THE IDEAL-THEORY OF THE PARTIALLY ORDERED SET

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M. H. STONE has developed systematically the ideal-theory of the distributive lattice. Macneille in his fundamental work on partially ordered sets has casually suggested a definition of an 'ideal' in a partially ordered set. This paper has for its object the formulation of three concepts, namely *semi-complete ideal* (which corresponds to Macneille's suggestion), *complete ideal*, and *comprincipal ideal,* necessary for the ideal-theory of an ordered set. It is the last concept of *comprincipal ideal* which leads to the full clarification of the idea of 'cuts' which is used in Macneille's paper without an explicit recognition of its bearings.

I. An abstract set P of any sort of elements is a 'partially ordered set', if it is the field of a binary, reflexive, transitive relation '$<$', with the property, $a < b$ and $b < a$ imply $a = b$. An element a of P such that $a < x$ for every x in P, is called 0 (zero) if it exists. Similarly an element b such that $x < b$ for every x in P is called 1 (one) if it exists. 0 and 1 are necessarily unique if they exist. The converse relation of '$<$' is written '$>$'.

The sum (or lattice-sum) s and the product (or lattice-product) p of any subset π of P are defined to be elements of P satisfying the following conditions:

(1) $x < s$ for every x in π, and for all z, $x < z$ for every x in π implies $s < z$.

(2) $p < x$ for every x in π, and for all z, $z < x$ for every x in π implies $z < p$.

The elements s, p may not exist, but if they exist they are necessarily unique. The sum $s = \sum\limits_{x \epsilon \pi} x$ and the product $p = \prod\limits_{x \epsilon \pi} x$ are said to be *distributive,* if for every element a of P, the equalities

$$a \cdot \sum_{x \epsilon \pi} x = \sum_{x \epsilon \pi} a x$$

$$a + \prod_{x \epsilon \pi} x = \prod_{x \epsilon \pi} (a + x)$$

hold whenever either side exists.

The ordered set P is said to be a *lattice* if all finite sums and products exist, and a *distributive lattice* if the finite sums and products are distributive. It is said to be a *complete lattice* if all sums and products exist, and a *completely distributive complete lattice*, if all the sums and products are distributive. A complete lattice necessarily contains the units 0, 1.

THEOREM I.—*If the ordered set P contains* 1 *and if every product exists in P, then every sum also exists and P is a complete lattice.*

For if π is any subset of P, then the set Π of elements $>$ every x in π is not null since it contains 1. The product of the elements of Π exists in virtue of our hypothesis, and is easily seen to be the sum, as defined, of the elements of π.

Cor. Dually if P contains 0, and every sum exists in P, then P is a complete lattice.

II. *C-ideals and S-ideals of the ordered set P.*

A subset L of P, which has the property:

$$a \text{ is in L and } x < a \text{ imply } x \text{ is in L,}$$

is called a *complete ideal* or *C-ideal* of P if it contains every existing sum of its elements, and a *semi-complete ideal* or S-ideal of P, if it contains every existing distributive sum of its elements.

It is clear from the definition that every C-ideal is an S-ideal. These definitions refer to the ordering relation $<$, and may be called definitions of *lower C-* and *S-ideals*, or in the terminology of M. H. Stone, of S_μ- and C_μ-ideals. Dual definitions may be given of *upper C-* and *S-ideals*, or of S_α- and C_α-ideals ; namely—

A subset L of P, with the property that 'a is in L and $x > a$' implies 'x is in L' is called an upper C-ideal or C_α-ideal if it contains every existing product of its elements, and an upper S-ideal or S_α-ideal if it contains every existing distributive product of its elements.

We shall adopt the convention that the terms C-ideal and S-ideal used without any qualification refer to *lower* or μ-ideals, and that where upper or α-ideals are meant there shall be a specific mention to that effect.

A special class of C-ideals are the *principal* ideals which are defined by any element a of the ordered set. The principal ideal P (a) is defined to be the set of elements $x < a$. This is evidently a lower C-ideal or C_μ-ideal, conformably with our convention. Similarly the upper principal ideal P_α (a) defined by a is the set of elements $x > a$. This is an upper C-ideal or C_α-ideal.

Remarks on the above definitions :

The conception of complete ideal appears to be new, and is precisely the conception necessary for the elucidation of Macneille's Theory of ' cuts ' in the next paragraph.

