A FUNDAMENTAL PROPERTY OF HART SYSTEMS OF CIRCLES.

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Two sets of four circles in a plane such that each circle of either set is touched by all the four circles of the other set, are said to form two *complementary Hart Systems*, in the case in which neither set admits a common orthogonal circle. The configuration of two such Hart Systems is fairly complicated and has been extensively studied; an account of the theory may be found in Baker's *Principles of Geometry*, Vol. IV, and in Coolidge's *Treatise on the Circle and Sphere.*

The purpose of this communication is to draw attention to a very significant and hitherto unnoticed property of two complementary Hart Systems and to discuss some of its implications. The property in question is:

THEOREM I.—*The eight circles of two complementary Hart Systems belong to a one-parameter cubic family of circles.*

I. It is well known that the plane geometry of circles under the Inversion-group is equivalent to a projective Non-Euclidean geometry of three dimensions, with a hyperbolic metric determined by a real closed quadric Q. This is seen by establishing a correspondence between the points (or planes) of a three-space S_3 and the circles of the plane; the locus of points in S_3 which correspond to the point-circles of the plane is then a real closed quadric Q. The points in S_3 which are without Q correspond to real circles and the points within Q to imginary circles (meaning circles with a real centre, the square of whose radius is a negative number). The angle between two intersecting circles is the projective separation (with Q as absolute quadric) of the corresponding points in S_3. We shall here follow the usual practice and use the language of three-dimensional geometry, so that our symbols denote both points in S_3, and the corresponding circles of the plane.

Let A, B, C be three points in S_3, and O the pole of the plane ABC with respect to Q. Let q be the conic of intersection of Q with the plane ABC, and let A′B′C′ be the polar triangle of ABC with respect to q. The conics in the plane ABC which are outpolar to q and have A′B′C′ as self-polar triangle must pass through four fixed points $\alpha, \beta, \gamma, \delta$. It is well known that the four lines $O\alpha$, $O\beta$, $O\gamma$, $O\delta$ correspond to the four coaxial systems of circles which cut the given circles A,B,C at equal angles. The eight circles $c_1\ c_1'\ c_2\ c_2'\ c_3\ c_3'\ c_4\ c_4'$ which touch A, B, C, must accordingly be found in these four coaxial systems. It is well known that they are distributed in pairs c_1c_1', c_2c_2', c_3c_3', c_4c_4' lying respectively on the four lines $O\alpha$, $O\beta$, $O\gamma$, $O\delta$. It is also known that, to form a Hart System (*i.e.*, a set of four of these eight circles, which are touched by a fourth circle in addition to A, B, C), we must choose one circle from each of the four pairs. Since there are eight ways of choosing a Hart System from the 8 circles, it follows that three circles $c_1\, c_2\, c_3$ chosen in any manner one from each of three pairs, can be associated with one and only one circle c_4 of the fourth pair, so as to form a Hart System.

We now see that *our theorem (Theorem I) furnishes the selective principle which determines c_4 when $c_1\, c_2\, c_3$ are given.* For, in the language of three-dimensional geometry, Theorem I amounts to the statement that the seven points A, B, C, c_1, c_2, c_3, c_4 lie on a twisted cubic. Since a twisted cubic is determined by six points, we see that the twisted cubic which passes through A, B, C and through three points chosen in any manner, one from each of three pairs, necessarily passes (by Theorem I) through a definite point of the fourth pair, which is precisely the point which completes the Hart System.

II. We now proceed to prove (or rather verify) the truth of Theorem I.

Let A, B, C be three circles and O their common orthogonal circle. If in S_3 we choose the corresponding points O, A, B, C as the vertices of the tetrahedron of reference, the tangential equation of the Absolute Quadric is

$$aL^2 + bM^2 + cN^2 + 2fMN + 2gNL + 2hLM = T^2.$$

Four of the circles which touch A B C and form a Hart System are then given by:[1]

$$\begin{array}{llll} [a^{\frac{1}{2}} \cos (s-\lambda), & b^{\frac{1}{2}} \cos (s-\mu), & c^{\frac{1}{2}} \cos (s-\nu), & -1] \\ [a^{\frac{1}{2}} \cos (s-\nu), & b^{\frac{1}{2}} \cos s, & c^{\frac{1}{2}} \cos (s-\lambda), & -1] \\ [a^{\frac{1}{2}} \cos (s-\mu), & b^{\frac{1}{2}} \cos (s-\lambda), & c^{\frac{1}{2}} \cos s, & -1] \\ [a^{\frac{1}{2}} \cos s, & b^{\frac{1}{2}} \cos (s-\nu), & c^{\frac{1}{2}} \cos (s-\mu), & -1]; \end{array}$$

where $s = \frac{1}{2}(\lambda + \mu + \nu)$, $\cos\lambda = (bc)^{-\frac{1}{2}} f$; etc.

