ON THE ARITHMETICO-LOGICAL SYMMETRIC FUNCTIONS OF n ATTRIBUTES.

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IN THE statistical theory of attributes[1] a double order of ideas is used. In the first place the logical calculus is applied to the attributes, so that the symbol (AB) represents the logical product of the attributes A and B; that is, (AB) means, by definition, the attribute possessed by, and only by, those elements of the universe of discourse, that possess simultaneously the attributes A and B. In a similar manner, $A \oplus B$ representing the logical sum of A and B, is the attribute possessed by, and only by, those elements of the universe which possess at least one of the attributes A, B. These two fundamental operations of the logical calculus are associative and commutative[2]; further each of them is distributed by the other, so that, contrary to what happens in the case of the arithmetical sum and the arithmetical product, there is a dual symmetry between the properties of these two operations. For example, the fundamental law of absorption $a \oplus a\, b = a$, in the logical calculus is paralleled by $a\,(a \oplus b) = a$, which is obtained by interchanging the two operations.

In the second place, in the statistical theory each attribute-symbol A is made to represent also the number (supposed to be finite) of elements of the universe of discourse, which possess the attribute A. This permits attribute-symbols to be subjected to the arithmetical operations of addition and subtraction, in addition to the logical operations of addition and multiplication. By this double-edged use of the symbolism, the formulæ of the theory of attributes are able to represent definite numerical relations, instead of being merely qualitative, as those of the logical calculus.

In this paper this double use of the symbolism will be adopted, and the methods of the logical calculus will be utilised to study the properties of what I call the *logico-symmetric functions*, and the *arithmetico-logical symmetric functions* of n attributes $A_1, A_2, \cdots, A_n$. Particular cases of these properties are known, but the general results and their applications I

[1] See Yule's *Introduction to the Theory of Statistics*, Part I.

[2] See Whitehead's *Universal Algebra*, or any book on Symbolic Logic.

believe to be new. We shall begin by a compact formulation and proof of two known results.

§1. The logical sum of n attributes $A_1, A_2, \cdots, A_n$ can be expressed in terms of their logical products, by means of arithmetical operations. The precise expression is:

THEOREM I. $A_1 \oplus A_2 \oplus \cdots \oplus A_n = \Sigma A_1 - \Sigma (A_1A_2) + \Sigma (A_1A_2A_3) - \cdots,$

where Σ's represent always arithmetical sums.

Proof. An element of the universe which possesses exactly t of the n attributes is enumerated $\binom{t}{1}$ times in the first term of the right-hand side, $\binom{t}{2}$ times in the second term, $\binom{t}{3}$ times in the third and so on. Hence the number of times it is enumerated on the right-hand side is:

$$\binom{t}{1} - \binom{t}{2} + \binom{t}{3} - \cdots = 1 - (1-1)^t = \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } t > 0. \end{cases}$$

Thus any element which actually possesses one or more of the attributes A is enumerated exactly once on the right, and also exactly once in the logical sum of the n attributes. Hence the result.

From the symmetric relation between logical addition and logical multiplication, we may expect an analogous expression for the logical product of the n attributes in terms of their logical sums. We have in fact:

THEOREM II. $(A_1A_2 \cdots A_n) = \Sigma A_1 - \Sigma S_{12} + \Sigma S_{123} - \Sigma S_{1234} + \cdots,$

where $S_{pqr\ldots}$ denotes $A_p \oplus A_q \oplus A_r \oplus \cdots$

Proof. An element which possesses exactly t of the attributes is enumerated t times in the first term on the right, $\binom{n}{2} - \binom{n-t}{2}$ times in the second term, $\binom{n}{3} - \binom{n-t}{3}$ times in the third term, and so on. Hence the number of times it is enumerated on the right is:

$$\left[\binom{n}{1} - \binom{n}{2} + \cdots\right] - \left[\binom{n-t}{1} - \binom{n-t}{2} + \cdots\right]$$
$$= (1-1)^{n-t} - (1-1)^n$$
$$= \begin{cases} 1, & \text{if } t = n \\ 0, & \text{otherwise.} \end{cases}$$

Thus every element is enumerated the same number of times on the right as in the logical product on the left. Hence the result.

