ON THE LATTICE OF OPEN SETS OF A TOPOLOGICAL SPACE

By R. Vaidyanathaswamy

(*Madras University*)

Received November 18, 1942

Any space S can be topologised by assigning arbitrarily the family (Γ_1) of its open sets, subject to the requirements:

(1) 0 and 1 (*i.e.*, the null set and the whole space S) are in $\{\Gamma_1\}$,

(2) Any set-sum and any finite set-product of members of $\{\Gamma_1\}$ are in $\{\Gamma_1\}$.

Since set-sums and set-products are distributive, it follows that Γ_1 is a general distributive lattice of subsets of S, containing 0, 1, closed for all set-sums, and with all sums distributive. From known theorems in lattice-theory, it results that Γ_1 is a *complete* distributive lattice, with completely distributive sums, in which all lattice-sums and finite lattice products are identical with the corresponding set-operations. The lattice-product of an infinite number of open sets will in fact be the *interior* of their set-product (which will not in general be open).

As a simple example to shew the non-distributivity of infinite lattice-products in Γ_1, consider the Cantor-discontinuum C in the interval (0, 1). Then C is the set-product of a sequence $G_1, G_2, \cdots$ of open sets, and being non-dense, has no interior. If C′ is the dense open set which is the complement of C, it follows that $C' + \text{Interior}\,(\Pi G_i) = C' \neq 1$. But $C' + G_i = 1$ for each i, and therefore,

$$\text{Interior}\,\Pi\,(C' + G_i) = \Pi\,(C' + G_i) = 1 \neq C' + \text{Interior}\,(\Pi\, G_i).$$

Since the topology is completely determined by Γ_1, the topological properties of S may be interpreted as structural lattice-properties of Γ_1. This is true in particular of the separation-postulates T_0, T_1, T_2, T_3, T_4*

---

* These postulates are :

T_0—Of any two distinct points, one at least has a neighbourhood not containing the other.

T_1—Any point has a neighbourhood not containing any other assigned point.

T_2—Two distinct points possess disjoint neighbourhoods.

T_3—Two disjoint closed sets, of which one consists of a single point, possess disjoint neighbourhoods.

T_4—Two disjoint closed sets possess disjoint neighbourhoods.

In these, 'neighbourhood' may be understood as 'open neighbourhood'.—*Cf.* Alexandroff and Hopf, *Topologie*, pages 58, 59, 67, 68.

by which we pass from general topological spaces to the special ones which are nearer metrical space. The present paper is an attempt to view these separation-postulates from the lattice-structure of Γ_1, and to shew in particular that the last two, namely T_3, T_4 are connected with *the theory of the last residue class.*[4]

We begin by certain preliminaries relating to distributive lattices.

§1. *The ideals Π_α, Π_μ of a distributive lattice Γ with* 0, 1

Even though Γ is not closed for product- or sum-complements, there exists a set Π_α (Π_μ) of elements whose product- (sum-) complement is 0 (1). Neither Π_α nor Π_μ is empty since $1 \in \Pi_\alpha$, $0 \in \Pi_\mu$. It is clear that Π_α is an α-ideal, and Π_μ is a μ-ideal, and that Π_α, Π_μ are important structural elements of Γ.

In the particular case in which Γ is a complete lattice with completely distributive sums (products), we can characterise in terms of Π_α (Π_μ) the μ-ideals (α-ideals) of Γ whose product-complement is 0 (1). Namely,

THEOREM I: *If Γ is a complete lattice with completely distributive sums, the μ-ideals whose product-complement is* 0, *are precisely those whose comprincipal envelope is a principal ideal of the form P_μ (t) where $t \in \Pi_\alpha$.*

For, in a complete lattice any μ-ideal P_μ is *convergent*, since the sum t of its elements exists, and the comprincipal envelope[3] of P_μ is the principal ideal P_μ (t). If the product-complement $P_\mu' = 0$, it follows that $\{P_\mu(t)\}' \subset P_\mu'$ and is therefore 0. It results that $t \in \Pi_\alpha$. Conversely let the comprincipal envelope of a μ-ideal P_μ be the principal ideal $P_\mu(t)$, where $t \in \Pi_\alpha$. To prove that $P_\mu' = 0$, we observe that since t is the distributive sum of the elements of P_μ, $ty = 0$ implies and is implied by $xy = 0$ for every x in P_μ. Hence $P_\mu' = \{P_\mu(t)\}' = 0$.

Properties of a lattice in which one of the ideals Π_α, Π_μ is 0

Assume for instance that in Γ the ideal $\Pi_\mu = 0$, *i.e.*, consists of the single element 0. Dual properties will hold for a lattice in which Π_α is 0, *i.e.*, consists of the single element 1.