The conception of Semi-complete ideal is advanced in Macneille's paper where it is called simply, ' ideal '. I have avoided this name for the obvious reason that it will cause confusion when we come to the distributive lattice. It is of course possible to define ' ideal ' in an ordered set on the analogy of the distributive lattice, namely—

A *subset* L *of the* ordered set P is a (lower) *ideal* if, ' a is in L and $x < a$ ' implies 'x is in L' and if L contains every existing finite sum of its elements. But this concept is not very significant for the theory of the ordered set, and I shall avoid it, at any rate in this paper.

III. *Properties of C-ideals and S-ideals.*

It is clear that the S-ideals of the ordered set P form a partially ordered set S_μ under the ordering relation of set-inclusion. Since any C-ideal is an S-ideal, the C-ideals form a subset C_μ of the ordered set S_μ. Since the whole set P is an S-ideal (and C-ideal) which contains every other S-ideal, it follows that S_μ has a unit 1 which is also an element of C_μ. We shall assume that P has a zero ; then since 0 by itself constitutes a C-ideal, it follows that S_μ as well as C_μ has a zero—namely the C-ideal constituted by the element 0. We next shew that S_μ, C_μ are not only ordered sets, but *complete lattices*.

The intersection or set-product of any number of S-ideals is not null, since 0 is an element of every S-ideal, and is obviously an S-ideal. Thus the set-product of any family of S-ideals satisfies the definition of the lattice-product of the family. Thus S_μ is an ordered set with 0, 1, in which every product exists and is therefore by theorem I a complete lattice.

Similar reasoning shews that the intersection of any family of C-ideals is not null, and is a C-ideal. Therefore in C_μ all products exist; since C_μ contains 0, 1, it follows by theorem I that C_μ is also a complete lattice. Further since the lattice-product both in C_μ and in S_μ is identical with the set product, it follows that the lattice-product in C_μ is the same as that in S_μ ; in other words, *C_μ is a multiplicative sub-system of S_μ*. But the lattice-sums in C_μ and in S_μ are not the same. The lattice addition in S_μ and C_μ must be elucidated by the concept of the S-ideal or C-ideal *generated* by a set of elements of P.

If π is any subset of P, the S-ideal *generated* by π is defined as the set of all elements x such that $x < \sigma$ for some distributive sum σ of elements chosen from π. For, it is easy to see that this set is an S-ideal. Further it is clear that any S-ideal which contains π must contain this set. This justifies the definition. Similarly the C-ideal generated by π is the set of all elements x of P, such that $x < \sigma$ for some sum σ of elements chosen from π. It is clear that the C-ideal generated by π contains the S-ideal generated by the same set π.

We may now identify the lattice-sum in S_μ of any family (S_1, S_2,...) of S-ideals, as the S-ideal generated by the set-sum of (S_1, S_2,...); for, this S-ideal contains S_1, S_2,...and is contained in any S-ideal containing them. If (S_1, S_2,...) are all C-ideals, then their lattice-sum in the lattice C_μ is the C-ideal generated by the set-sum of (S_1, S_2,...). Since the C-ideal generated by a set π is in general different from the S-ideal generated by the same set, it follows that the lattice-sum in C_μ of a set of elements in C_μ is not the same as their lattice-sum in S_μ, while their lattice-product in C_μ is the same as in S_μ.

We now proceed to study the relations between principal ideals and complete or C-ideals. It is obvious in the first place that any principal ideal is complete; for the lattice-sum of any set of elements $< a$ must, if it exists, be $< a$. Further it has been shewn that the product of any family of complete ideals is complete; therefore the intersection of any family of principal ideals is complete. The converse, *viz.*, that any complete ideal is the intersection of some family of principal ideals, is not in general true. We have therefore to introduce the concept of *comprincipal ideal* or *$\overline{C}$-ideal* to denote the inter-section of an family of principal ideals. It is clear that the $\overline{C}$-ideals form a sub-set $\overline{C}_\mu$ of the complete lattice C_μ. Further it is a consequence of our definition that the intersection of any family of comprincipal ideals is comprincipal; hence $\overline{C}_\mu$ is closed for unrestricted multiplication. Further P (which has been assumed to possess the unit 1), as a principal ideal is comprincipal and therefore belongs to and is the unit of $\overline{C}_\mu$. Hence by Theorem I $\overline{C}_\mu$ is a complete lattice, though it is not a sublattice of C_μ (but only a multiplicative sub-system) since addition in $\overline{C}_\mu$ differs from that in C_μ. The fundamental property of the $\overline{C}$-ideal may be stated as :