§2. *The elementary logico-symmetric functions of $A_1, A_2, \cdots, A_n$.*

By analogy with the algebraic theory of symmetric functions, we may define *the* rth *elementary logico-symmetric function* α_r of $A_1, A_2, \cdots, A_n$, to be the logical sum of their $\binom{n}{r}$ logical products r at a time. Thus the logical sum and the logical product of $A_1, A_2, \cdots, A_n$ are respectively their first and nth logico-symmetric functions. The following simple properties of the logico-symmetric functions α_r should be noted:

(*a*) *The* rth *symmetric function* α_r *of* $A_1, A_2, \cdots, A_n$ *is not only the logical sum of their logical products* r *at a time, but also the logical product of their logical sums* $n-r+1$ *at a time.*

For, the logical sum α_r of the logical products r at a time of $A_1, A_2, \cdots, A_n$, enumerates exactly once every element e of the universe which possesses at least r of the attributes $A_1, A_2, \cdots, A_n$. It is clear that these elements e are precisely those that possess one attribute at least out of every set of $n-r+1$ attributes chosen from $A_1, A_2, \cdots, A_n$; (for example, if an element e' possesses less than r of the attributes A, it is obvious that we can choose at least one set of $n-r+1$ attributes A, none of which is possessed by e'). Thus the elements e are precisely those that are enumerated without repetition and exception by the logical product of the logical sums $n-r+1$ at a time of $A_1, A_2, \cdots, A_n$, which proves our statement.

(*b*) There is a relation of implication between any two attributes represented by the symmetric functions α, so that the logical sum and product of any two α's may be immediately written down. We have in fact;

$$\left.\begin{array}{l} \alpha_p \oplus \alpha_q = \alpha_p \\ (\alpha_p\, \alpha_q) \;\; = \alpha_q \end{array}\right\} \text{ if } p \not> q.$$

These are obvious from the meaning of the logical operations and the definition of the α's. If we use the ordinary geometric interpretation adopted in the logical calculus, the attributes $\alpha_1, \alpha_2, \cdots, \alpha_n$ will correspond to n regions each of which contains the next.

It follows from these equations that the distinct values of the logical products r at a time of $\alpha_1, \alpha_2, \cdots, \alpha_n$ are $\alpha_r, \alpha_{r+1}, \cdots, \alpha_n$. Hence the logical sum of these logical products is α_r. In other words the attributes $\alpha_1, \alpha_2, \cdots, \alpha_n$ are in order their own successive symmetric functions. It follows that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are the only symmetric functions of $A_1, A_2, \cdots, A_n$ that can be derived purely by means of the two logical operations,

§3. *The Arithmetico-logical symmetric functions of $A_1, A_2, \cdots, A_n$.*

If however we are permitted to use arithmetical addition over and above the two logical operations, then every expression of the form:

$$a_1\, \alpha_1 + a_2\, \alpha_2 + \cdots + a_n\, \alpha_n,$$

where $a_1, \cdots, a_n$ are positive or negative integers, is a symmetric function of $A_1, A_2, \cdots, A_n$; strictly speaking, it does not represent an attribute but an enumerating process carried out by means of the attributes. Conversely, we may shew that any arithmetico-logical symmetric function S of the attributes $A_1, A_2, \cdots, A_n$ can be represented as a linear function with integer coefficients, of $\alpha_1, \alpha_2, \cdots, \alpha_n$. For since by hypothesis S is a symmetric function of $A_1, A_2, \cdots, A_n$, the number of times S (t) which it enumerates an element e possessing t of the attributes A, does not depend on which of the attributes A are possessed by e, but only on the number t. The symmetric function S is therefore completely characterised[3] by the n non-negative integers S(1), $\cdots$, S(n). Now, the number of times which an element possessing t of the attributes A is enumerated by $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$ is $a_1 + a_2 + \cdots + a_t$. Hence S must be identical with $S(1)\ \alpha_1 + \{S(2)-S(1)\}\ \alpha_2 + \cdots + \{S(n)-S(n-1)\}\ \alpha_n$. We have thus expressed an arbitrary arithmetico-logical symmetric function S as a linear form with integer-coefficients in $\alpha_1, \alpha_2, \cdots, \alpha_n$.

