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SPECTRAL INVARIANCE, K-THEORY ISOMORPHISM AND AN APPLICATION TO THE DIFFERENTIAL STRUCTURE OF C^*-ALGEBRAS

S.J. BHATT, A. INOUE and H. OGI

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Abstract. The notion of spectral invariance of a locally convex $*$-algebra is defined by constructing the enveloping C^*-algebra and is characterized. It is shown that the spectral invariance induces K-theory isomorphism at a general level. As an application the differential structure of C^*-algebras is studied.

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1. INTRODUCTION

Recent developments in non-commutative geometry ([11]) demand the search for, and the investigations of, smooth structures associated with a C^*-algebra $\mathcal{B}$, usually (but not always) manifested as dense $*$-subalgebras $\mathcal{A}$ of $\mathcal{B}$ ([9]). Differential seminorms provide a general mean to construct a differential structure associated with a dense subalgebra $\mathfrak{A}$ of $\mathcal{B}$ ([9]). The differential Fréchet $*$-algebra $\mathfrak{A}_\tau$ and the differential Banach $*$-algebra $\mathfrak{A}_T$ defined by a differential $*$-seminorm on $\mathfrak{A}$ are generally not subalgebras of $\mathcal{B}$, though there exist surjective $*$-homomorphisms $\mathfrak{A}_\tau \to \mathcal{B}$, $\mathfrak{A}_T \to \mathcal{B}$. Now besides completeness in an appropriate locally convex $*$-algebra topology, spectral invariance and closure under appropriate functional calculus have been recongnized as important attributes of smooth subalgebras ([9]). One says that $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$ if $\forall\, x \in \mathcal{A}$, $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{\mathcal{B}}(x)$. This is known to imply the K-theory isomorphism $K_*(\mathcal{B}) = K_*(\mathcal{A})$. We aim to discuss spectral invariance and the closure under holomorphic functional calculus in a situation where there is a natural homomorphism from $\mathcal{A}$ to $\mathcal{B}$ instead of inclusion with a view to understand the structure of the differential algebras $\mathfrak{A}_\tau$ and $\mathfrak{A}_T$.

In fact, spectral invariance of a locally m-convex algebra $\mathcal{A}$ in its homomorphic image has been considered in [19] to give a short proof of the assertion that $M_n(\mathcal{A})$ is local if $\mathcal{A}$ is local and Fréchet, whereas the spectral invariance of a Banach $*$-algebra $\mathcal{A}$ in its enveloping C^*-algebra has been considered in [3] to discuss the discretized version of CCR algebras. First we shall characterize the spectral invariance by the spectrality of submultiplicative $*$-seminorms or of C^*-seminorms on a pseudo-complete locally convex $*$-algebra $\mathcal{A}$ in which every element is bounded in a natural sense ([1]). A submultiplicative $*$-seminorm p on $\mathcal{A}$ is said to be *spectral* if $\gamma_{\mathcal{A}}(x) \leqslant p(x)$ for each $x \in \mathcal{A}$, where $\gamma_{\mathcal{A}}(x)$ is the spectral radius of x in $\mathcal{A}$. If there exists a non-zero continuous spectral submultiplicative $*$-seminorm (respectively C^*-seminorm) on $\mathcal{A}$, then $\mathcal{A}$ is said to be *spectral* (respectively *C^*-spectral*). The spectral invariance of $\mathcal{A}$ is defined as follows: Let $\mathrm{CRep}(\mathcal{A})$ be the family of all non-zero (automatically, bounded ([5])) continuous $*$-representations of $\mathcal{A}$. Suppose $\mathrm{CRep}(\mathcal{A}) \neq \emptyset$, then a C^*-seminorm $|\cdot|_u$ on $\mathcal{A}$ called a *Gelfand-Naimark C^*-seminorm* is defined by

$$|x|_u = \sup\{\|\pi(x)\| : \pi \in \mathrm{CRep}(\mathcal{A})\}, \quad x \in \mathcal{A}$$

and the C^*-algebra $E(\mathcal{A})$ obtained by completion of the normed C^*-algebra $\mathcal{A}/\ker|\cdot|_u$ is called an *enveloping C^*-algebra.* If $\mathrm{CRep}(\mathcal{A}) \neq \emptyset$ and $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{E(\mathcal{A})}(j(x))$ for each $x \in \mathcal{A}$, where j is a natural map of $\mathcal{A}$ onto $\mathcal{A}|\ker(|\cdot|_u)$, then $\mathcal{A}$ is said to be *spectral invariant.* We define $\mathcal{A}$ to be *local* (or closed under the holomorphic functional calculus (of $E(\mathcal{A})$)) if given $x \in \mathcal{A}$ and a function f holomorphic on $\mathrm{Sp}_{E(\mathcal{A})}(j(x))$, there exists $y \in \mathcal{A}$ such that $f(j(x)) = j(y)$. This refines the usual notion of local subalgebras ([19]). By Lemma 1.2 of [19], if $\mathcal{A}$ is a Fréchet subalgebra of an m-convex Fréchet Q-algebra $\mathcal{B}$ (in particular, of a C^*-algebra $\mathcal{B}$), then $\mathcal{A}$ is closed under the holomorphic functional calculus of $\mathcal{B}$ if and only if $\mathcal{A}$ is spectrally invariant in $\mathcal{B}$. Here we shall incorporate this at the generality of the present paper where $\mathcal{A}$ is not a subalgebra of a C^*-algebra $\mathcal{B}$, but there exists the continuous $*$-homomorphism $j : \mathcal{A} \to E(\mathcal{A}) = \mathcal{B}$. In Theorem 2.11, it is shown that $\mathcal{A}$ is spectral and hermitian ($\mathrm{Sp}_{\mathcal{A}}(x) \subset \mathbb{R}$ for each $x^* = x \in \mathcal{A}$) if and only if $\mathcal{A}$ is C^*-spectral if and only if $\mathcal{A}$ is spectrally invariant if and only if $\mathcal{A}$ is local and $\mathrm{rad}\,\mathcal{A} = \mathrm{srad}\,\mathcal{A}$ (the strong radical of $\mathcal{A}$). Speaking the proofs roughly, suppose $\mathcal{A}$ is spectral and hermitian, then $s_{\mathcal{A}}$ defined by $s_{\mathcal{A}}(x) \equiv \gamma_{\mathcal{A}}(x^*x)^{1/2}$, $x \in \mathcal{A}$, becomes a continuous spectral C^*-seminorm on $\mathcal{A}$, that is, $\mathcal{A}$ is C^*-spectral. The converse is trivial. Suppose that $\mathcal{A}$ is C^*-spectral, that is, there exists a non-zero continuous spectral C^*-seminorm p on $\mathcal{A}$. Then it can be shown that $p = |\cdot|_u = s_{\mathcal{A}}$, which implies that $\mathcal{A}$ is spectrally invariant. The necessary and sufficient condition of the spectral invariance of $\mathcal{A}$ and of the locality of $\mathcal{A}$ and $\mathrm{rad}\,\mathcal{A} = \mathrm{srad}\,\mathcal{A}$ is based on the holomorphic functional calculus in pseudo-complete locally convex algebras ([1]).

It is also known that the spectral invariance plays an important rule for the structure theory and for the representation theory of locally convex $*$-algebras ([6], [7]) and so in Theorem 2.15 we shall characterize the spectral invariance by the properties of $*$-representations (the existence of spectral $*$-representations, the dilation property of $*$-representations etc.) though they are not used in this paper.

In Section 3 we shall consider the K-theory isomorphisms of Fréchet $*$-algebras and the differential structure of a C^*-algebra as applications of Theorem 2.11. Given a dense $*$-subalgebra $\mathcal{A}$ of a C^*-algebra $\mathcal{B}$, the significance of the spectral invariance of $\mathcal{A}$ in $\mathcal{B}$ lies in the fact that it induces K-theory isomorphism $K_*(\mathcal{A}) = K_*(\mathcal{B})$ (Chapter III, Appendix C, [11]). This can be extended to more general Fréchet $*$-algebras applying Theorem 2.11 and the K-theory for Fréchet algebras developed by Phillips ([15]). In Theorem 3.1, it is shown that if $\mathcal{A}$ is a Fréchet locally m-convex $*$-algebra in which each element is bounded, then the spectral invariance of $\mathcal{A}$ implies the K-theory isomorphisms $K_*(\mathcal{A}) \simeq K_*(E(\mathcal{A}))$. As an application of Theorems 2.11 and 3.1, we investigate the properties of the C^*-spectrality and the spectral invariance of a Fréchet $*$-algebra defined by a differential seminorm. Let $\mathcal{A}$ be a C^*-algebra and $\mathfrak{A}$ a dense $*$-subalgebra of $\mathcal{A}$. Given a differential $*$-seminorm $T \sim (T_k)_0^\infty$ on $\mathfrak{A}$ in the sense of [9], let $p_k(x) = \sum_{i=0}^{k} T_i(x)$. Then $(p_k)_0^\infty$ is a separating increasing sequence of submultiplicative $*$-seminorms. Let τ be a locally convex $*$-algebra topology on $\mathfrak{A}$ defined by $(p_k)_0^\infty$. The completion $\mathfrak{A}_\tau$ of $\mathfrak{A}$ with respect to τ is a Fréchet $*$-algebra which is an inverse limit $\varprojlim \mathfrak{A}_{(k)}$ of the Banach $*$-algebras $\mathfrak{A}_{(k)}$ obtained by the completion of $\mathfrak{A}$ with respect to p_k. Let $\mathcal{B}$ denote $\mathfrak{A}_{(k)}$ or $\mathfrak{A}_\tau$. In Theorem 3.3, it is shown that $\mathcal{B}$ is a C^*-spectral and spectral invariant hermitian Q-algebra such that $E(\mathcal{B}) = \mathcal{A}$ and $K_*(\mathcal{B}) = K_*(\mathcal{A}) = K_*(\mathfrak{A}_{(k)})$ for all k.

