Chaos and information entropy production

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Abstract

We consider a general N-degree-of-freedom nonlinear system which is chaotic and dissipative and show that the nature of chaotic diffusion is reflected in the correlation of fluctuation of linear stability matrix for the equation of motion of the dynamical system whose phase space variables behave as stochastic variables in the chaotic regime. Based on a Fokker-Planck description of the system in the associated tangent space and an information entropy balance equation a relationship between chaotic diffusion and the thermodynamically-inspired quantities like entropy production and entropy flux is established. The theoretical propositions have been verified by numerical experiments.

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I. INTRODUCTION

Several authors have enquired recently about the relationship between phase space dynamics of a dynamical system and thermodynamics [1–9]. The question acquires a special relevance for the dissipative system when the phase space volume contracts by virtue of possessing the attractors and also when the system is nonlinear and comprises a few-degrees-of-freedom. Thus even when these systems are not truly statistical in the thermodynamic sense, it is possible that chaotic diffusion due to intrinsic deterministic chaos or stochasticity plays a significant role in the dynamics. It is therefore worthwhile to enquire about the relationship between *chaotic diffusion* in a dynamical system and the *thermodynamically-inspired quantities like entropy production and entropy flux*. Our purpose in this paper is to address this specific issue.

In what follows we shall be concerned with the nonlinear dynamical systems which are chaotic and dissipative. We do not consider any stochasticity due to thermal environment or external nonthermal noise. The "deterministic stochasticity" (i.e, chaos) has a purely dynamical basis and its emergence in nonlinear dynamical system is essentially due to loss of correlation of initially nearby trajectories. This is reflected in the linear stability matrix or Jacobian of the system [10]. When chaos has fully set in, the time dependence of this matrix can be described as a stochastic process, since the phase space variables behave as stochastic variables [11]. It has been shown that this fluctuation is amenable to a theoretical description in terms of the theory of multiplicative noise [12]. Based on this consideration a number of important results of nonequilibrium statistical mechanics, like Kubo relations, fluctuation-decoherence relation, fluctuation-dissipation relation and exponential divergence of quantum fluctuations have been realized in chaotic dynamics of a few-degree-of-freedom system [13–18]. In the present paper we make use of this stochastic description of chaotic dynamics to formulate a Fokker-Planck equation of probability density function for the relevant dynamical variables, the "stochasticity" (i.e, chaoticity) being incorporated through the fluctuations of the time-dependent linear stability matrix. Once the drift and chaotic

diffusion terms are appropriately identified the thermodynamic-like quantities can be derived with the help of the suitable information entropy balance equation.

The outlay of the paper is as follows: In Sec. II we introduce a Fokker-Planck description of the dynamical system and identify the chaotic drift and diffusion terms. This is followed by setting up of an information entropy balance equation in Sec. III. We then look for the entropy flux and entropy production-like terms in the steady state. The shift of stationary state due to additional external forcing and the associated change in entropy production is considered in Sec. IV. We illustrate the theory in detail with the help of an example in Sec. V. The paper is concluded in Sec. VI.

II. A FOKKER-PLANCK EQUATION FOR DISSIPATIVE CHAOTIC DYNAMICS

We are concerned here with a general N-degree-of-freedom system whose Hamiltonian is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m_i} + V(\{q_i\}, t) \ , \ \ i = 1 \cdots N \tag{1}$$

where $\{q_i, p_i\}$ are the co-ordinate and momentum of the i-th degree-of-freedom, respectively, which satisfy the generic form of equations

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \ , \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i} \ . \end{aligned} \tag{2}$$

We now make the Hamiltonian system dissipative by introducing $-\gamma p_i$ on the right hand side of the second of Eqs.(2). For simplicity we assume γ to be the same for all the N degrees of freedom. By invoking the symplectic structure of the Hamiltonian dynamics as

$$z_i = \begin{cases} q_i & \text{for } i = 1 \cdots N \ , \\ p_{i-N} & \text{for } i = N+1, \cdots 2N \ , \end{cases}$$

and defining I as

$$I = \begin{bmatrix} 0 & E \\ -E & -\gamma E \end{bmatrix}$$

where E is an $N \times N$ unit matrix, and 0 is an $N \times N$ null matrix, the equation of motion for the dissipative system can be written as

$$\dot{z}_i = \sum_{j=1}^{2N} I_{ij} \frac{\partial H}{\partial z_j} \, . \tag{3}$$