Among the arithmetico-logical symmetric functions of $A_1, A_2, \cdots, A_n$, the functions A_r^p are of fundamental importance. *We define A_r^p to be the arithmetic sum of the $\binom{n}{r}$ pth logico-symmetric functions of sets of r attributes chosen from $A_1, A_2, \cdots, A_n$.* This definition implies that $p \not> r$. It is clear that $A_n^p = \alpha_p$; apart from the functions α, there are $\binom{n}{2}$ functions A_r^p. Since A_r^p can be expressed as a linear form in the α's, we may write

$$A_r^p = \sum_s a_{rs}^p\ \alpha_s.$$

THEOREM III. $a_{rt}^p = \binom{t-1}{p-1} \cdot \binom{n-t}{r-p}$, so that $a_{rt}^p = 0$, if $t < p$ or $> n-r+p$.

Proof. The number of times which an element possessing exactly t of the attributes A, is enumerated by A_r^p is equal to the number of r-combinations of the n attributes which contain at least p of the t attributes. This number is evidently:

$$A_r^p\ (t) = \binom{t}{p}\binom{n-t}{r-p} + \binom{t}{p+1}\binom{n-t}{r-p-1} + \cdots$$

[3] It is assumed that the symmetric function does not enumerate the elements of the universe, which possess none of the attributes A.

Also:

$$a^p_{rt} = A^p_r(t) - A^p_r(t-1)$$
$$= \binom{t-1}{p-1}\binom{n-t}{r-p}.$$

It follows that A^p_r can be expressed in terms of $n-r+1$ of the logico-symmetric functions, namely, $a_p, a_{p+1}, \cdots, a_{n-r+p}$.

In particular,

$$A^r_r = a_r + \binom{r}{1} a_{r+1} + \binom{r+1}{2} a_{r+2} + \binom{r+2}{3} a_{r+3} + \cdots + \binom{n-1}{n-r} a_n.$$
$$A^1_r = \binom{n-1}{r-1} a_1 + \binom{n-2}{r-1} a_2 + \cdots + a_{n-r+1}.$$

It follows that the determinant of the $n-r+1$ functions $A^r_r, A^{r+1}_{r+1}, \cdots, A^n_n$, *qua* linear forms in $a_r, a_{r+1}, \cdots, a_n$ is unity. It follows that a_p can be expressed, for $p \geqslant r$ as an integral linear form in $A^r_r, A^{r+1}_{r+1}, \cdots, A^n_n$.

The actual expression is given by:

THEOREM IV. $a_r = A^r_r - \binom{r}{1} A^{r+1}_{r+1} + \binom{r+1}{2} A^{r+2}_{r+2} - \cdots$

For, the coefficient of a_{r+t} on the right is

$$\binom{r+t-1}{t} - \binom{r}{1}\binom{r+t-1}{t-1} + \binom{r+1}{2}\binom{r+t-1}{t-2} - \cdots$$
$$= \text{coefficient of } x^t \text{ in } (1+x)^{r+t-1}(1+x)^{-r} = (1+x)^{t-1}$$
$$= \begin{cases} 0, & \text{if } t > 0 \\ 1, & \text{if } t = 0, \end{cases} \text{ which proves the result.}$$

Similarly, $A^1_r, A^1_{r+1}, \cdots, A^1_n$ are linear forms in $a_1, a_2, \cdots, a_{n-r+1}$, with determinant ± 1. Hence a_{n-r+1} is an integral linear form in $A^1_r, \cdots, A^1_n$. We have:

THEOREM V. $a_{n-r+1} = A^1_r - \binom{r}{1} A^1_{r+1} + \binom{r+1}{2} A^1_{r+2} - \cdots$