2. SPECTRAL INVARIANCE

We begin with the basic definitions and properties about locally convex $*$-algebras. For more details refer to [1] and [2]. The term locally convex $*$-algebra means a $*$-algebra $\mathcal{A}$ equipped with a topology τ such that

(i) $\mathcal{A}[\tau]$ is a Hausdorff locally convex space;
(ii) the multiplication of $\mathcal{A}$ is separately continuous;
(iii) the involution on $\mathcal{A}$ is continuous.

We may essentially restrict our considerations in this paper to the case in which $\mathcal{A}$ has an identity $\mathbb{1}$ by considering the adjunction $\mathcal{A}_{\mathbf{1}}$ of an identity if $\mathcal{A}$ has no identity. Henceforth it will be assumed, without further mention that $\mathcal{A}$ has an identity $\mathbb{1}$.

Let $\mathcal{A}$ be a locally convex $*$-algebra. An element x of $\mathcal{A}$ is *bounded* if, for some non-zero $\lambda \in \mathbb{C}$, the set $\{(\lambda^{-1}x)^n : n \in \mathbb{N}\}$ is bounded. The set of all bounded elements of $\mathcal{A}$ is denoted by $\mathcal{A}_0$. We write $\mathcal{B}$ for the collection of all absolutely convex, bounded and closed subsets B of $\mathcal{A}$ such that $\mathbb{1} \in \mathrm{B}$ and $\mathrm{B}^2 \subset \mathrm{B}$. For each $\mathrm{B} \in \mathcal{B}$, let $\mathcal{A}[\mathrm{B}]$ denote the subspace of $\mathcal{A}$ generated by B. Then $\mathcal{A}[\mathrm{B}] = \{\lambda x : \lambda \in \mathbb{C}, x \in \mathrm{B}\}$ and the equation: $\|x\|_{\mathrm{B}} = \inf\{\lambda > 0 : x \in \lambda\mathrm{B}\}$ defines a norm on $\mathcal{A}[\mathrm{B}]$, which makes $\mathcal{A}[\mathrm{B}]$ a normed algebra. If $\mathcal{A}[\mathrm{B}]$ is complete for each $\mathrm{B} \in \mathcal{B}$, then $\mathcal{A}$ is said to be *pseudo-complete*. We remark that if $\mathcal{A}$ is sequentially complete, then $\mathcal{A}$ is pseudo-complete. Throughout this paper we consider only a locally convex $*$-algebra $\mathcal{A}$ with $\mathcal{A} = \mathcal{A}_0$.

We define the spectrum $\mathrm{Sp}_{\mathcal{A}}(x)$ and the spectral radius of x in $\mathcal{A}$ as follows:

$$\mathrm{Sp}_{\mathcal{A}}(x) = \{\lambda \in \mathbb{C} : \nexists(\lambda\mathbb{1} - x)^{-1} \text{ in } \mathcal{A}\}, \quad r_{\mathcal{A}}(x) = \sup\{|\lambda| : \lambda \in \mathrm{Sp}_{\mathcal{A}}(x)\}.$$

Then it is known in [1] that

$$\begin{aligned}\gamma_{\mathcal{A}}(x) = \beta(x) &\equiv \inf\{\lambda > 0 : \{(\lambda^{-1}x)^n : n \in \mathbb{N}\} \text{ is bounded}\}\\ &= \sup\{\varlimsup_{n\to\infty} |f(x^n)|^{\frac{1}{n}} : f \in \mathcal{A}'\}\\ &= \sup\{\varlimsup_{n\to\infty} p(x^n)^{\frac{1}{n}} : p \in P\},\end{aligned} \tag{2.1}$$

where $\mathcal{A}'$ is the dual space of $\mathcal{A}$ and P is a family of seminorms which define the topology.

Definition 2.1. A (continuous) seminorm p on $\mathcal{A}$ is said to be *spectral* if $\gamma_{\mathcal{A}}(x) \leqslant p(x)$ for each $x \in \mathcal{A}$.

An element x of $\mathcal{A}$ is said to be *quasi-regular* if $(\mathbb{1} - x)$ has the inverse belonging to $\mathcal{A}$. Let $\mathcal{A}^{\mathrm{qr}}$ be the set of all quasi-regular elements of $\mathcal{A}$.

By Lemma 4.1 of [5]) we have the following

Lemma 2.2. *Let $\mathcal{A}$ be pseudo-complete and p a seminorm on $\mathcal{A}$. Then the following statements are equivalent:*

(i) *p is spectral;*

(ii) *$\{x \in \mathcal{A} : p(x) < 1\} \subset \mathcal{A}^{\mathrm{qr}}$.*

A locally convex $*$-algebra $\mathcal{A}$ is said to be *Q-algebra* if $\mathcal{A}^{\mathrm{qr}}$ is open.

By Theorem 4.2 of [5] we have the following

Lemma 2.3. *Let $\mathcal{A}$ be a pseudo-complete locally convex $*$-algebra. Consider the following statements:*

(i) *$\mathcal{A}$ has continuous quasi-inverse, that is, there exists a neighbourhood U of 0 such that $U \subset \mathcal{A}^{\mathrm{qr}}$ and the quasi-inverse $x \to x^q$ is continuous at 0;*

(ii) *$\mathcal{A}$ is a Q-algebra;*

(iii) *there exists a continuous spectral seminorm on $\mathcal{A}$.*

Then the following implications hold: (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii).

In particular, if $\mathcal{A}$ has jointly continuous multiplication, then (i), (ii) *and* (iii) *are equivalent.*

We next define the notions of C^*-spectrality, spectral invariance and stability of locally convex $*$-algebra. A seminorm p on a (locally convex) $*$-algebra $\mathcal{A}$ is said to be a *m^*-seminorm* (respectively a *C^*-seminorm*) if it is $*$-submultiplicative, that is, $p(xy) \leqslant p(x)p(y)$ and $p(x^*) = p(x)$, $\forall\, x, y \in \mathcal{A}$ (respectively $p(x^*x) = p(x)^2$, $\forall\, x \in \mathcal{A}$). Let p be a m^*-seminorm on $\mathcal{A}$. Then $\mathcal{N}_p \equiv \ker p = \{x \in \mathcal{A} : p(x) = 0\}$ is a $*$-ideal of $\mathcal{A}$ and the quotient space $\mathcal{A}/\mathcal{N}_p$ is a normed $*$-algebra equipped with the multiplication $(x+\mathcal{N}_p)(y+\mathcal{N}_p) \equiv xy+\mathcal{N}_p$, the involution $(x+\mathcal{N}_p)^* \equiv x^*+\mathcal{N}_p$ and the norm $\|x+\mathcal{N}_p\|_p \equiv p(x)$. We denote by $\mathcal{A}_p$ the Banach $*$-algebra which is the completion of $\mathcal{A}/\mathcal{N}_p$. In particular, if p is a C^*-seminorm on $\mathcal{A}$, then $\mathcal{A}_p$ is a C^*-algebra.

LEMMA 2.4. *Let p be a (continuous) m^*-seminorm on a locally convex $*$-algebra $\mathcal{A}$. Then the following statements are equivalent:*

(i) *p is spectral;*

(ii) $\gamma_{\mathcal{A}}(x) = \lim\limits_{n\to\infty} p(x^n)^{\frac{1}{n}}$, $\forall x \in \mathcal{A}$*;*

(iii) $\gamma_{\mathcal{A}}(x) = \gamma_{\mathcal{A}_p}(x_p)$, $\forall x \in \mathcal{A}$, *where* $x_p \equiv x + \mathcal{N}_p$*;*

(iv) $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{\mathcal{A}_p}(x_p)$, $\forall x \in \mathcal{A}$.

In particular, if p is a C^*-seminorm on $\mathcal{A}$, then the above statements (i)–(iv) *are equivalent to*

(v) $\gamma_{\mathcal{A}}(x) = p(x)$, $\forall x \in \mathcal{A}_h \equiv \{a \in \mathcal{A} : a^* = a\}$.

Proof. (i) $\Rightarrow$ (iv) Let $x \in \mathcal{A}$. It is clear that $\mathrm{Sp}_{\mathcal{A}_p}(x_p) \subset \mathrm{Sp}_{\mathcal{A}}(x)$. We show the converse. Take an arbitrary $\lambda \in \mathbb{C}$ such that $(\lambda \mathbb{1}_p - x_p)^{-1}$ exists in $\mathcal{A}_p$. Since $\mathcal{A}_p$ is the completion of the normed $*$-algebra $\mathcal{A}\lceil\mathcal{N}_p$, there exists an element y of $\mathcal{A}$ such that $\|\mathbb{1}_p - (\lambda\mathbb{1}_p - x_p)y_p\|_p = p(\mathbb{1} - (\lambda\mathbb{1} - x)y) < 1$ and $\|\mathbb{1}_p - y_p(\lambda\mathbb{1}_p - x_p)\|_p = p(\mathbb{1} - y(\lambda\mathbb{1} - x)) < 1$, which implies by the spectrality of p that $(\lambda\mathbb{1} - x)y$ and $y(\lambda\mathbb{1} - x)$ are invertible. Hence, $(\lambda\mathbb{1} - x)$ is invertible, and so $\lambda \notin \mathrm{Sp}_{\mathcal{A}}(x)$. Thus we have $\mathrm{Sp}_{\mathcal{A}}(x) \subset \mathrm{Sp}_{\mathcal{A}_p}(x_p)$.