THEOREM II: *If $\Pi_\mu = 0$, the most general α-ideal with the product-complement* 0 *has* 1 *for its comprincipal envelope (so that the product of all its elements exists and is equal to* 0).

(*Note.*—This is not a special case of Theorem (I), since we do not assume here that Γ is complete or has completely distributive sums or products.)

PROOF: *If $P_\alpha \subset$ the principal ideal $P_\alpha(t)$, and if $P_\alpha' = 0$, then $\{P_\alpha(t)\}' = 0$.* Hence, since $\Pi_\mu = 0$, it follows that $t = 0$. Thus the only principal α-ideal containing P_α is 1.

THEOREM III: *If $\Pi_\mu = 0$, and a is any element of the lattice, the ideal $\{P_\mu(a)\}'$ is always comprincipal and is the cut-complement of $\{P_\alpha(a)\}'$; dashes denoting product-complements.*

Corollary. If $\Pi_\mu = 0$, the product-complement of any μ-ideal P_μ is comprincipal and is the cut-complement of its last residue class.

PROOF: Let t and x be arbitrary elements such that $ta = 0, a + x = 1$.

Then $t = t(a+x) = ta + tx = tx$. Hence every $t \subset$ every x. Conversely if an element $t \subset$ every x such that $a + x = 1$, $tx = t = t(a+x) = ta + tx$.

Hence $ta \subset tx$ for every x. Therefore $ta = tax$, so that $ta \subset$ every ax. In other words $P_\alpha(ta)$ contains $P_\alpha(a) + \{P_\alpha(a)\}'$. Since the product-complement of this latter ideal is 0, it follows from Theorem (II), that $ta = 0$. This proves that $\{P_\mu(a)\}'$ is the cut-complement of $\{P_\alpha(a)\}'$, and is therefore comprincipal.

It follows that $\Pi_a \{P_\mu(a)\}'$ is the cut-complement of $\Sigma_a \{P_\alpha(a)\}'$, where the product and sum extend over all the elements a of a μ-ideal P_μ. But the product is P_μ' and the sum is the last residue class of P_μ; as the product of the comprincipal ideals $\{P_\mu(a)\}'$, the product-complement P_μ' is comprincipal. This proves the corollary.

§2. *Semisimple ideals of a distributive lattice*

An ideal P_μ (P_α) of a distributive lattice Γ with units will be said to be *semisimple*, if for each element x in P we can find an element y in P such that there exists an element t with $tx = 0$, $t + y = 1$ ($t + x = 1$, $ty = 0$).

In other words a μ- or α-ideal is semisimple, if for each element x of the ideal, $P_\mu + \{P_\mu(x)\}' = 1$, $P_\alpha + \{P_\alpha(x)\}' = 1$. It is clear from this that every semisimple ideal contains the double-product-complement of every principal ideal that it contains.*

It is known that a simple ideal of Γ must necessarily be principal. Conversely it follows immediately from the definition that a principal ideal is semisimple only if it is simple.

THEOREM II: *The semisimple ideals are those which are identical with the last residue-class of their last residue-class.*

---

* This is also a property of normal ideals. It is not known whether the normal and the semisimple ideals are the only ones which possess this property.

For the *l.r.c.* of a μ-ideal P is the set of t's for which there is y in P with $t+y=1$. The necessary and sufficient condition that P may be the *l.r.c.* of its *l.r.c.* is that for each x in P there exists a t with $tx=0$. This is clearly the same as the condition that P be semisimple. We also note that if P be semisimple, its *l.r.c.* is also semisimple.

THEOREM III: *The sum of semisimple ideals is semisimple; the product of two semisimple ideals is semisimple.*

For any element of the sum of a family (P_i) of semisimple μ-ideals is of the form $x_1+x_2+\ldots+x_n$; $x_i \in P_i$. Also for each x_i there is y_i in P_i and t_i, such that $t_i x_i=0$, $t_i+y_i=1$. Hence $t_1 t_2 \ldots t_n (x_1+x_2+\ldots+x_n) = 0$ and $t_1 t_2 \ldots t_n + y_1 + y_2 + \ldots + y_n = 1$. This proves the first part.

Also any element of $P_1 P_2$ is of the form $x_1 x_2$ $(x_i \in P_i)$. For each such element, there is an element $y_1 y_2$ in $P_1 P_2$ and t_1+t_2, such that $(t_1+t_2)\,x_1 x_2=0$, $(t_1+t_2+y_1 y_2)=1$. This proves the second part.

THEOREM IV: *The product-complement of a semisimple ideal is the cut-complement of its last residue class.*

For let P_μ be a semisimple μ-ideal and let s be $\subset$ every element of the last residue class of P_μ. If possible let there be an element x in P_μ such that $sx \neq 0$. Since P_μ is semisimple there is y in P_μ and an element t in the last residue class, with $tx=0$, $t+y=1$. Hence $0 \neq sx \subset s \subset t$. But $sxt=0 \neq sx$. We have thus a contradiction. Therefore the product-complement of P_μ contains the cut-complement of its last residue-class. On the other hand any element x of P_μ' is clearly $\subset$ any element t of the last residue-class, since $t+y=1$ gives $xt=x$. This proves the theorem.