For the coefficient of $a_{n-r+1-t}$ on the right is:

$$\binom{r+t-1}{r-1} - \binom{r}{1}\binom{r+t-1}{r} + \cdots$$
$$= \text{Coefficient of } x^t \text{ in } (1+x)^{r+t-1}(1+x)^{-r} = (1+x)^{t-1}$$
$$= \begin{cases} 0, & \text{if } t = 0 \\ 1, & \text{if } t > 0 \end{cases}, \text{ which proves the result.}$$

More generally the forms $A^p_r, A^{p+1}_{r+1}, \cdots A^{n-r+p}_n$ are $n-r+1$ linear forms with non-vanishing determinant in $a_p, a_{p+1}, \cdots, a_{n-r+p}$. Hence a_p can be expressed as a linear form in $A^p_r, A^{p+1}_{r+1}, \cdots, A^{n-r+p}_n$. We have in fact:

THEOREM VI. $\binom{n-p}{r-p} a_p = A_r^p - \binom{p}{1} A_{r+1}^{p+1} + \binom{p+1}{2} A_{r+2}^{p+2} - \cdots$

Proof. The coefficient of a_{p+t} on the right is:

$$\binom{n-p-t}{r-p} \cdot \left[\binom{p+t-1}{t} - \binom{p}{1}\binom{p+t-1}{t-1} + \binom{p+1}{2}\binom{p+t-1}{t-2} - \cdots\right]$$

As before, the part within square brackets is equal to 1 or 0, according as $t = 0$ or > 0. This proves the result.

In the same way, the forms $A_r^p, A_{r+1}^p, \cdots, A_n^p$ are $n-r+1$ linear forms with non-vanishing determinant in $a_p, a_{p+1}, \cdots, a_{n-r+p}$. Hence a_{n-r+p} can be expressed linearly in terms of them. We have in fact:

THEOREM VII. $\binom{n-r+p-1}{p-1} a_{n-r+p} = A_r^p - \binom{r-p+1}{1} A_{r+1}^p + \binom{r-p+2}{2} A_{r+2}^p - \cdots$

Proof. The coefficient of $a_{n-r+p-t}$ on the right is:

$$\binom{n-r+p-t-1}{p-1}\left[\binom{r-p+t}{r-p} - \binom{r-p+1}{1}\binom{r-p+t}{r-p+1} + \cdots\right]$$

The part within square brackets is, as before, equal to 1 or 0, according as $t = 0$ or $t > 0$. This proves the result.

Theorem VI is an extension of Theorem IV, which is itself an extension of Theorem I; similarly Theorem VII is an extension of Theorem V, which is itself an extension of Theorem II.

§4. *Application to* n *real numbers.*

Let $A_1, A_2, A_3, \cdots, A_n$ be n real numbers represented by points $A_1, A_2, \cdots, A_n$ on a straight line. Choose a point O on the line to the left of all the points A, and let $d_i =$ the distance OA_i. Let the universe of discourse be a set of points to the right of O, whose distances from O are integral multiples of an infinitesimal (ϵ). Let now A_i represent also the attribute of lying between O and A_i. It is then clear that the number A_i is proportional to the distance d_i. Also the numbers corresponding to (A_iA_j) and $A_i \oplus A_j$ are clearly proportional to the lesser and the greater of the two distances d_i, d_j. Hence the logical product and the logical sum of A_i, A_j correspond respectively to the less distant from O, and the more distant from O, of the two points A_i, A_j. If we denote by $R_{pqr\cdots}^{\lambda}$, that one among the points $A_p, A_q, A_r, \cdots$ which is the λth in order from the right, it is clear that the point $R_{pqr\cdots}^{\lambda}$ corresponds to the λth logico-symmetric function of $A_p, A_q, A_r, \cdots$. Similarly $I_{pqr\cdots}^{\lambda}$, which is the λth in ascending order of magnitude of the k real numbers $A_p, A_q, \cdots$, or the λth in order from the left of the k points $A_p, A_q, \cdots$, corresponds to $(k-\lambda+1)$ symmetric function of $A_p, A_q, \cdots$

We shall now state the forms which our previous theorem this concrete instance of the calculus:

THEOREM I. $R^1_{12\ldots n}$ = the greatest of the numbers $A_1, A_2, \cdots$

$$= \Sigma A_1 - \Sigma L^1_{12} + L^1_{123} - \Sigma L^1_{1234} + \cdots$$

THEOREM II. $L^1_{12\ldots n}$ = the least of the numbers $A_1, A_2, \cdots,$

$$= \Sigma A_1 - \Sigma R^1_{12} + \Sigma R^1_{123} - \Sigma R^1_{1234} + \cdots$$

THEOREM IV. $R^r_{12\ldots n}$ = the rth in descending order of magnit numbers $A_1, A_2, \cdots, A_n$,

$$= \Sigma L^1_{12\ldots r} - \binom{r}{1} \Sigma L^1_{12\ldots(r+1)} + \binom{r+1}{2} \Sigma$$

THEOREM V. $L^r_{12\ldots n}$ = the rth in ascending order of magni numbers $A_1, A_2, \cdots, A_n$

$$= \Sigma R^1_{12\ldots r} - \binom{r}{1} \Sigma R^1_{12\ldots(r+1)} + \binom{r+1}{2} \Sigma$$

THEOREM VI. $\binom{n-p}{r-p} R^p_{12\ldots n} = \binom{n-p}{r-p} \times$ the pth in descendir the numbers A,

$$= \Sigma L^{r-p+1}_{12\ldots r} - \binom{p}{1} \Sigma L^{r-p+1}_{12\ldots(r+1)} + \binom{p+1}{2}$$

$$\Sigma L^{r-p+}_{12\ldots}$$

THEOREM VII. $\binom{n-p}{r-p} L^p_{12\ldots n} = \binom{n-p}{r-p} \times$ the pth in ascen of the numbers,

$$= \Sigma R^{r-p+1}_{12\ldots r} - \binom{p}{1} \Sigma R^{r-p+1}_{12\ldots(r+1)} + \binom{p+1}{2}$$

$$\Sigma R^{r-p+}_{12\ldots}$$

We may make now a formal extension of these results. Let set of m real numbers $(a_1, a_2, \cdots, a_m)$ be called an m-dimensic with the components $a_1, a_2, \cdots$. Given n vectors $A_1, A_2, \cdots,$ components $A_i = (a_{i1}, a_{i2}, \cdots, a_{im})$, we can determine a vector B_r $\cdots, b_{rm})$ whose kth component b_{rk} is the rth in descendin magnitude of the numbers $a_{1k}, a_{2k}, \cdots, a_{nk}$ $(k = 1, 2, \cdots, m)$. term the vector B_r in a purely formal manner, *the* rth *descendin* of the n vectors $A_1, A_2, \cdots, A_n$. We similarly define *the* rth *ascending composite* of $A_1, A_2, \cdots, A_n$ as the vector whose kth component is the rth in ascending order of magnitude of the numbers $a_{1k}, a_{2k}, \cdots, a_{nk}$. We may denote the rth descending and ascending composites of $A_p, A_q, \cdots$ by $R^r_{pq\ldots}$, $L^r_{pq\ldots}$ Then, *the following vector equations hold,* which are of the same form as those established for numbers:

THEOREM VI $\binom{n-p}{r-p} R^p_{12\ldots n} = \Sigma\, L^{r-p+1}_{12\ldots r} - \binom{p}{1} \Sigma\, L^{r-p+1}_{12\ldots(r+1)} + \cdots$

THEOREM VII $\binom{n-p}{r-p} L^p_{12\ldots n} = \Sigma\, R^{r-p+1}_{12\ldots r} - \binom{p}{1} \Sigma\, R^{r-p+1}_{12\ldots(r+1)} + \cdots$

For, the corresponding equations have been established for any particular component of all the vectors involved.

These equations and the concept of the rth ascending and descending composites of n vectors will find application in the theory of the g.c.d. and l.c.m. of a set of integers.