(iv) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (ii) This follows from the equalities:

$$\gamma_{\mathcal{A}}(x) = \gamma_{\mathcal{A}_p}(x_p) = \lim_{n\to\infty} \|x_p^n\|_p^{\frac{1}{n}} = \lim_{n\to\infty} p(x^n)^{\frac{1}{n}}, \quad x \in \mathcal{A}.$$

(ii) $\Rightarrow$ (i) This follows from the submultiplicativity of p. Suppose p is a C^*-seminorm on $\mathcal{A}$. Then the equivalence of (ii) and (v) is clear. ∎

DEFINITION 2.5. A locally convex $*$-algebra $\mathcal{A}$ is said to be *spectral* (respectively *C^*-spectral*) if there exists a non-zero continuous spectral m^*-seminorm (respectively C^*-seminorm) on $\mathcal{A}$.

We define the Gelfand-Naimark C^*-seminorm $|\cdot|_u$ on $\mathcal{A}$ and the enveloping C^*-algebra $E(\mathcal{A})$ of $\mathcal{A}$. We state the definition of $*$-representation of $\mathcal{A}$. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and $\mathcal{L}^\dagger(\mathcal{D})$ the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra under the usual operations and the involution $X \mapsto X^\dagger \equiv X^*\lceil\mathcal{D}$. A $*$-homomorphism π of $\mathcal{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$ satisfying $\pi(\mathbb{1}) = I$ is a *$*$-representation* of $\mathcal{A}$ on $\mathcal{H}$ with domain $\mathcal{D}$, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. For more details refer to [16] and [18]. If $\pi(x) \in \mathcal{B}(\mathcal{H}_\pi)$ for each $x \in \mathcal{A}$, equivalently $\mathcal{D}(\pi) = \mathcal{H}_\pi$, then π is said to be *bounded.* By Corollary 3.13 from [5] we have the following

LEMMA 2.6. *Every $*$-representation π of $\mathcal{A}$ is bounded and* $\|\pi(x)\| \leqslant s_{\mathcal{A}}(x) \equiv \gamma_{\mathcal{A}}(x^*x)^{1/2}$ *for each $x \in \mathcal{A}$.*

It is natural to consider unbounded $*$-representations for general locally convex $*$-algebras, but by Lemma 2.6 it is here sufficient to consider only bounded $*$-representations. We denote by $\mathrm{CRep}(\mathcal{A})$ the family of all continuous (automatically, bounded by Lemma 2.6) $*$-representations of $\mathcal{A}$.

Definition 2.7. If $\mathrm{CRep}(\mathcal{A}) \neq \emptyset$, then $\mathcal{A}$ is said to be *representable*.

We remark that even a Banach $*$-algebra is not necessarily representable (Example 37.16, [10]). Suppose $\mathcal{A}$ is representable. By Lemma 2.6, a C^*-seminorm on $\mathcal{A}$ is defined by

$$|x|_u = \sup\{\|\pi(x)\| : \pi \in \mathrm{CRep}(\mathcal{A})\}, \quad x \in \mathcal{A},$$

and it is said to be the *Gelfand-Naimark C^*-seminorm* on $\mathcal{A}$. The C^*-algebra $\mathcal{A}_{|\cdot|_u}$ constructed from the C^*-seminorm $|\cdot|_u$ is said to be an *enveloping C^*-algebra* of $\mathcal{A}$ and denoted by $E(\mathcal{A})$. The natural map $j : x \in \mathcal{A} \mapsto x + \mathcal{N}_{|\cdot|_u} \in E(\mathcal{A})$ is a $*$-homomorphism.

Definition 2.8. If $\mathcal{A}$ is representable and $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{E(\mathcal{A})}(j(x))$ for each $x \in \mathcal{A}$, then $\mathcal{A}$ is said to be *spectrally invariant*.

The family $C^*N(\mathcal{A})$ of all C^*-seminorms on $\mathcal{A}$ is a partially ordered family with order $r_1 \leqslant r_2$ defined by $r_1(x) \leqslant r_2(x)$, $\forall x \in \mathcal{A}$.

Lemma 2.9. *Let r be a spectral C^*-seminorm on $\mathcal{A}$. Then $r = s_{\mathcal{A}}$ and r is the largest element in the partially ordered family $C^*N(\mathcal{A})$. Thus, a spectral C^*-seminorm is unique. Further, if r is continuous then $\mathcal{A}$ is representable and $r = |\cdot|_u$.*

Proof. It follows from Lemma 2.4, (v) that $r(h) = r_{\mathcal{A}}(h)$ for $\forall h \in \mathcal{A}_h$, which implies that $r = s_{\mathcal{A}}$. Thus, a spectral C^*-seminorm is unique. Let $p \in C^*N(\mathcal{A})$. Then it follows that $q \equiv \max(r, p) \in C^*N(\mathcal{A})$ and q is spectral. By the uniqueness of a spectral C^*-seminorm we have $q = r$, which implies that r is the largest in $C^*N(\mathcal{A})$. Suppose that r is continuous. Then the continuous $*$-representation π_r of $\mathcal{A}$ is defined by $\pi_r(x) = \Pi_r(x + N_r)$, $x \in \mathcal{A}$, where Π_r is a faithful $*$-representation of the C^*-algebra $\mathcal{A}_r$ on a Hilbert space. Hence it follows that $\mathcal{A}$ is representable and

$$r(x) = \|\pi_r(x)\| \leqslant \Big\| \bigoplus_{\pi \in \mathrm{CRep}(\mathcal{A})} \pi(x) \Big\| = |x|_u$$

for all $x \in \mathcal{A}$. On the other hand, since r is the largest in $C^*N(\mathcal{A})$, we have $|\cdot|_u \leqslant r$. Thus, we have $r = |\cdot|_u$. This completes the proof. ∎

We define the locality of $\mathcal{A}$.

Definition 2.10. $\mathcal{A}$ is said to be *local* if for $x \in \mathcal{A}$ and a function f holomorphic on $\mathrm{Sp}_{E(\mathcal{A})}(j(x))$, there exists $y \in \mathcal{A}$ such that $f(j(x)) = j(y)$.

This refines the usual definition of local subalgebras ([19]).

The spectral invariance of $\mathcal{A}$ can be characterized by the (C^*-)spectrality and the locality of $\mathcal{A}$ as follows:

THEOREM 2.11. *The following statements are equivalent:*
(i) $\mathcal{A}$ *is spectrally invariant;*
(ii) $\mathcal{A}$ *is* C^**-spectral;*
(iii) $\mathcal{A}$ *is spectral and hermitian;*
(iv) $\mathcal{A}$ *is local and* $\operatorname{rad}\mathcal{A} = \operatorname{srad}\mathcal{A}$.

Proof. The equivalence of (i) and (ii) follows from Lemmas 2.4 and 2.9.

(ii) $\Rightarrow$ (iii) This is trivial.

(iii) $\Rightarrow$ (ii) We can show in a slight change of proof of Theorem 41.7 from [10] that $s_{\mathcal{A}}$ is a spectral C^*-seminorm on $\mathcal{A}$. We first show the Ptak inequality:

$$\gamma_{\mathcal{A}}(a) \leqslant s_{\mathcal{A}}(a), \quad \forall a \in \mathcal{A}. \tag{2.2}$$

In fact, take an arbitrary $a \in \mathcal{A}$ such that $s_{\mathcal{A}}(a) < 1$. Since $\mathcal{A} = \mathcal{A}_0$ and $\mathcal{A}$ is pseudo-complete, there exists $\mathrm{B}^* = \mathrm{B} \in \mathcal{B}$ such that a^*a is an element of the Banach $*$-algebra $\mathcal{A}[\mathrm{B}]$. By Proposition 12.11 of [10] there is a unique element x of $\mathcal{A}[\mathrm{B}]_h$ such that $2x - x^2 = a^*a$ and $\gamma_{\mathcal{A}[\mathrm{B}]}(x) < 1$. Since $\gamma_{\mathcal{A}}(x) \leqslant \gamma_{\mathcal{A}[\mathrm{B}]}(x) < 1$, it follows that $h \equiv \mathbb{1} - x$ is invertible and $\mathbb{1} - a^*a = h^2$. Since $(\mathbb{1} + a^*)(\mathbb{1} - a) = h\{\mathbb{1} + h^{-1}(a^* - a)h^{-1}\}h$, $\mathrm{i}h^{-1}(a^* - a)h^{-1} \in \mathcal{A}_h$ and $\mathcal{A}$ is hermitian, it follows that $(\mathbb{1} + h^{-1}(a^* - a)h^{-1})$ is invertible, which implies that $\mathbb{1} - a$ is left invertible. Similarly, $\mathbb{1} - a$ is right invertible. Hence $1 \notin \operatorname{Sp}_{\mathcal{A}}(a)$ and so $\gamma_{\mathcal{A}}(a) < 1$. Thus we have $\gamma_{\mathcal{A}}(a) \leqslant s_{\mathcal{A}}(a)$. We next show the inequalities:

$$\gamma_{\mathcal{A}}(hk) \leqslant \gamma_{\mathcal{A}}(h)\gamma_{\mathcal{A}}(k), \quad \forall h, k \in \mathcal{A}_h \tag{2.3}$$

$$s_{\mathcal{A}}(xy) \leqslant s_{\mathcal{A}}(x)s_{\mathcal{A}}(y), \quad \forall x, y \in \mathcal{A}. \tag{2.4}$$

In fact, since $\mathcal{A}$ is spectral, there exists a continuous m^*-seminorm p on $\mathcal{A}$ such that

$$\gamma_{\mathcal{A}}(x) \leqslant p(x), \quad \forall x \in \mathcal{A}. \tag{2.5}$$

Take arbitrary $h, k \in \mathcal{A}_h$. Then we have

$$\begin{aligned}
\gamma_{\mathcal{A}}(hk) &\leqslant \gamma_{\mathcal{A}}(kh^2k)^{\frac{1}{2}} && \text{by 2.2}\\
&= \gamma_{\mathcal{A}}(h^2k^2)^{\frac{1}{2}} \\
&\leqslant \gamma_{\mathcal{A}}(h^{2^n}k^{2^n})^{\frac{1}{2^n}} && \text{by repeating this}\\
&\leqslant p(h^{2^n})^{\frac{1}{2^n}} p(k^{2^n})^{\frac{1}{2^n}} && \text{by 2.5}\\
&= \gamma_{\mathcal{A}}(h)\gamma_{\mathcal{A}}(k), && \text{by 2.1}
\end{aligned}$$

which implies immediately the inequality (2.4). We can prove the same way as Theorem 41.7 in [10] that $s_{\mathcal{A}}$ is a seminorm on $\mathcal{A}$. It is clear that $s_{\mathcal{A}}(x)^2 = s_{\mathcal{A}}(x^*x)$, $\forall x \in \mathcal{A}$. Thus $s_{\mathcal{A}}$ is a spectral C^*-seminorm on $\mathcal{A}$. Further, it follows from (2.5) that $s_{\mathcal{A}}$ is continuous. Thus, $\mathcal{A}$ is C^*-spectral.

