ON CONTINUOUS FUNCTIONS OF A REAL VARIABLE

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Received October 26, 1938

In a recent* paper " On the Zeroes of Weierstrass ' Non-differentiable Function ", Dr. C. N. Srinivasiengar proves that if a knot-point of Weierstrass' function be a zero of the function, it must necessarily be a limit point of zeroes. The principle involved in this result is elucidated in Theorems VI and VII of this paper, which obtain a statement of it in a more general form.

Dr. Srinivasiengar also raises certain questions regarding the set of zeroes of Weierstrass' function, and quotes Dr. A. N. Singh's statement that ' it is desirable to find some *special* characteristic of the set of zeroes of Weierstrass' function, as for instance, whether it is closed or open, enumerable or not '. The answer to one at least of these questions is easy and well known. Namely, the set of points at which a continuous function $f(x)$ takes an assigned value α—or briefly the set S (α) of the α-points of $f(x)$—is necessarily a closed set. I concern myself here with these questions, in so far as they relate to the *general* continuous function.

I. The Closed Sets Associated with a Continuous Function

Let $f(x)$ be a continuous function defined in a closed internal (0, 1) say. Let L and U be the lower and upper bounds of $f(x)$; and let α be any number such that $L \leqslant \alpha \leqslant U$. Then the set S (α) of solutions of the equation

$$f(x) = \alpha$$

is not empty, in virtue of the continuity of $f(x)$.

Theorem I. *S (α) is a closed set.*

This is an immediate consequence of the definition of continuity ; namely, $f(x)$ is continuous if $\lim_{y \to x} f(y)$ exists and is equal to $f(x)$.

We now make the assumption that the set of points of non-differentiability of $f(x)$ is everywhere dense in the interval (0, 1) ; this includes the case

* *Journal of the Indian Mathematical Society*, Sep. 1938.

in which $f(x)$ is non-differentiable throughout the interval. With this assumption, we have

THEOREM II. *$S(\alpha)$ is closed and non-dense.*

For, if $S(\alpha)$ contains any subinterval of the interval $(0, 1)$, then $f(x)$ would be equal to α throughout the subinterval, and hence be differentiable in it. This contradicts the hypothesis that the set of points of non-differentiability of $f(x)$ is everywhere dense. Hence $S(\alpha)$ cannot contain any sub-interval and is therefore non-dense.

By a classical theorem in the theory of sets of points, a closed non-dense set can be decomposed into an enumerable set, and a perfect (non-dense) set. Since a perfect set has the cardinal number of the continuum, it follows that $S(\alpha)$ will be enumerable if it has no perfect component, and will have the cardinal number of the continuum if it has. The following considerations shew that no better result than this can be stated for the general continuous function:

(1) Given any enumerable set E in $(0, 1)$, it is not difficult to shew that we can construct a continuous function, which vanishes (or takes an assigned value α) at all points of the set E and at no other points.

(2) Given any perfect (non-dense) set, it is shewn below under Theorem III how we can construct a continuous function which vanishes at all points of the set.

(3) Even for special values of α—say $\alpha = L$ or U—no better result can be stated. A proof that the *proper* maxima and minima of a continuous function form an enumerable set will be found in Hobson's *Functions of a Real Variable.* But this by no means implies that the set $S(L)$ or $S(U)$ is enumerable.

THEOREM III. *The set $S(\alpha)$ is of measure zero, except possibly for an enumerable set of values of α in (L, U).*

Proof.—Since $f(x)$ is continuous, the set $S(\alpha)$ is measurable. From the property of measurable sets, it follows that the sum of the measures of the sets $S(\alpha)$ for any sequence of values of α must be equal to or less than 1 [which is the measure of the whole interval $(0, 1)$]. Hence if ϵ is any positive quantity, the measure of $S(\alpha)$ can be greater than ϵ only for a finite number of values of α. Hence it follows at once that the set of values of α for which $S(\alpha)$ has a measure greater than zero, must be enumerable.