§5. *The greatest common divisor and least common multiple.*

Every integer which does not involve prime factors other than a given finite set of primes $(p_1, p_2, \cdots, p_m)$, may be represented in the form $p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$, where a_1, a_2, $\cdots$, a_m are positive integers or zero. Thus each such integer may be represented by a vector $(a_1, a_2, \cdots, a_m)$ in such a way, that the product of two such integers is represented by the vector-sum of the corresponding vectors. Also it is clear that the g.c.d. and l.c.m. of a number of such integers $A_1, A_2, \cdots, A_n$ (say), is represented the first ascending composite, and the first descending composite of the corresponding vectors. Let g_1 be the g.c.d. of $A_1, A_2, \cdots, A_n$ and let g_r be the g.c.d. of the l.c.m.'s r at a time of $A_1, A_2, \cdots, A_n$. The series of numbers $g_1, g_2, \cdots, g_n$ may be called the first, second, $\cdots$ rth, $\cdots$ g.c.d.'s of $A_1, A_2, \cdots, A_n$. It is clear that g_n is the l.c.m. l_1 of $A_1, A_2, \cdots, A_n$. We may similarly define l_r to be the l.c.m. of the g.c.d.'s r at a time, and call l_r the rth l.c.m. of $A_1, A_2, \cdots, A_n$. It is clear that the vector corresponding to g_r is the rth ascending composite of the vectors corresponding to $A_1, A_2, \cdots, A_n$, and that the vector corresponding to l_r is the rth descending composite of the same vectors.

Since the rth ascending composite of n vectors is the $n-r+1$ descending composite of the same vectors, it follows that $g_r = l_{n-r+1}$.

Denote the kth g.c.d. and kth l.c.m. of A_p, A_q, $A_r \cdots$ by $g^k_{pqr\ldots}$ and $l^k_{pqr\ldots}$ respectively. Since the addition and subtraction of vectors correspond to the multiplication and division of the corresponding numbers, we see that the two vector relations of last para would lead to multiplicative expressions for l_p and g_p in terms of the $(r-p+1)$th g.c.d.'s and the $(r-p+1)$th l.c.m.'s respectively of subsets of the n numbers. The actual expressions are:

$$\text{THEOREM VI.}\quad (l_p)^{\binom{n-p}{r-p}} = \frac{\pi\, g^{r-p+1}_{12\ldots r} \cdot \left(\pi\, g^{r-p+1}_{12\ldots(r+2)}\right)^{\binom{p+1}{2}} \cdot \left(\pi\, g^{r-p+1}_{12\ldots r+4}\right)^{\binom{p+3}{4}} \cdots}{\left(\pi\, g^{r-p+1}_{12\ldots(r+1)}\right)^{p} \cdot \left(\pi\, g^{r-p+1}_{12\ldots(r+3)}\right)^{\binom{p+2}{3}} \cdots}$$

$$\text{THEOREM VII.}\quad (g_p)^{\binom{n-p}{r-p}} = \frac{\pi\, l^{r-p+1}_{12\ldots r} \cdot \left(\pi\, l^{r-p+1}_{12\ldots(r+2)}\right)^{\binom{p+1}{2}} \cdot \left(\pi\, l^{r-p+1}_{12\ldots(r+4)}\right)^{\binom{p+3}{4}} \cdots}{\left(\pi\, l^{r-p+1}_{12\ldots r+1}\right)^{p} \cdot \left(\pi\, l^{r-p+1}_{12\ldots r+3}\right)^{\binom{p+2}{3}} \cdots}$$

These relations are very general. Their particular cases are:

$$\text{THEOREM IV.}\quad l_p = \frac{\pi\, g^{1}_{12\ldots p} \cdot \left(\pi\, g^{1}_{12\ldots p+2}\right)^{\binom{p+1}{2}} \cdots}{\left(\pi\, g^{1}_{12\ldots(p+1)}\right)^{p} \cdot \left(\pi\, g^{1}_{12\ldots(p+3)}\right)^{\binom{p+2}{3}} \cdots}$$

$$\text{THEOREM I.}\quad l_1 = \frac{A_1 A_2 \cdots A_n \cdot \pi\, g^{1}_{123} \cdot \pi\, g^{1}_{12345} \cdots}{\pi\, g^{1}_{12} \cdot \pi\, g^{1}_{1234} \cdots}$$

and relations of the same form with l and g interchanged.