(i) $\Rightarrow$ (iv) Assume (i). Then $\operatorname{Sp}_{\mathcal{A}}(x) = \operatorname{Sp}_{E(\mathcal{A})}(j(x))$ $(\equiv \mathcal{K})$ for an arbitrary fixed $x \in \mathcal{A}$, and $\mathcal{K} \subset \mathbb{C}$ is compact. Let f be a function holomorphic on an open set U containing $\mathcal{K}$. Let Γ be a rectifiable Jordan curve in $U \setminus \mathcal{K}$ enclosing $\mathcal{K}$. Put $z = j(x)$. Then by the holomorphic functional calculus in $E(\mathcal{A})$,

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} f(\lambda)(\lambda I - z)^{-1}\, \mathrm{d}\lambda = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma} j(f(\lambda)(\lambda \mathbb{1} - x)^{-1})\, \mathrm{d}\lambda. \tag{2.6}$$

Now $\mathcal{A} = \mathcal{A}_0 = \cup\{\mathcal{A}[\mathrm{B}] : \mathrm{B} \in \mathcal{B}\}$, and by the pseudo-completeness of $\mathcal{A}$, each $(\mathcal{A}[\mathrm{B}], \|\cdot\|_{\mathrm{B}})$ is a Banach algebra. Also, by Proposition 5.1 of [1], there exists $\mathrm{B} \in \mathcal{B}$ such that both the resolvent $R_\lambda = (\lambda\mathbb{1} - x)^{-1}$, $\forall\lambda \in \Gamma$, and $f(x) = \frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\lambda)(\lambda\mathbb{1} - x)^{-1}\,\mathrm{d}\lambda$ exist in $\mathcal{A}[\mathrm{B}]$ in the sense of the norm convergence in $\mathcal{A}[\mathrm{B}]$. Consider the following diagram:

$$\begin{array}{ccc} (\mathcal{A}[\mathrm{B}], \|\cdot\|_{\mathrm{B}}) & & \\ \mathrm{id}\big\downarrow & \searrow \widetilde{j} & \\ (\mathcal{A}, \tau) & \xrightarrow{j} & (E(\mathcal{A}), \|\cdot\|). \end{array}$$

Here τ is a given topology of $\mathcal{A}$. The map $\mathrm{id} : (\mathcal{A}[\mathrm{B}], \|\cdot\|_{\mathrm{B}}) \to (\mathcal{A}, \tau)$ is continuous. Also $\mathcal{A}$ is C^*-spectral. Let p be a continuous spectral C^*-seminorm on $\mathcal{A}$. Let τ' be the topology defined by τ and p. Then $(\mathcal{A}, \tau')$ is a Q-algebra, and the map $\mathrm{id} : (\mathcal{A}[\mathrm{B}], \|\cdot\|_{\mathrm{B}}) \to (\mathcal{A}, \tau')$ is also continuous. Since j is a $*$-homomorphism from the Q-algebra $(\mathcal{A}, \tau')$ to the C^*-algebra $E(\mathcal{A})$, j is τ'-continuous. It follows that the map $\widetilde{j} = j\lceil_{\mathcal{A}[\mathrm{B}]}$ is $\|\cdot\|_{\mathrm{B}}$-continuous. This is used in (2.6) to show that

$$f(z) = j\bigg(\frac{1}{2\pi\mathrm{i}}\int_\Gamma f(\lambda)(\lambda\mathbb{1} - x)^{-1}\,\mathrm{d}\lambda\bigg).$$

Thus we have $f(j(x)) = j(f(x))$. Also $\mathcal{A}$ is hermitian and C^*-spectral. Then for all $x \in \mathcal{A}$,

$$\gamma_{\mathcal{A}}(x) \leqslant p(x) = |x|_u = s_{\mathcal{A}}(x).$$

Hence, $\operatorname{srad}\mathcal{A} \subset \operatorname{rad}\mathcal{A} \subset \operatorname{srad}\mathcal{A}$. Therefore we have $\operatorname{rad}\mathcal{A} = \operatorname{srad}\mathcal{A}$.

(iv) $\Rightarrow$ (i) Assume that (iv) holds. It is clear that $\mathrm{Sp}_{\mathcal{A}}(x) \supset \mathrm{Sp}_{E(\mathcal{A})}(j(x))$ for each $x \in \mathcal{A}$. Let $\lambda \notin \mathrm{Sp}_{E(\mathcal{A})}(j(x))$. Then $R_\lambda = (\lambda I - j(x))^{-1} \in E(\mathcal{A})$. The function $f(\mu) = (\lambda - \mu)^{-1}$ is holomorphic on a neighborhood of the closed set $\mathrm{Sp}_{E(\mathcal{A})}(j(x))$ and $R_\lambda = f(j(x))$. Hence by (vi), there exists $y \in \mathcal{A}$ such that $(\lambda I - j(x))^{-1} = f(j(x)) = j(y)$, and so $j(y(\lambda\mathbb{1} - x)) = j((\lambda\mathbb{1} - x)y) = I$. This implies that $\lambda \notin \mathrm{Sp}_{j(\mathcal{A})}(j(x))$. Thus we have $\mathrm{Sp}_{j(\mathcal{A})}(j(x)) \subset \mathrm{Sp}_{E(\mathcal{A})}(j(x)) \subset \mathrm{Sp}_{j(\mathcal{A})}(j(x))$. Hence the $*$-subalgebra $j(\mathcal{A})$ is spectrally invariant in the C^*-algebra $E(\mathcal{A})$, and so $j(\mathcal{A})$ is hermitian. Since $\operatorname{rad}\mathcal{A} = \operatorname{srad}\mathcal{A}$ by the assumption, it follows that $j(\mathcal{A}) = \mathcal{A}/\operatorname{rad}\mathcal{A}$ and $j(x) = x + \operatorname{rad}\mathcal{A}$. Hence we have $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{j(\mathcal{A})}(x + \operatorname{rad}\mathcal{A}) = \mathrm{Sp}_{E(\mathcal{A})}(j(x))$. Thus (i) follows. This completes the proof. ∎

We give an example of a C^*-spectral locally convex $*$-algebra.

Example 2.12. The Schwartz spaces $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ and $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ equipped with the Volterra convolution and the involution:

$$(f \circ g)(x, y) = \int_{\mathbb{R}^n} f(x, z)g(z, y)\,\mathrm{d}z \quad\text{and}\quad f^*(x, y) = \overline{f(y, x)}$$

are C^*-spectral. In fact, let $f \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$. We put

$$[\pi_0(f)\varphi](x) = \int_{\mathbb{R}^n} f(x, y)\varphi(y)\,\mathrm{d}y, \quad \varphi \in \mathcal{D}(\mathbb{R}^n).$$

Then we can show that $\pi_0(f)$ can be extended to a bounded linear operator $\pi(f)$ on $L^2(\mathbb{R}^n)$ and π is a continuous bounded $*$-representation of $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ on $L^2(\mathbb{R}^n)$. We show that the continuous C^*-seminorm r on $\mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ defined by $r(f) = \|\pi(f)\|$, $f \in \mathcal{D}(\mathbb{R}^n \times \mathbb{R}^n)$ is spectral. By the simple calculation we have

$$f^{[n]} \equiv \overbrace{f \circ \cdots \circ f}^{n} = \Bigg(\int\limits_{\mathbb{R}^n} f(x,x)\,\mathrm{d}x \Bigg)^{n-1} f, \quad n \in \mathbb{N}$$

and

$$\Bigg| \int\limits_{\mathbb{R}^n} f(x,x)\,\mathrm{d}x \Bigg| < 1 \quad \text{if } r(f) < 1,$$

which implies that r is spectral. Similarly, we can show that the Schwartz space $\mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n)$ is also C^*-spectral.

Next we shall consider to characterize the spectral invariance of $\mathcal{A}$ by the notions different from (ii), (iii) and (iv) in Theorem 2.11.

We define the spectrality of $*$-representations and the stability of $\mathcal{A}$.

Definition 2.13. A continuous $*$-representation π of $\mathcal{A}$ is said to be *spectral* if $\mathrm{Sp}_{\mathcal{A}}(x) = \mathrm{Sp}_{C^*(\pi)}(\pi(x))$ for each $x \in \mathcal{A}$, where $C^*(\pi)$ is a C^*-algebra generated by $\pi(\mathcal{A})$.