Corollary. In a complete distributive lattice the product-complement of a semisimple ideal is principal.

§ 3. *Applications to Γ_1. Product-complements of μ-ideals*

Consider now the lattice Γ_1 of open sets of the topological space S. Since Γ_1 is a complete lattice with completely distributive sums, it is closed for product-complements. Define the *exterior* (*interior*) of any set X as the sum of all the open sets disjoint with (contained in) X. Then the product-complement in Γ_1 of any open set is then its exterior. The *open domains*[1] (as the *normal* open sets may be called) are then the open sets which are identical with the exterior of their exterior, or alternatively, with the interior of their closure. The lattice-product (that is, the interior of the set-product) of any number of open domains is an open domain. The sum of two open domains is not necessarily an open domain. (Example: If from a circular area in the Cartesian plane, the circumference and a diameter be removed, what remains is an open set which is not an open domain, even though it is the sum of two open domains, namely the two semicircular areas.)

The open sets whose exterior is null (namely, the elements of the ideal Π_α of Γ_1) are the dense open sets, which are obtained by removing a non-dense closed set from the space S. From Theorem I, the μ-ideals whose product-complement is 0 are precisely those which are generated from an open covering of a dense open set (in particular, of space).

If the space S is *bicompact*[2] (that is, if every open covering of S contains a finite covering), it is easy to see that every μ-ideal of Γ_1 is principal, and conversely.

§ 4. *Product-complements of α-ideals of* Γ_1.

If the space S is metrical, or even if it is simply a T_1-space, it is easy to see that the ideal $\Pi_\mu = 0$. But $\Pi_\mu = 0$ is not a consequence of the general topological postulates. We have accordingly to introduce a postulate (which I shall call Π_0) of the same nature as the separation-postulates T, to ensure this.

Postulate Π_o. *Any non-null open set contains a non-null closed set.*

If the open set $g \neq 0$, contains the closed set $c \neq 0$, then $g + c' = 1$; $c' \neq 1$. Hence the product-complement of P_α (g) $\neq 0$, since it contains an element different from 1, namely c'. Hence $\Pi_\mu = 0$ and conversely $\Pi_\mu = 0$ implies the poslulate Π_0.

It is clear that T_1 implies Π_0 ; on the other hand Π_0 is independent of T_0, that is, neither implies the other. This is seen from the following examples :

Ex. (1). Adjoin a point a to a T_1-space M, and define a T_0-topology in $M + a$ by:

Closure of X in $M + a$ = closure of X in M, if $X \subset M$; closure of $a = a + b$ ($b \in M$).

This topology does not satisfy Π_0, since the non-null open set (a) contains no non-null closed set. Hence T_0 does not imply Π_0.

Ex. (2). Consider the resolution-topology (studied by Miss Mary Thomas)[5] defined as one in which every closed set is open, and *vice versa*. It follows that the closures of two points are either identical or mutually disjoint, and that the topology resolves the space into mutually disjoint sets, which are the closures of single points. This topology is not T_0, but satisfies Π_0. Hence Π_0 does not imply T_0.

As Π_0 is related to the structure of Γ_1 more directly than T_0, we substitute Π_0 for T_0 and consider the sequence of separation-postulates $(\Pi_0, T_1, T_2, T_3, T_4)$.

From Theorem (III) corollary, it follows that for a Π_0-space, the product-complement X′ of any μ-ideal X of the Γ_1 is the cut-complement of the last residue class of X. That the product-complement X′ is comprincipal follows already from the fact that Γ_1 is closed for product-complements.

For a T_1-space, every one-pointic set is closed. Hence Γ_1 has a maximal basis, consisting of all open sets obtained by removing a single point from space. These maximal sets are all dense, and hence belong to Π_α, if no point is an isolated point of space, that is, if space is *dense-in-itself.*

THEOREM V: *In the case of a T_1-space, every principal α-ideal of Γ_1 is normal.*

For, in a T_1-space S, if a set X does not contain the point a, there is an open set containing X and not containing a, for example the open set $(S-a)$. Now the product-complement $\{P_\alpha(g)\}'$ of the principal α-ideal $P_\alpha(g)$ evidently consists of all open sets which contain the closed set g' (the complement of g). To prove that $\{P_\alpha(g)\}''$ cannot be different from $P_\alpha(g)$, we have only to shew that for any open set g_1 smaller than g, we can find an open set g_2 in $\{P_\alpha(g)\}'$, such that $g_1+g_2 \neq 1$. This is evident, since if a is a point in g which is not in g_1, we can take $g_2 = S - a$.