We have therefore to shew that the points A, B, C, whose co-ordinates are (1,0,0,0), (0,1,0,0), (0,0,1,0) and the four points whose co-ordinates are written above, lie on a twisted cubic. It is clear that for our purpose, we can omit the factors $a^{\frac{1}{2}}$, $b^{\frac{1}{2}}$, $c^{\frac{1}{2}}$ in the above co-ordinates. Further, if we prove that the projections from A on any plane, of the remaining six points lie on a conic, the same will follow by symmetry for the projections from B and C, and it will result that the seven points in question, and by symmetry the eighth point which with A B C completes the Hart System, lie on a twisted cubic.

[1] Baker, *Principles of Geometry*, Cambridge University Press, 1925, Vol. IV, p. 73, 74.

Now the projections from A on the plane OBC, of the remaining six points have the co-ordinates

$$\begin{array}{lll} 1, & 0, & 0 \\ 0, & 1, & 0 \\ \cos(s-\mu), & \cos(s-\nu), & -1 \\ \cos s, & \cos(s-\lambda), & -1 \\ \cos(s-\lambda), & \cos s, & -1 \\ \cos(s-\nu), & \cos(s-\mu), & -1. \end{array}$$

It is obvious that these six points lie on a conic. For, the cross ratios of the pencils subtended at the first and second points by the remaining four are

$$\{\cos(s-\nu),\ \cos(s-\lambda),\ \cos s,\ \cos(s-\mu)\} \text{ and}$$
$$\{\cos(s-\mu),\ \cos s,\ \cos(s-\lambda),\ \cos(s-\nu)\}.$$

These cross ratios are equal, since either of them is derived from the other by reversal of order. The theorem is thus proved.

III. The twisted cubic of Theorem I may be called a *Hart cubic* with respect to the given absolute quadric Q. To study the relation of a Hart cubic H to the absolute quadric Q, consider the correspondence Γ between pairs of points p, q on H such that pq touches Q (so that the *circles* corresponding to p, q touch each other). This correspondence is obviously a symmetric (4, 4) correspondence, which (remembering that the twisted cubic is a rational curve) is expressed by a relation of degree (4, 4) between the binary parameters of p, q. But from the definition of the Hart cubic, it contains two tetrads, such that any two points chosen one from each tetrad correspond in Γ. Hence Γ must be *sub-rational*, that is, it is determined by a relation of the form $f(x)\,f(y) + \phi(x)\,\phi(y) = 0$, where f, ϕ are quartics and x, y the parameters of corresponding points. It also follows that Γ must admit an infinity of pairs of tetrads of the same type. Calling the family of circles corresponding to a Hart cubic, *a Hart family*, we have therefore:

THEOREM II.—*A Hart family of circles (that is, a family of the type given by Theorem I) contains an infinity of pairs of complementary Hart tetrads of circles. All these Hart tetrads belong to a pencil, i.e., the linear family* $f(x) + \lambda\phi(x)$.

We next proceed to shew that the pencil $f(x)+\lambda\phi(x)$ associated with the sub-rational correspondence Γ on the Hart cubic is a *syzygetic pencil.* To see this consider the six intersections $a_1\ a_2\ a_3\ a_4\ a_5\ a_6$ of the Hart cubic H with the absolute quadric Q. Let the tangent plane at a_1 to Q cut H again in $p_1\ q_1$. It will follow then that the pencil $f(x)+\lambda\phi(x)$ contains *six* perfect squares of the type $(x-p_1)^2\ (x-q_1)^2$. This is impossible as the maximum number of perfect squares that can be contained in a pencil of quartics is three. It follows that more than one a must give rise to the same $p_1\ q_1$.