Definition 2.14. If for any closed $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$, any continuous $*$-representation π of $\mathcal{B}$ on a Hilbert space $\mathcal{H}_\pi$ admits a dilation to a continuous $*$-representation $\widetilde{\pi}$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}_{\widetilde{\pi}}$ in the sense that there exist a Hilbert space $\mathcal{H}_{\widetilde{\pi}}$ containing $\mathcal{H}_\pi$ as a closed subspace and a continuous $*$-representation $\widetilde{\pi}$ of $\mathcal{A}$ on $\mathcal{H}_{\widetilde{\pi}}$ such that $\pi(x) = \widetilde{\pi}(x)\lceil\mathcal{H}_\pi$ for each $x \in \mathcal{B}$, $\mathcal{A}$ is said to be *stable.*

Theorem 2.15. *The following statements are equivalent:*

(i) $\mathcal{A}$ *is spectrally invarinat;*

(ii) $\mathcal{A}$ *is spectral and stable;*

(iii) *there exists a spectral continuous $*$-representation of $\mathcal{A}$ into bounded linear operators on a Hilbert space;*

(iv) *every algebraically irreducible representation of $\mathcal{A}$ on a vector space is similar to an algebraically irreducible continuous bounded operator $*$-representation on a pre-Hilbert space;*

(v) *every algebraicaly irreducible representation of $\mathcal{A}$ on a vector space extends to an irreducible $*$-representation of the C^*-algebra $E(\mathcal{A})$ on a Hilbert space.*

Proof. The equivalence of (i), (ii) and (iii) is shown similarly to Theorem 6.10, Proposition 6.12 and Theorem 6.8 in [7], and Theorem 1.6 in [6].

(i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) Assume (i). Let $\pi : \mathcal{A} \mapsto \mathcal{L}(V)$ be an algebraically irreducible representation of $\mathcal{A}$ on a vector space V. Here $\mathcal{L}(V)$ is the algebra of all linear operators on the vector space V. Let $v \neq 0$ in V, $\mathfrak{N} = \{x \in \mathcal{A} : \pi(x)v = 0\}$. Then $\pi(\mathcal{A})v = V$, and $\mathfrak{N}$ is a maximal modular left ideal of $\mathcal{A}$. We define a representation $\sigma : \mathcal{A} \mapsto \mathcal{L}(\mathcal{A}/\mathfrak{N})$ by $\sigma(x)(y + \mathfrak{N}) = xy + \mathfrak{N}$. Then we show that

$$\pi \text{ is similar to } \sigma. \tag{2.7}$$

Indeed, the similarity is implemented by the bijective linear map $U : \mathcal{A}/\mathfrak{N} \mapsto V$, $U(y + \mathfrak{N}) = \pi(y)v$ satisfying $U\sigma(x)\xi = \pi(x)U\xi$, $\forall \xi \in \mathcal{A}/\mathfrak{N}$, $\forall x \in \mathcal{A}$. We show that there exists a pure state f on the enveloping C^*-algebra $E(\mathcal{A})$ such that π_f is an extension of π in the sense that there exists an injection $W : V \mapsto \mathcal{H}_f$ with dense range such that

$$\pi_f(j(x))W\xi = W\pi(x)\xi, \quad x \in \mathcal{A}, \xi \in V. \tag{2.8}$$

This is proved as follows. Since $\mathfrak{N}$ is a maximal left ideal of $\mathcal{A}$, we have $\mathfrak{N} \cap \mathcal{A}^{\mathrm{r}} = \varphi$, where $\mathcal{A}^{\mathrm{r}}$ is the set of all regular elements of $\mathcal{A}$. By the assumption (i), $j(\mathfrak{N}) \cap E(\mathcal{A})^{\mathrm{r}} = \varphi$. Now $E(\mathcal{A})$ being a C^*-algebra, $E(\mathcal{A})^{\mathrm{r}}$ is an open set in $E(\mathcal{A})$. Hence $\overline{j(\mathfrak{N})}$ (= closure in $E(\mathcal{A})$) is a proper subset of $E(\mathcal{A})$, and so

$$\overline{j(\mathfrak{N})} \text{ is a closed left ideal of } E(\mathcal{A}) \tag{2.9}$$

and there exists a maximal left ideal $\mathfrak{M}$ of $E(\mathcal{A})$ containing $\overline{j(\mathfrak{N})}$. Since $E(\mathcal{A})$ is a C^*-algebra, $\mathfrak{M}$ is closed, and by p. 56 of [12] there exists a pure state f on $E(\mathcal{A})$ such that $\mathfrak{M} = \mathcal{N}_f \equiv \{x \in E(\mathcal{A}) : f(x^*x) = 0\}$, and by p. 53 of [12] the pre-Hilbert $\mathcal{H}_f \equiv E(\mathcal{A})/\mathcal{N}_f$ is complete. Since $j(\mathfrak{N}) \subset \mathcal{N}_f$, we can define a linear map $\widetilde{j} : \mathcal{A}/\mathfrak{N} \mapsto \mathcal{H}_f$ by

$$\widetilde{j}(x + \mathfrak{N}) = \widetilde{j}(x) + \mathcal{N}_f, \quad x \in \mathcal{A}.$$

Since $\overline{j(\mathfrak{N})}$ is a left ideal of $E(\mathcal{A})$ and $j(\mathcal{A})$ is dense in $E(\mathcal{A})$, it follows that $j^{-1}(\overline{j(\mathfrak{N})}) \neq \mathcal{A}$. Thus $j^{-1}(\overline{j(\mathfrak{N})})$ is a proper left ideal of $\mathcal{A}$ containing $\mathfrak{N}$. By the maximality of $\mathfrak{N}$ we have $\mathfrak{N} = j^{-1}(\overline{j(\mathfrak{N})})$. Further, $j^{-1}(\mathcal{N}_f)$ is a proper left ideal of $\mathcal{A}$. In fact, suppose $j^{-1}(\mathcal{N}_f) = \mathcal{A}$. Then, $f(j(x)^*j(x)) = 0$ for all $x \in \mathcal{A}$, and so $f(j(x)) = 0$ by the Cauchy-Schwartz inequality. Since f is continuous and $j(\mathcal{A})$ is dense in $E(\mathcal{A})$, we have $f = 0$, which is contradition. Hence $j^{-1}(\mathcal{N}_f) \neq \mathcal{A}$. Since $\mathfrak{N} = j^{-1}(\overline{j(\mathfrak{N})}) \subset j^{-1}(\mathcal{N}_f) \neq \mathcal{A}$, it follows from the maximality of $\mathfrak{N}$ that $\mathfrak{N} = j^{-1}(\overline{j(\mathfrak{N})}) = j^{-1}(\mathcal{N}_f)$, which implies that $\widetilde{j}$ is an injection. Here we put $W = \widetilde{j} \circ U^{-1}$. Then W is an injection from V onto a dense subspace of $\mathcal{H}_f$ satisfying (2.8). We put $g = f \circ j$. Then g is a pure state on $\mathcal{A}$ satisfying

$$g(y^*x^*xy) \leqslant |x|_u^2 g(y^*y), \quad \forall x, y \in \mathcal{A}.$$

Hence, π_g is continuous. Further, since $\mathfrak{N} \subset \mathcal{N}_g \subset j^{-1}(\mathcal{N}_f) \neq \mathcal{A}$, it follows from the maximality of $\mathfrak{N}$ that $\mathfrak{N} = \mathcal{N}_g = j^{-1}(\mathcal{N}_f)$, which implies that the restriction π_g° of π_g to the pre-Hilbert space $\mathcal{A}/\mathcal{N}_g$ coincides with σ. Hence it follows from (2.7) that π is similar to the algebraically irreducible continuous bounded $*$-representation π_g° of $\mathcal{A}$ on a pre-Hilbert space $\mathcal{A}/\mathcal{N}_g$. We have thus shown that (i) $\Rightarrow$ (iv) and (i) $\Rightarrow$ (v) hold.

(iv) $\Rightarrow$ (i) Let $x \in \mathcal{A}$. This follows from

$$\begin{aligned}
\mathrm{Sp}_{\mathcal{A}}(x) &= \bigcup\{\mathrm{Sp}(\pi(x)) : \pi \text{ is an algebraically irreducible representation of } \mathcal{A} \text{ on a vector space}\} \\
&\qquad\qquad \text{(by [17], Theorem 2.2.9)} \\
&= \bigcup\{\mathrm{Sp}(\pi(x)) : \pi \text{ is an algebraically irreducible continuous bounded } *\text{-representation on a pre-Hilbert space}\} \\
&\qquad\qquad \text{(by assumption (iv))} \\
&\subset \bigcup\{\mathrm{Sp}(\pi(x)) : \pi \text{ is a topologically irreducible continuous bounded } *\text{-representation on a pre-Hilbert space}\} \\
&= \bigcup\{\mathrm{Sp}(\sigma(j(x))) : \sigma \text{ is a topologically irreducible } *\text{-representation of } E(\mathcal{A}) \text{ on a Hilbert space}\} \\
&= \bigcup\{\mathrm{Sp}(\sigma(j(x))) : \sigma \text{ is an algebraically irreducible } *\text{-representation on a Hilbert space}\} \\
&\qquad\qquad \text{(by Kadison's transitivity in the } C^*\text{-algebra } E(\mathcal{A}) \text{ ([12]))} \\
&= \mathrm{Sp}_{E(\mathcal{A})}(j(x)) \\
&\subset \mathrm{Sp}_{\mathcal{A}}(x).
\end{aligned}$$

This completes the proof. ∎

Some comments on the relavance of Theorems 2.11 and 2.15 are in order. At the level of general $*$-algebras, Theorems 2.11 and 2.15 supplements Theorem 1.6 in [6] and Theorem 6.10 in [7]. At the level of Banach $*$-algebras, it supplements Corollary 2.7 in [6]. Further, it follows from Theorems 2.11 and 2.15 that a Banach $*$-algebra $\mathcal{A}$ is hermitian if and only if every algebraically irreducible representation of $\mathcal{A}$ on a vector space extends to a topologically irreducible $*$-representation of $E(\mathcal{A})$ on a Hilbert space. This is a non-commutative analogue of the well known result that a commutative Banach $*$-algebra $\mathcal{A}$ is hermitian if and only if $\varphi(x^*) = \overline{\varphi(x)}$ for all $x \in \mathcal{A}$, for all complex homomorphisms φ on $\mathcal{A}$ ([10], Theorem 35.3). Also, Theorem 4.7.11 from [17] implies that if $\mathcal{A}$ is hermitian, then any $*$-representation of a closed $*$-subalgebra $\mathcal{B}$ of $\mathcal{A}$ dilates to a $*$-representation of $\mathcal{A}$. In Theorem 2.15 (ii) implies (v) provides a converse of this.