THEOREM II.—*A complete or semi-complete ideal L is comprincipal if, and only if, L is the intersection of all principal ideals containing it.*

The first part follows from the definition. To prove the second part we have only to observe that if L is comprincipal, and therefore the intersection of a family (F) of principal ideals, and if L′ is the intersection of *all* principal ideals containing L, then $L' < L$, and $L' > L$.

III. Macneille's Theory of Cuts.

According to Macneille, a *cut* of an ordered set P with 0, 1, is a pair of subsets (A, B) of P, satisfying the following conditions :

(1) every element of A $<$ every element of B,

(2) $x <$ every element of B implies x is in A.

(3) $y >$ every element of A implies y is in B.

It follows from these conditions that A is the intersection of principal ideals determined by the elements of B, and therefore comprincipal. Similarly B is the intersection of the principal upper ideals determined by the elements of A, and therefore a comprincipal upper ideal. Thus the two elements of any cut are comprincipal ideals standing in a certain relation to each other which we shall call 'cut-complementarity'.

If π is any subset of P, we define the '*upper cut-complement*' of π to be the set of all elements $>$ every element of π. Similarly the lower cut-complement of π is the set of all elements $<$ every element of π. It follows that the lower and upper cut-complements of π are lower and upper comprincipal ideals, since they are *defined* as intersections of principal ideals. The key-theorem which is necessary to elucidate the theory of cuts, may be stated thus:

THEOREM III.—*Let π be any subset of P, π_1 the upper cut-complement of π, and π_2 the lower cut complement of π_1. Then $\pi = \pi_2$ if and only if π is a comprincipal ideal.*

For it is clear that π_2 is a comprincipal ideal containing π. Further, π_2 as the lower cut-complement of π_1, is the intersection of all principal ideals determined by elements of π_1. But these latter principal ideals are precisely the principal ideals containing π. Thus π_2 as the intersection of all principal ideals containing π is the *comprincipal envelope* of π. It is clear that a set π can be identical with its comprincipal envelope if and only if it is itself a comprincipal ideal.

It is obvious that the theorem dual to Theorem III holds, by interchanging 'upper' and 'lower' ideals.

THEOREM IV.—If $\{C_i\}$ is any family of comprincipal ideals, whose upper cut-complements are $\{C'_i\}$, then (1) the upper cut-complement of the product ΠC_i is the comprincipal envelope of $\{C'_i\}$, {in other words, the lattice-sum $\Sigma C'_i$ in the complete lattice C'_a} and (2) the upper cut-complement of the comprincipal envelope of $\{C_i\}$ (*i.e.*, the lattice-sum ΣC_i in the complete lattice $\overline{C}_\mu$) is the product $\Pi C'_i$.

For it follows from Theorem III that the upper cut-complement of any set K is the same as the upper cut-complement of the comprincipal envelope of K. Therefore, the upper cut-complement of the comprincipal envelope of $\{C_i\}$ is the same as the cut-complement of the set-sum of the C_i's which is clearly the product ΠC_i's. This proves the second part of the theorem. We shew in the same manner that the lower cut-complement of the comprincipal envelope of $\{C'_i\}$ is the product ΠC_i. The first part of the theorem follows from this, by using Theorem III.

The result of these two theorems is (1) the correspondence in structure between the cuts of the ordered set P, and the comprincipal (lower or upper) ideals which are elements of the cuts and (2) the *dual* isomorphism in respect of all sums and products effected by cut-complementation between the two complete lattices $\{\overline{C}_\mu, \overline{C}_\alpha\}$. Hence the *canonical extension of the partially ordered set P to a complete lattice which Macneille reaches by the theory of cuts is abstractly identical with the complete lattice* $\{\overline{C}_\mu\}$ *(or dually the same as* $\{\overline{C}_\alpha\}$.

The mutual relations between semicomplete, complete and comprincipal ideals in a distributive lattice is a problem which awaits research.

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