Corollary. For a T_1-space, all comprincipal α-ideals of Γ_1 are normal.

It was shewn in Theorem (II), that if $\Pi_\mu = 0$, the comprincipal envelope of any α-ideal whose product-complement is zero, must be 1. In the case of Γ_1 (for which $\Pi_\mu = 0$), we can specify precisely the product-complement of an α-ideal P_α whose comprincipal envelope is 1 (so that the lattice-product of the elements of P_α is 0, and therefore their set-product is a non-dense set N).

THEOREM VI: *If the set-product of the elements of P_α be a non-dense set N (so that the comprincipal envelope of P_α is* 1), *the product-complement P_α' is the family of open sets containing N′, the set-complement of* N. *Hence $P_\alpha' = 0$ if and only if $N = 0$.*

As a matter of fact, a similar theorem is true of the product-complement of any α-ideal, P_α, the set-product of whose elements is N. For, an open set g whose sum with every element of P_α is 1 cannot exclude any point of the set-complement N′ of N; for, if it did, there are elements of P_α which exclude the same point, and the sum of g with these elements would not be 1.

As a particular case, we observe that in a T_1-space, there are dense open sets which exclude any limit-point of space. Hence the set-product of all dense open sets (that is, of all elements of Π_α) is the set I of isolated points of space. Hence

THEOREM VII: *In a T_1-space the product-complement Π_α' of Π_α is the family of open sets containing all limit-points of space. In particular, if space is dense-in-itself, $\Pi_\alpha' = 0$.*

§ 5. *The separation-postulates T_2, T_3, T_4.*

To get the background of these separation postulates, we begin by considering the last residue class of the product-complement of a principal α-ideal $P_\alpha(g)$. If g' is the closed set which is the complement of g, this product-complement is the family of open sets containing g'.

THEOREM VIII: *The last residue class of $\{P_\alpha(g)\}'$ is the α-ideal generated by the exteriors of open sets containing g'; the last residue class of this last residue class is the family of open sets containing the closure of an open set containing g'.*

The first part follows from the definition of the last residue class. If g_1 is an open set containing g', and g_2 is an open set such that $g_2 + \text{Ext. } g_1 = 1$, it follows that g_2 should contain $\bar{g}_1$, since Ext. g_1 is the complement of $\bar{g}_1$. Hence the second part.

It is known that T_4 and T_3 can be formulated alternatively thus:

T_4 Every open neighbourhood of a closed set g' contains the closure of another neighbourhood of g'.

T_3 Every neighbourhood of a point contains the closure of another neighbourhood of the same point.

Hence:

THEOREM IX: *The regularity postulate T_3 states that every α-ideal Q_α* (a) *composed of all open sets containing a point a is semisimple; the normality postulate T_4 states that every α-ideal Q_α* (g') *composed of all open sets containing a closed set* g' *is semisimple.*

Consider now the postulate T_2; it states that two distinct points a, b possess disjoint open neighbourhoods g_a, g_b. Hence the closed set g_a' contains g_b and therefore the *closed domain* $\bar{g}_b$. Hence the exterior of a can be covered by closed domains of the form $\bar{g}_b$ or by open domains Int. $\bar{g}_b$ or Ext. g_a. Hence if we denote the exterior of a by A we see that T_2 states that the last residue-class of $Q_\alpha(a) = \{P_\alpha(A)\}'$ has $P_\alpha(A) = \{P_\alpha(A)\}''$ for its cut-complement. Conversely this property implies T_2. It is to be noticed that this property has the same form as the property of μ-ideals given by Theorem III. It will also be noticed that this does not imply the semi-simplicity of $\{P_\alpha(A)\}'$, asserted by T_3.

It is clear that T_3 can also be put in this form; namely, if g is any open set, T_3 asserts that the last residue class of $Q_\alpha(g') = \{P_\alpha(g)'\}$ has $\{P_\alpha(g)' = \{P_\alpha(g)\}''$ for its cut-complement. That this implies and is implied by the semisimplicity of every $\{P_\alpha(A)\}'$ should be capable of direct proof from lattice theoretic considerations.

BIBLIOGRAPHY

1. Kuratowski .. *Topologie.*
2. Alexandroff and Hopf .. *Ibid.*
3. R. Vaidyanathaswamy .. "The Ideal theory of the partially ordered set," *Proc. Ind. Acad. Sci.*, 1941, **13**, No. 2.
4. V. S. Krishnan .. "The Problem of the last residue class in the distributive lattice," *ibid.*, 1942, **16**, No. 3.
5. Mary Thomas .. "Topology of Resolution-spaces," unpublished thesis approved for the M.Sc. degree by the Madras University.