3. K-theory isomorphism and application to differential structure of C^*-algebras

We begin with considering the K-theory isomorphisms of Fréchet $*$-algebras as an application of Theorem 2.11.

Theorem 3.1. *Let $\mathcal{A}$ be a Fréchet locally m-convex $*$-algebra in which each element is bounded. Let $\mathcal{A}$ be spectrally invariant. Then the* K*-theory isomorphisms $K_*(\mathcal{A}) \cong K_*(E(\mathcal{A}))$ hold.*

Proof. Let $(p_n)_{n=1}^{\infty}$ be a sequence of submultiplicative $*$-seminorms defining the given topology τ of $\mathcal{A}$. We have $\mathcal{A} = \mathcal{A}_0$. Assume that $\mathcal{A}$ is spectrally invariant. By Theorem 2.11, $\mathcal{A}$ is hermitian and C^*-spectral. Let q be a continuous spectral C^*-seminorm on $\mathcal{A}$. By Lemma 2.9, $q = s_{\mathcal{A}} = |\cdot|_u$. By Lemma 2.3, $(\mathcal{A}, \tau)$ is a Fréchet Q-algebra. Notice that $\operatorname{rad}\mathcal{A} = \operatorname{srad}\mathcal{A}$. Let $\mathcal{A}_q = \mathcal{A}/\operatorname{rad}\mathcal{A}$, which is a dense $*$-subalgebra of the C^*-algebra $E(\mathcal{A})$ and is a Fréchet Q-algebra in the quotient topology τ_q. The C^*-norm $|\cdot|_u$ of $E(\mathcal{A})$ is spectral. Thus $\mathcal{A}_q$ is spectrally invariant in $E(\mathcal{A})$. By Corollary 7.9 in [15], $K_*(\mathcal{A}_q) = K_*(E(\mathcal{A}))$. The maps

$$\mathcal{A} \xrightarrow{j} \mathcal{A}_q \xrightarrow{\text{id}} E(\mathcal{A})$$

induces the $*$-homomorphisms for each $n \in \mathbb{N}$,

$$M_n(\mathcal{A}) \xrightarrow{j_n} M_n(\mathcal{A}_q) = [M_n(\mathcal{A})]_q \to M_n(E(\mathcal{A})) = E(M_n(\mathcal{A})).$$

By the spectral invariance of $\mathcal{A}$ in $\mathcal{A}_q$, $j(\operatorname{inv}(\mathcal{A})) = \operatorname{inv}(\mathcal{A}_q)$. Let $\operatorname{inv}_0(\cdot)$ denote the principle component of $\operatorname{inv}(\cdot)$. Since $\mathcal{A}$ and also $\mathcal{A}_q$ are Fréchet Q-algebras, $\operatorname{inv}_0(\mathcal{A})$ (respectively $\operatorname{inv}_0(\mathcal{A}_q)$) is the subgroup generated by the range $\exp\mathcal{A}$ (respectively $\exp(\mathcal{A}_q)$) of the exponential function. This gives $j(\operatorname{inv}_0(\mathcal{A})) = \operatorname{inv}_0(\mathcal{A}_q)$, and in view of the spectral invariance of $M_n(\mathcal{A})$ in $M_n(\mathcal{A}_q)$ via the map j_n, analogous arguments give $j_n(\operatorname{inv}_0(M_n(\mathcal{A}))) = \operatorname{inv}_0(M_n(\mathcal{A}_q))$. Now the surjective group homomorphisms

$$\operatorname{inv}(M_n(\mathcal{A})) \to \operatorname{inv}(M_n(\mathcal{A}_q)) \to \operatorname{inv}(M_n(\mathcal{A}_q))/\operatorname{inv}_0(M_n(\mathcal{A}_q))$$

give the isomorphism of groups

$$\operatorname{inv}(M_n(\mathcal{A}_q)) \setminus \operatorname{inv}_0(M_n(\mathcal{A}_q)) \cong \operatorname{inv}(M_n(\mathcal{A})) \setminus \operatorname{inv}_0(M_n(\mathcal{A})).$$

Hence by the definition of K_1,

$$K_1(\mathcal{A}) = \lim_{n\to\infty} \frac{\operatorname{inv}(M_n(\mathcal{A}))}{\operatorname{inv}_0(M_n(\mathcal{A}))} \cong \lim_{n\to\infty} \frac{\operatorname{inv}(M_n(\mathcal{A}_q))}{\operatorname{inv}_0(M_n(\mathcal{A}_q))} = K_1(\mathcal{A}_q). \tag{3.1}$$

Further, for $\mathcal{B}$ to be $\mathcal{A}$ or $\mathcal{A}_q$, the suspension of $\mathcal{B}$ is

$$S\mathcal{B} = \{f \in C([0,1], \mathcal{B}) : f(0) = f(1) = 0\} \cong C_0(\mathbb{R}, \mathcal{B}).$$

We use the Bott periodicity theorem $K_0(\mathcal{B}) \cong K_1(S\mathcal{B})$ to show that $K_0(\mathcal{A}) \cong K_0(\mathcal{A}_q)$. We have $\operatorname{rad}(SA) \cong \operatorname{rad} C_0(\mathbb{R}, \mathcal{A}) \cong C_0(\mathbb{R}, \operatorname{rad}\mathcal{A})$, hence

$$\begin{aligned} S\mathcal{A}_q &\cong C_0(\mathbb{R}, \mathcal{A}_q) = C_0(\mathbb{R}, \mathcal{A}/\operatorname{rad}\mathcal{A}) \\ &\cong C_0(\mathbb{R}, \mathcal{A})/C_0(\mathbb{R}, \operatorname{rad}\mathcal{A}) \cong S\mathcal{A}/\operatorname{rad} S\mathcal{A}. \end{aligned}$$

Hence by applying (3.1) to $S\mathcal{A}$,

$$\begin{aligned} K_0(\mathcal{A}_q) &\cong K_1(S\mathcal{A}_q) \cong K_1(S\mathcal{A}/\operatorname{rad} S\mathcal{A}) \\ &\cong K_1(S\mathcal{A}) \cong K_0(\mathcal{A}). \end{aligned}$$

Therefore we have $K_*(\mathcal{A}) = K_*(\mathcal{A}_q)$. This completes the proof of Theorem 3.1. ∎

In [9], Blackadar and Cuntz have developed an abstract theory of differential structure in a C^*-algebra based on the notion of differential seminorm. Next we investigate the properties of C^*-spectrality and spectral invariance of the Fréchet algebra defined by a differential seminorm as a typical application of Theorem 2.11 and Theorem 3.1.

Let $\mathfrak{A}$ be a $*$-algebra and $\|\cdot\|$ a C^*-seminorm on $\mathfrak{A}$. Let $\mathcal{A} = (\mathfrak{A}, \|\cdot\|)^{\sim}$ be the Hausdorff completion of $\mathfrak{A}$. Following [9], a map $T : \mathfrak{A} \to l^1(\mathbb{N})$ is said to be a *differential seminorm* on $\mathfrak{A}$ if $T(x) = (T_k(x))_0^\infty \in l^1(\mathbb{N})$ satisfies the following (i)–(iv):

(i) $T(x) \geqslant 0$, i.e. $T_k(x) \geqslant 0$ for $\forall x$, $\forall k$.

(ii) $T(x+y) \leqslant T(x) + T(y)$ for $\forall x, y \in \mathfrak{A}$; $T(\lambda x) = |\lambda| T(x)$ for $\forall \lambda \in \mathbb{C}$, $\forall x \in \mathfrak{A}$.

(iii) $T(xy) \leqslant T(x)T(y)$ (convolution) for $\forall\, x, y \in \mathfrak{A}$, i.e. for $\forall\, k \in \mathbb{N}$ we have $T_k(xy) \leqslant \sum\limits_{i+j=k} T_i(x)T_j(y)$.

(iv) There exists some constant $c > 0$ such that $T_0(x) \leqslant c\|x\|$ for $\forall x \in \mathfrak{A}$.

By (ii) each T_k is a seminorm. We say that T is a *differential $*$-seminorm* if further

(v) $T_k(x^*) = T_k(x)$ for $\forall x \in \mathfrak{A}$, $\forall k \in \mathbb{N}$.

