partial A, the Silov boundary of A. By Theorem 3.1, A_e has UUNP. Note that \Delta(A) = \partial A \subseteq \Delta_e, \Delta(A) is dense in \Delta(A_e) and \partial A_e is closed. These imply \partial A_e = \Delta(A_e). Hence, again by [2, Corollary 2.4(II)], A_e is weakly regular. Conversely, assume that A_e
is weakly regular. The proof of Lemma C will work for the following statement; If \( A \) is weakly regular and \( I \) is a closed ideal of \( A \) such that \( I = k(h(I)) \), then \( I \) is also weakly regular. Since \( A \) is a closed ideal of \( A_c \) with \( k(h(A)) = A \), \( A \) is weakly regular.

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