To see that Theorem III is a best possible result even if $f(x)$ is non-differentiable, we may shew how to construct a continuous function which [illegible] all points of an [illegible] perfect set irrespective of whether the [illegible] is equal to or greater than zero, and is non differenti[illegible]

[illegible], we start with a sequence $n_1, n_2, \cdots$ of [illegible] 1. We remove symmetrically from the middle of the interval [illegible] an interval of length $\frac{1}{n_1}$, we then remove [illegible] from the middle of each of the two intervals left, a fraction $\frac{1}{n_2}$ [illegible]. What remains after the sequence of removals is the [illegible] of the intervals removed, and the limit points of these endpoints. [illegible] a perfect non dense set of measure θ given by:

$$\theta = \left(1 - \frac{1}{n_1}\right)\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n_3}\right)\cdots$$

We can [illegible] the sequence $n_1, n_2, \cdots$ so that θ is zero or any positive number [illegible] than 1.

We now define a continuous function [illegible] as follows: We define $f(x) = 0$ if x is any point of the perfect set thus constructed. If x is not a point of the perfect set, it is an interior point of a definite one of the intervals removed, say, the sth interval (α_s, β_s). We then define $f(x)$ by:

$$[illegible] \cdot \left(x - 1 + \frac{[illegible]}{[illegible]}\right)(\alpha_s - x - \beta_s),$$

so that within each of the intervals removed, $f(x)$ is a Weierstrassian non differentiable function [illegible], at the endpoints of the interval. The [illegible] defined by an enumerable set of specifications in (0, 1) is [illegible] and non differentiable and vanishes at points of the perfect set [illegible]

II. [illegible] of a Continuous Function

[illegible] to study the relation of the sets [illegible] to the concept [illegible] the subject matter of Dr. Srinivasiengar's [illegible] of the notion of 'derivate' which I give here seems [illegible] treatment to be found, e.g., in Hobson [illegible]

[illegible] function whatever, defined in the open [illegible] as h → 0 is a closed set.

A proof [illegible] will be found in Hobson [illegible]. As a matter of fact, no special proof [illegible], as this theorem is only a particular statement of the fact that the derived set of any set whatever is a closed set.

THEOREM V. *If the ϕ (h) of Theorem IV is continuous in the open interval* (0, k), *the closed set of limits of ϕ (h) as h $\rightarrow 0$ is a closed interval.**

To prove this, let ϕ_1, ϕ_2 be any two functional limits of ϕ (h) as h $\rightarrow +0$, and let the corresponding sequences of values of h be $(h_1, h_2, \cdots)$, $(h_1', h_2', \cdots)$, so that:

$$\lim_{n\to\infty} h_n = \lim_{n\to\infty} h_n' = +0 \, ; \; \phi_1 = \lim_{n\to\infty} \phi\,(h_n) \, ; \; \phi_2 = \lim_{n\to\infty} \phi\,(h_n').$$

Let $\phi = t\,\phi_1 + (1-t)\,\phi_2$ $(0 < t < 1)$ be any number between ϕ_1, ϕ_2. To prove that ϕ is also a limit of ϕ (h), we observe that $t\,\phi\,(h_n) + (1-t)\,\phi\,(h_n')$ is a number between the two values $\phi\,(h_n)$, $\phi\,(h_n')$ of the continuous function ϕ and is therefore a value of ϕ at some point H_n of the interval (h_n, h_n'). It is clear that as $n \rightarrow \infty$ $H_n \rightarrow 0$, and Lim $\phi\,(H_n) = \phi$. This proves that when ϕ is continuous, the closed set of functional limits of ϕ (h) is a closed interval.

THEOREM VI. *If in Theorem V*, t *is an interior point of the closed interval of functional limits of ϕ* (h), *we can find two sequences* (h_n), (h_n') *tending to* $+0$, *such that* $[\phi\,(h_n)]$ *tends to* (t $+$ 0), *and* $\phi\,(h_n')$ *tends to* (t $-$ 0).