Since, however, not more than two tangent planes to Q can pass through a given line $p_1 q_1$, not more than two a's can give rise to the same $p_1 q_1$. It follows that the six intersections fall exactly into three pairs in such a way that the tangent planes at the points of each pair intersect in a chord $p_1 q_1$ of H. Thus the pencil $f(x)+\lambda\phi(x)$ contains three perfect squares and is therefore a syzygetic pencil. We have thus:

THEOREM III.—*The Hart tetrads of circles contained in a Hart family belong to a syzygetic pencil.*

An enumerative argument[2] shews that Theorems II and III are *complete*, in the sense that there are no further limitations on the projective relation of a Hart cubic to the Absolute Quadric. In other words, if H is any twisted cubic, and Δ_1, Δ_2, Δ_3, Δ_4 *any* four members of *any* pencil of syzygetic tetrads on H, then

(1) there are ∞^1 quadrics which touch the 16 joins of the vertices of Δ_1 with those of Δ_2;

(2) among these ∞^1 there is precisely one, say Q, which touches in addition the 16 joins of the vertices of Δ_3 with those of Δ_4;

(3) H is a Hart cubic in respect of Q.

We may verify directly one particular aspect of these statements. Among the Hart tetrads on a Hart cubic, there would be two which coalesce with their complementary Hart tetrads. A tetrad of this kind corresponds to four mutually tangent circles. *If Δ is a tetrahedron inscribed in a twisted cubic, we may shew that there are just ∞^1 quadrics which touch the six edges of Δ and the four tangents to the curve at the vertices of Δ.* For take Δ as the tetrahedron of reference. The equation of a quadric touching its six edges is of the form:

$$a_0^2x_0^2 + a_1^2x_1^2 + a_2^2x_2^2 + a_3^2x_3^2 - 2a_0x_0\Sigma a_1x_1 - 2\Sigma a_2a_3x_2x_3 = 0.$$

The tangent-cone to this from x_0 is seen to be:

$$\frac{1}{a_1x_1} + \frac{1}{a_2x_2} + \frac{1}{a_3x_3} = 0.$$

Now, if the tangents to the twisted cubic cut the opposite faces in $(0, \alpha_1, \alpha_2, \alpha_3)$, $(\beta_0, 0, \beta_2, \beta_3)$, $(\gamma_0, \gamma_1, 0, \gamma_3)$, $(\delta_0, \delta_1, \delta_2, 0)$, the rank of the matrix

[2] The number of cubic curves which are Hart cubics of a given quadric is $\infty^{9-1}=\infty^8$. There are ∞^9 quadrics and ∞^{12} cubic curves in S_3. If each cubic curve is a Hart cubic of ∞^r quadrics, it follows that $12+r=8+9$ or $r=5$. But the number of syzygetic pencils on the cubic curve is ∞^3, and the number of involutions between tetrads of a syzygetic pencil is therefore $\infty^3 \times \infty^2 = \infty^5$. Thus each quadric of which the curve is a Hart cubic is to be associated with an arbitrary involution between tetrads of a syzygetic pencil on the curve.

$$\begin{vmatrix} 0, & \frac{1}{a_1}, & \frac{1}{a_2}, & \frac{1}{a_3} \\ \frac{1}{\beta_0}, & 0, & \frac{1}{\beta_2}, & \frac{1}{\beta_3} \\ \frac{1}{\gamma_0}, & \frac{1}{\gamma_1}, & 0, & \frac{1}{\gamma_3} \\ \frac{1}{\delta_0}, & \frac{1}{\delta_1}, & \frac{1}{\delta_2}, & 0 \end{vmatrix}$$

is just 2. Hence out of the four conditions of the type

$$\frac{1}{a_1\alpha_1}+\frac{1}{a_2\alpha_2}+\frac{1}{a_3\alpha_3}=0,$$

which the quadric has to satisfy for touching the four tangents, only two are independent. Hence there are ∞^1 quadrics satisfying the given conditions. Since the twisted cubic is a Hart cubic of each of these ∞^1 quadrics, we infer that :

If $\Delta_1\,\Delta_2$ are two syzygetic tetrads on a twisted cubic H, *there is a unique quadric* Q *which touches the* 12 *edges of $\Delta_1\,\Delta_2$ and the eight tangents to the curve at their vertices.* H *is a Hart cubic of* Q.

Application to the theory of the Pedal and Contact circles of a triangle.

IV. The most familiar example of complementary Hart tetrads of circles is furnished by the sides and Feuerbach circle (N.P. circle) of a triangle ABC, and the tetrad of its in- and ex-circles. Accordingly, by Theorem I, there is a cubic family H of circles comprising as members, the sides, the Feuerbach circle and the in- and ex-circles. By Theorem II, H contains an infinity of pairs of complementary Hart tetrads of circles, and by Theorem III, all these Hart tetrads form a syzygetic family in H. These results have a close bearing on two theories of triangle-geometry, the theory of the pedal circle, and the theory of the contact circle.