T is said to be a *differential norm* if $T(x) = 0$ implies $x = 0$, i.e. $(T_k)_0^\infty$ is a separating family of seminorms. Following [9], the total seminorm of T is $T_{\text{tot}}(x) = \sum\limits_{k=0}^{\infty} T_k(x)$, $x \in \mathfrak{A}$. Throughout this section we assume that T is a differential $*$-norm. Then T_{tot} is a $*$-norm. Let $\mathfrak{A}_T = (\mathfrak{A}, T_{\text{tot}})^{\sim}$ be the completion of $\mathfrak{A}$ with respect to T_{tot}. $\mathfrak{A}_T$ is a Banach $*$-algebra. We construct a Fréchet $*$-algebra as follows. For each $k \in \mathbb{N}$, we put $p_k(x) = \sum\limits_{i=0}^{k} T_i(x)$, $x \in \mathfrak{A}$. Then each p_k is a submultiplicative $*$-seminorm. On $\mathfrak{A}$, we have

$$p_0 \leqslant p_1 \leqslant p_2 \leqslant \cdots \leqslant p_k \leqslant p_{k+1} \leqslant \cdots,$$

and $(p_k)_0^\infty$ is a separating family of submultiplicative $*$-seminorms on $\mathfrak{A}$. Let τ be a locally convex $*$-algebra topology defined on $\mathfrak{A}$ by $(p_k)_0^\infty$. We denote $\mathfrak{A}_\tau = (\mathfrak{A}, \tau)^{\sim}$ the completion of $\mathfrak{A}$ with respect to τ and $\mathfrak{A}_{(k)} = (\mathfrak{A}, p_k)^{\sim}$ the completion od $\mathfrak{A}$ with respect to p_k. $\mathfrak{A}_\tau$ is a Fréchet $*$-algebra and $\mathfrak{A}_{(k)}$ is a Banach $*$-algebra. Then we have

$$\mathfrak{A}_T = b(\mathfrak{A}_\tau) : \text{the bounded part of } \mathfrak{A}_\tau = \{x \in \mathfrak{A}_\tau : T_{\text{tot}}(x) = \sup_n p_n(x) < \infty\}.$$

By the difinitions, there exist continuous surjective $*$-homomorphisms $\varphi_k : \mathfrak{A}_{(k)} \to \mathcal{A}$, $\varphi : \mathfrak{A}_\tau \to \mathcal{A}$. Notice that even if each p_k is a norm on $\mathfrak{A}$, p_k need not be a norm on $\mathfrak{A}_\tau$. Also, the identity map $\mathfrak{A} \to \mathfrak{A}$ extends uniquely as continuous surjective $*$-homomorphism $\varphi_k : \mathfrak{A}_{(k+1)} \to \mathfrak{A}_{(k)}$ such that

$$\mathfrak{A}_{(0)} \xleftarrow{\varphi_0} \mathfrak{A}_{(1)} \xleftarrow{\varphi_1} \mathfrak{A}_{(2)} \xleftarrow{\varphi_2} \mathfrak{A}_{(3)} \longleftarrow \cdots$$

is a dense inverse limit sequence of Banach $*$-algebras. Hence by the abstract Mittag-Laffer theorem (e.g. [18]), the projections $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)} \to \mathfrak{A}_{(k)}$ have dense ranges. We summarize above discussion in the following.

Proposition 3.2. *Let T be a differential $*$-norm on a C^*-normed algebra $(\mathfrak{A}, \|\cdot\|)$. The following hold:*

(i) *$\mathfrak{A}_\tau$ is a Fréchet $*$-algebra and $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)}$;*

(ii) *the projections $\mathfrak{A}_\tau \to \mathfrak{A}_{(k)}$ have dense ranges;*

(iii) *if $T_0(x) = c\|x\|$ for $\forall x \in \mathfrak{A}$, then $\mathfrak{A}_T = b(\mathfrak{A}_\tau)$ is dense in $\mathfrak{A}_\tau$.*

Proof. (i) and (ii) were shown in the above. We prove (iii) only. Since each p_k is a norm, $\mathfrak{A}$ is dense in $\mathfrak{A}_{(k)}$ for $\forall k$. Hence if $\pi_k : \mathfrak{A}_\tau \to \mathfrak{A}_{(k)}$ be the projections, then $\overline{\pi_k(\mathfrak{A}_T)} = \mathfrak{A}_{(k)}$ by above. Hence $\mathfrak{A}_T$ is dense in $\bigcap_k \pi_k^{-1}(\overline{\pi_k(\mathfrak{A}_T)}) = \bigcap_k \pi_k^{-1}(\mathfrak{A}_{(k)}) = \mathfrak{A}_\tau$. Therefore $\mathfrak{A}_T$ is dense in $\mathfrak{A}_\tau$. This completes the proof. ∎

We define

$$\mathcal{I}_k = \{x \in \mathfrak{A}_{(k)} : \varphi_k(x) = 0\},$$
$$\mathcal{I} = \{x \in \mathfrak{A}_\tau : \varphi(x) = 0\},$$
$$\mathcal{I}_{\mathrm{tot}} = \{x \in \mathfrak{A}_T : \varphi_T(x) = 0\},$$

where $\varphi_T = \varphi\lceil_{\mathfrak{A}_T}$.

Theorem 3.3. *Let $(\mathfrak{A}, \|\cdot\|)$ be a C^*-normed algebra and $\mathcal{A}$ the completion of $(\mathfrak{A}, \|\cdot\|)$. Let $\mathcal{B}$ denote $\mathfrak{A}_{(k)}$ or $\mathfrak{A}_\tau$ with respective topologies. The following hold:*

(i) *$\mathcal{B}$ is a hermitian Q-algebra;*

(ii) *$E(\mathcal{B}) = \mathcal{A}$;*

(iii) *$\mathcal{B}$ is C^*-spectral and spectrally invariant;*

(iv) *$K_*(\mathcal{B}) = K_*(\mathcal{A}) = K_*(\mathfrak{A}_{(k)})$ for all k.*

Proof. We have the following diagram:

$$\begin{array}{ccc}
\mathfrak{A}_{(k)} & \xrightarrow{j} & E(\mathfrak{A}_{(k)}) \\
\uparrow & \searrow^{\varphi_k} & \downarrow \\
\mathfrak{A} & \hookrightarrow & \mathcal{A} \\
\downarrow & \nearrow^{\varphi} & \uparrow \\
\mathfrak{A}_\tau & \longrightarrow & E(\mathfrak{A}_\tau) \\
{\scriptstyle \mathrm{id}}\uparrow & & \uparrow \\
\mathfrak{A}_T & \longrightarrow & E(\mathfrak{A}_T).
\end{array}$$

Case 1. Assume that T is of finite order, say n, so that $T_i(x) = 0$ for $\forall x \in \mathfrak{A}$, $\forall i > n$. Then $T_{\mathrm{tot}} = p_n$. Hence $\mathfrak{A}_\tau = \mathfrak{A}_{(n)}$ is a Banach $*$-algebra denoted by $\mathfrak{A}_T$. Since $\mathcal{I} = \ker \varphi = \{z \in \mathfrak{A}_T : T_0(z) = 0\}$, $\mathcal{I}^n = \{0\}$ by [9]. Then $\mathcal{I}$ is a nilideal, hence $\mathcal{I} \subset \operatorname{rad} \mathfrak{A}_T$. Thus $\psi : \mathfrak{A}_T/\mathcal{I} \to \mathfrak{A}_T/\operatorname{rad} \mathfrak{A}_T$, $\psi(x + \mathcal{I}) = x + \operatorname{rad} \mathfrak{A}_T$, is a well defined $*$-homomorphism. By standard Banach algebra arguments, for any $z \in \mathfrak{A}_T$,

$$\mathrm{Sp}_{\mathfrak{A}_T}(z) = \mathrm{Sp}_{\mathfrak{A}_T/\operatorname{rad}\mathfrak{A}_T}(z + \operatorname{rad}\mathfrak{A}_T) = \mathrm{Sp}_{\mathfrak{A}_T/\mathcal{I}}(z + \mathcal{I}) = \mathrm{Sp}_{\mathrm{Image}\,\varphi}(\varphi(z)).$$

Now let $\mathcal{K} = \mathfrak{A}_T/\mathcal{I}$, and let $\widetilde{\varphi} : \mathcal{K} \to \mathcal{A}$ be the surjective $*$-homomorphism induced by $\varphi : \mathfrak{A}_T \to \mathcal{A}$. Since T is of finite order, T_{tot} is analytic (p. 264, [9]), hence the quotient norm α on $\mathcal{K}$ is also analytic (p. 264, [9]). Further $(\mathcal{K}, \alpha)$ is a Banach algebra and is dense in $\mathcal{A}$ via $\widetilde{\varphi}$. Since $(\mathfrak{A}, \|\cdot\|)$ is assumed to be a C^*-normed algebra, Proposition 3.12 in [9] applies showing that $\mathcal{K}$ is spectrally invariant in $\mathcal{A}$ via $\widetilde{\varphi}$. This with above equalities implies that $\mathrm{Sp}_{\mathfrak{A}_T}(z) = \mathrm{Sp}_{\mathcal{A}}(\varphi(z))$ for $\forall z \in \mathfrak{A}_T$.