If t *is an endpoint of the interval of functional limits, it may or may not be possible to find two sequences* (h_n), (h_n') *with this property.*

The theorem will be proved, if we can shew that are values of ϕ (h) in each of the intervals $(t, t \pm \epsilon)$ for every positive ϵ. Since $t \pm \frac{\epsilon}{2^m}$ $(m = 1,2,\cdots)$ are all limits of ϕ (h), we can find h_n, h_n' so that

$$\left.\begin{array}{l} \left|\phi\,(h_n) - \left(t + \frac{\epsilon}{2^n}\right)\right| < \frac{\epsilon}{2^{n+1}} \\ \left|\phi\,(h_n') - \left(t - \frac{\epsilon}{2^n}\right)\right| < \frac{\epsilon}{2^{n+1}} \end{array}\right\} n = 1, 2, \cdots$$

The sequences $\phi\,(h_n)$, $\phi\,(h_n')$ tend to t from different sides as $n \rightarrow \infty$, so that the first part of the theorem is proved.

This proof is no longer valid if t is an endpoint of the interval of functional limits. But it is easy to construct examples which shew that t may still be approachable as a functional limit from both sides.

We now apply theorems V and VI to the function ϕ (h) which is equal to the incrementary ratio $\dfrac{f\,(x+h) - f\,(x)}{h}$ of the continuous function f (x) at the point x. This function ϕ (h) is continuous in open intervals on either

* For theorems of this sort in Anaysis, it is convenient to consider the real number continuum as a closed interval with the endpoints $+\infty$ and $-\infty$.

side of the endpoint $h = 0$. By Theorem V, the functional limits of $\phi(h)$ as $h \to +0$ and -0, constitute respectively two closed intervals. I shall call these intervals the *right* and *left derivate-intervals* of the continuous function $f(x)$ at the point x.

Suppose now that $f(x) = \alpha$, so that x belongs to the closed set S (α). If a right neighbourhood $(x, x+h)$ of x can be found, throughout which $f(x) > \alpha$ $(< \alpha)$, we say that $f(x)$ is *increasing* (*decreasing*) on the right of x. If in every small right neighbourhood $(x, x+h)$ $f(x) \geqslant \alpha$ $(\leqslant \alpha)$, we say that $f(x)$ is *non-decreasing* (*non-increasing*) on the right of x. If in every small right neighbourhood $(x, x+h)$ of x there are points at which $f(x) > \alpha$ as well as points at which $f(x) < \alpha$, we say that $f(x)$ is *oscillatory* on the right of x.

It is obvious that if 0 be external to the right derivate interval of $f(x)$ at x, then $f(x)$ must be either increasing or decreasing to the right of x. If however 0 is an endpoint of the right derivate-interval, all possibilities are open for the behaviour of $f(x)$ on the right of x.

If x is a *limitpoint* of the set S (α), approached say from the right, it is clear that 0 must belong to the right derivate-interval. Is the converse of this theorem true? Theorem VI shows that the converse can be asserted categorically, only if it is known that 0 is an *interior* point of the derivate-interval.

THEOREM VII. *If* 0 *is an interior point of the right derivate-interval of* f (x) *at* x, *then* f (x) *is necessarily oscillatory on the right of* x. *Also* x *is necessarily a limitpoint of* S (α) *approached from the right, where* α = f (x).

Since, by hypothesis, 0 is an interior point of the derivate-interval, it follows from Theorem VI, that the incrementary ratio can approach zero from either side as $h \to +0$. Hence $f(x)$ must be oscillatory to the right of x. Hence in every right neighbourhood of x, there are points at which $f(x) > \alpha$, as well as points at which $f(x) < \alpha$. Hence, since f is continuous, there are in every right neighbourhood of x, points at which $f(x) = \alpha$; in other words x is a limitpoint of S (α) approached from the right.

The point x is said to be a *knot-point* of $f(x)$, if both the derivate-intervals at x extend from $-\infty$ to $+\infty$. In such a case zero is an interior point of both the derivate-intervals. Hence a knot-point x of f is necessarily a limitpoint on both sides of the set S (α) $[\alpha = f(x)]$—which is Dr. Srinivasiengar's result.

It is clear that a result corresponding to Theorem VII can be stated for any other value of the derivate.