If P P′ are a pair of isogonal conjugate points of ABC, the six feet of the perpendiculars from P, P′ on the sides of the triangle lie on a circle which is usually called the pedal circle of P (or P′). We shall, however, find it more appropriate to call it *the pedal circle of the line* PP′ ; this alteration in nomenclature will cause no inconvenience as there is a unique pair of isogonally conjugate points on every line other than the bisectors of the angles A, B, C.

It may be shewn that the totality of pedal circles is a two-dimensional quadrinodal cubic manifold in circle-space, the in- and ex-circles i_0, i_1, i_2, i_3 being in fact the nodal circles. Now a quadrinodal cubic surface contains (1) the six joins of the nodes, two and two, (2) three other straight lines, and (3) ∞^2 twisted cubics passing through the nodes. Thus the coaxial systems determined by any two circles i are composed entirely of pedal circles. Corresponding to (2) we observe that the three coaxial systems determined by the circles on BC, CA, AB as diameters, are composed entirely of pedal circles. It is easy to identify the cubic families of circles which correspond to (3) ; they are in fact the families of pedal circles of lines which pass through a fixed point. All these families contain the four circles i. *Our cubic family* H *is one of these families, and indeed, the only one among them, which is a Hart family.* For, if O′ is the reflection of the orthocentre O in the circumcentre S of ABC, it is easily seen that BC is the pedal circle of the perpendicular from O′ on BC, and that the Feuerbach circle is the pedal circle of O′ S. Hence :

THEOREM IV.—*The cubic family* H *is the family of pedal circles of lines through* O′.

Since H is a Hart family we can conclude that the pedal circle of any line O′L is touched by the pedal circles of four other lines $O'M_1$, $O'M_2$, $O'M_3$, $O'M_4$, and that the latter four circles are all touched also by the pedal circles of three other lines $O'L_2$, $O'L_3$, $O'L_4$. Here the two tetrads of lines ($O'L_1$, $O'L_2$, $O'L_3$, $O'L_4$), ($O'M_1$, $O'M_2$, $O'M_3$, $O'M_4$) are syzygetic tetrads of ($O'I_o$, $O'I_1$, $O'I_2$, $O'I_3$), where the I's are the in- and ex-centres. The tetrad (O′S, O′α, O′β, O′γ), where O′α, O′β, O′γ are perpendicular to BC, CA, AB also belongs to the same syzygetic family.[3]

If P, P′ are isogonally conjugate points, the six points of contact with the sides, of the two inscribed conics whose centres are P, P′ lie on a circle, which is *the contact circle* of the line PP′. The totality of contact-circles forms also a quadrinodal cubic manifold, with the circles i as nodal circles. The three coaxial systems of contact-circles are lines through the mid-points of BC, CA, AB. As in the case of the pedal circle, the cubic families of contact-circles containing the circles i, are the contact-circles of lines through a fixed point. *Among these families our Hart family* H *is included.* For, if G be the centroid of ABC, the contact-circle of GA is BC, and the contact-circle of GS is the Feuerbach circle. Hence :

THEOREM V.—*The family* H *is also the family of contact-circles of lines through* G.

As before we may express the implications of the fact that H is a Hart family.[3]

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[3] Mr. M. Bhimasena Rao has noticed the existence of an infinity of pairs of complementary Hart tetrads in the system of pedal circles of lines through O′. In the postscript to his paper 'An Extension of Feuerbach's Theorem' (*Jour. Ind. Math. Soc.*, Dec. 1919, page 219), he says 'Of the system of pedal-contact circles, each circle touches four circles, which latter are touched by three more circles of the system', and adds that his proof is incomplete. He does not appear to have published any proof subsequently.

The complete intersection of two quadrinodal cubic surfaces with the same nodes consists of the six joins of the nodes, and a twisted cubic through the nodes. Hence the pedal circles which are also contact-circles must either belong to a coaxial system determined by two of the circles i, or to the Hart family H.

From the two-fold character of the family H, it follows that the pedal circle of any line O′X is the contact-circle of a line GX. Clearly the rays GX, O′X are homographically related, and therefore the locus of their intersection X is a conic R. The pedal circle as well as the contact-circle of any line through an in-centre I is evidently the corresponding in-circle. Hence the conic R passes through the in- and ex-centres and through G and O′. Thus finally, the rectangular hyperbola through the in- and ex-centres and the centroid G passes through the point O′; and if P be any point on it, the contact-circle of GP is identical with the pedal circle of O′P.