Now let $|z| = \|\varphi(z)\|$ with $z \in \mathfrak{A}_T$ be the C^*-seminorm induced by φ on $\mathfrak{A}_T$. Then $|\cdot|$ is continuous in T_{tot}. Further, let q be any C^*-seminorm on $\mathfrak{A}_T$, and let $\pi_q : \mathfrak{A}_T \to \mathcal{B}(\mathcal{H})$ be the $*$-representation defined by q identifying $(\mathfrak{A}_T/\ker q, \|\cdot\|_q)^\sim$ with an operator algebra on an appropriate Hilbert space $\mathcal{H}$. Then for $\forall z \in \mathfrak{A}_T$,

$$\begin{aligned} q(z)^2 = q(z^*z) = \|\pi_q(z)^*\pi_q(z)\| &= \gamma_{\mathcal{B}(\mathcal{H})}(\pi_q(z)^*\pi_q(z)) \\ &\leqslant \gamma_{\mathrm{Image}\,\varphi}(\pi_q(z^*z)) \leqslant \gamma_{\mathfrak{A}_T}(z^*z) = \gamma_{\mathcal{A}}(\varphi(z)^*\varphi(z)) \\ &= \|\varphi(z)\|^2 = |z|^2. \end{aligned}$$

It follows that $|\cdot|$ is the greatest C^*-seminorm on $\mathfrak{A}_T$ (the Gelfand-Naimark pseudo-norm) and $\mathrm{srad}\,\mathfrak{A}_T = \ker\varphi$. Hence $E(\mathfrak{A}_T) = \mathcal{A}$.

Case 2. Let $T = (T_i)_0^\infty$ be not necessarily of finite order. For each $k \in \mathbb{N}$, let ${}^{(k)}T' = (T_0, T_1, T_2, \ldots, T_k, 0, 0, \ldots)$ which is a differential $*$-seminorm of order k, for which ${}^{(k)}T'_{\mathrm{tot}} = p_k$, hence $\mathfrak{A}_{{}^{(k)}T'_{\mathrm{tot}}} = (\mathfrak{A}, p_k)^\sim = \mathfrak{A}_{(k)}$. By the Case 1, $\mathfrak{A}_{(k)}$ is spectrally invariant in $\mathcal{A}$ via φ_k and $E(\mathfrak{A}_{(k)}) = \mathcal{A}$. Thus $\mathfrak{A}_{(k)}$ is a hermitian Banach $*$-algebra and $E(\mathfrak{A}_{(k)}) = C^*(\mathfrak{A}_{(k)}) = \mathcal{A}$. Therefore the C^*-seminorm induced on $\mathfrak{A}_{(k)}$ by φ_k is a spectral C^*-seminorm. Now $\mathfrak{A}_\tau = \varprojlim \mathfrak{A}_{(k)}$. Hence $\mathrm{Sp}_{\mathfrak{A}_\tau}(z) = \bigcup_k \mathrm{Sp}_{\mathfrak{A}_{(k)}}(\varphi(z))$ for $\forall z \in \mathfrak{A}_\tau$, and $E(\mathfrak{A}_\tau) = \varprojlim E(\mathfrak{A}_{(k)}) = \mathcal{A}$. Thus $\mathfrak{A}_\tau$ is spectrally invariant in $\mathcal{A}$ via φ, $\mathrm{srad}\,\mathfrak{A}_\tau = \ker\varphi$ and $\mathfrak{A}_\tau$ is a hermitian Q-algebra. That $\mathfrak{A}_\tau$ is a Q-algebra follows from the fact that on $\mathfrak{A}_\tau$, the C^*-seminorm $|\cdot|$ induced by the complete C^*-norm $\|\cdot\|$ on $\mathcal{A}$ is spectral, because for any $z \in \mathfrak{A}_\tau$,

$$\gamma_{\mathfrak{A}_\tau}(z) = \gamma_{\mathcal{A}}(\varphi(z)) \leqslant \|\varphi(z)\| = |z|,$$

and also from the fact that $|\cdot| \leqslant$ the Fréchet topology on $\mathfrak{A}_\tau$. We also have, for $\forall z \in \mathfrak{A}_\tau$,

$$\mathrm{Sp}_{\mathfrak{A}_\tau}(z) = \mathrm{Sp}_{\mathfrak{A}_{(k)}}(\pi_k(z)) = \mathrm{Sp}_{\mathcal{A}}(\varphi(z)) = \mathrm{Sp}_{\mathcal{A}}(\varphi_k(z)).$$

This prove (i) and (ii). Then Theorem 2.11 and Theorem 3.1 imply (iii) and (iv). This completes the proof. ∎

It ought to be true, in the notations of Theorem 3.3, that $\mathcal{B}$ is closed under the C^∞-functional calculus of $\mathcal{A}$ in the sense that given $h = h^*$ in $\mathcal{B}$ and a C^∞-function f on $\mathrm{Sp}_{\mathcal{A}}(\varphi(h))$, there exists $y \in \mathcal{B}$ such that $f(\varphi(h)) = \varphi(y)$. However, we leave it open.

As shown in the proof of Theorem 3.3, we have the following

COROLLARY 3.4. *The following equalities hold:*
(i) $\mathrm{rad}\,\mathfrak{A}_{(k)} = \mathrm{srad}\,\mathfrak{A}_{(k)}$*;*
(ii) $\mathrm{rad}\,\mathfrak{A}_\tau = \mathrm{srad}\,\mathfrak{A}_\tau$*.*

Following [9], a seminorm α on $\mathfrak{A}$ is *closable* if for any sequence (x_k) in $\mathfrak{A}$ such that $\|x_k\| \to 0$ as $k \to \infty$, $\alpha(x_k) \to 0$ as $k \to \infty$.

Corollary 3.5. *Suppose $T_0(x) = c\|x\|$ for $\forall x \in \mathfrak{A}$. The following hold:*
(i) *The following statements are equivalent:*
(1) *T is closable (in the sense that each T_k is closable);*
(2) *$\mathcal{I} = \{0\}$;*
(3) *$\mathfrak{A}_\tau \subset \mathcal{A}$;*
(4) *$\mathfrak{A}_\tau$ is semisimple.*
In this case, $\{\mathfrak{A}_{(k)}\}$ is an increasing sequence of Banach $*$-algebras and $\mathfrak{A}_\tau = \bigcap\limits_{k=0}^{\infty} \mathfrak{A}_{(k)}$.
(ii) *If T is closable, then T_{tot} is closable.*
(iii) *$\mathcal{I} \subset \operatorname{rad} \mathfrak{A}_\tau$.*

Proof. (i) It is clear that p_k is closable if and only if $\mathcal{I}_k = \{0\}$ if and only if $\mathfrak{A}_{(k)} \subset \mathcal{A}$ for $\forall k$. From this it follows that T is closable if and only if $\mathcal{I} = \{0\}$ if and only if $\mathfrak{A}_\tau \subset \mathcal{A}$. That $\mathfrak{A}_\tau$ is semisimple if and only if $\mathfrak{A}_\tau \subset \mathcal{A}$ we shall prove using (iii).

(iii) Let $x \in \mathcal{I}$. Then $x \in \mathfrak{A}_\tau$ and $\varphi(x) = 0$. Hence there exists a sequence $(x_n) \subset \mathfrak{A}$ such that $x_n \to x$ in τ. Therefore for all $k \in \mathbb{N}$, $p_k(x_n - x) \to 0$. Hence $\pi_k(x_n) \to \pi_k(x)$ in $\mathfrak{A}_{(k)}$. Then

$$\widetilde{T_0}(\pi_k(x)) = \lim_n T_0(\pi_k(x_n)) = \lim_n T_0(x_n) = c \lim_n \|x_n\| = c\|\varphi(x)\| = 0,$$

where $\widetilde{T_0}$ is the extension of T_0 to $\mathfrak{A}_\tau$. Hence for $\forall k$, $\pi_k(x) \in \mathcal{I}_k \subset \operatorname{rad} \mathfrak{A}_{(k)}$. Therefore $x \in \bigcap\limits_k \pi_k^{-1}(\operatorname{rad} \mathfrak{A}_{(k)}) \subset \operatorname{rad} \mathfrak{A}_\tau$. Thus we have $\mathcal{I} \subset \operatorname{rad} \mathfrak{A}_\tau$.

(i) We prove $\mathfrak{A}_\tau$ is semisimple if and only if $\mathfrak{A}_\tau \subset \mathcal{A}$. Let $\mathfrak{A}_\tau$ be semisimple. Then $\operatorname{rad} \mathfrak{A}_\tau = \{0\}$. Hence $\mathcal{I} = \{0\}$ and so $\mathfrak{A}_\tau \subset \mathcal{A}$. Conversely let $\mathfrak{A}_\tau \subset \mathcal{A}$. Then $(\mathfrak{A}_\tau)^- = \mathcal{A}$. Now $\mathfrak{A}_\tau$ is a hermitian Fréchet Q-algebra having $E(\mathfrak{A}_\tau) = \mathcal{A}$. Hence $\operatorname{Rep} \mathfrak{A}_\tau = \operatorname{Rep} \mathcal{A}$ for $*$-representations. Therefore $\operatorname{srad} \mathfrak{A}_\tau = \mathfrak{A}_\tau \cap \operatorname{srad} \mathcal{A} = \{0\}$ as $\mathcal{A}$ is a C^*-algebra. Hence $\operatorname{rad} \mathfrak{A}_\tau = \{0\}$ as $\operatorname{rad} \mathfrak{A}_\tau \subset \operatorname{srad} \mathfrak{A}_\tau$. Thus $\mathfrak{A}_\tau$ is semisimple.

(ii) follows from the definition. This completes the proof. ∎

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S.J. BHATT
Department of Mathematics
Sardar Patel University
Vallabh Vidyanagar 388120
INDIA

E-mail: sjb@spu.ernet.in

CURRENT ADDRESS
Department of Mathematics and Statistics
Case Western Reserve University
Cleveland, Ohio 43403
USA

A. INOUE
Department of Applied Mathematics
Fukuoka University
Nanakuma, Jonan-ku Fukuoka 814-0180
JAPAN

E-mail: a-inoue@fukuoka-u.ac.jp

H. OGI
Department of Functional Materials
Engineering Institute of Technology
Wazirohigash, Higashi-ku Fukuoka 811-0295
JAPAN

E-mail: ogi@fit.ac.jp

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