We now consider two nearby trajectories, $z_i, \dot{z}_i$ and $z_i + X_i$, $\dot{z}_i + \dot{X}_i$ at the same time t in $2N$ dimensional phase space. The time evolution of separation of these trajectories is then determined by

$$\dot{X}_i = \sum_{j=1}^{2N} J_{ij}(t) X_j \tag{4}$$

in the tangent space or separation co-ordinate space X_i, where

$$J_{ij} = \sum_k I_{ik} \frac{\partial^2 H}{\partial z_k \partial z_j} \quad . \tag{5}$$

Therefore the $2N \times 2N$ linear stability matrix $\underline{J}$ assumes the following form

$$\underline{J} = \begin{bmatrix} 0 & E \\ M(t) & -\gamma E \end{bmatrix} \tag{6}$$

where M is an $N \times N$ matrix. Note that the time dependence of stability matrix $\underline{J}(t)$ is due to the second derivative $\frac{\partial^2 H}{\partial z_k \partial z_j}$ which is determined by the equation of motion (3). The procedure for calculation of X_i and the related quantities like Lyapunov exponents is to solve the trajectory equation (3) simultaneously with Eq.(4). Thus when the dissipative system described by (3) is chaotic, $\underline{\mathrm{J}}$(t) becomes ("deterministically") stochastic phase space due to the fact that z_i-s behave as stochastic phase space variables and the equation of motion (4) in the tangent space can be interpreted as a stochastic equation [13–18].

In the next step we shall be concerned with a stochastic description of $\underline{J}(t)$ or $M(t)$. For convenience we split up M into two parts as

$$M = M_0 + M_1(t) \tag{7}$$

where M_0 is independent of variables $\{z_i\}$ and therefore behaves a constant part and M_1 is determined by the variables $\{z_i\}$ for $i = 1 \cdots 2N$. M_1 refers to the fluctuating part. We now rewrite the equation of motion (4) in the tangent space as

$$\begin{aligned} \dot{X} &= \underline{J} X \\ &= L\left(\{X_i\}, \{z_i\}\right) \end{aligned} \tag{8}$$

where X and L are the vectors with $2N$ components. Corresponding to (7) L in (8) can be split up again to yield

$$\begin{aligned} & \dot{X} = L^0(X) + L^1(X, \{z_i(t)\}) \\ \text{or} \quad & \dot{X}_i = {L_i}^0(\{X_i\}) + L_i^1(\{X_i\}, \{z_i\}) \ , \quad i = 1 \cdots 2N \ . \end{aligned} \tag{9}$$

Eq.(4) indicates that Eq.(8) is linear in $\{X_i\}$. Eqs.(4), (5) and (6) express the fact the first N components of L^1 are zero and the last N components of L^1 are the functions of $\{X_i\}$ for $i = 1 \cdots N$ only. The fluctuation in L_i^1 is caused by the chaotic variables $\{z_i\}$-s. By defining ∇_X as differentiation with respect to components of X, i. e. , $\{X_i\}$ (explicitly $X_i = \Delta q_i$ for $i = 1 \cdots N$ and $X_i = \Delta p_i$ for $i = N+1 \cdots 2N$) and since $L_i^1 = 0$ for $i = 1 \cdots N$ and $L_i^1 = L_i^1(X_1 \cdots X_N)$ for $i = N+1 \cdots 2N$ we have $\nabla_X \cdot L^1 = \sum_{i=1}^{N} (\frac{\partial}{\partial X_i} \cdot 0) + \sum_{i=N+1}^{2N} \frac{\partial}{\partial X_i} \cdot L_i^1(X_1 \cdots X_N) = 0$. This allows us to write the following relation (which will be used later on),

$$\nabla_X \cdot L^1 \phi(\{X_i\}) = L^1 \cdot \nabla_X \phi(\{X_i\}) \tag{10}$$

where $\phi(\{X_i\})$ is any function of $\{X_i\}$.

Note that Eq.(9) by virtue of (8) is a linear differential equation with multiplicative "noise" due to $\{z_i\}$ determined by equation of motion (3). This is the starting point of our further analysis.

Eq.(9) determines a stochastic process with some given initial conditions $\{X_i(0)\}$. We now consider the motion of a representative point X in $2N$ dimensional tangent space

$(X_1 \cdots X_{2N})$ as governed by Eq.(9). The equation of continuity, which expresses the conservation of points determines the variation of density function $\phi(X,t)$ in time as given by

$$\frac{\partial \phi(X,t)}{\partial t} = -\nabla_{\mathbf{X}} \cdot L(t)\phi(X,t) \ . \tag{11}$$

Expressing A_0 and A_1 as

$$\begin{aligned} A_0 &= -\nabla_X \cdot L^0 \\ \text{and} \quad A_1 &= -\nabla_X \cdot L^1 \end{aligned} \tag{12}$$

we may rewrite the equation of continuity as

$$\frac{\partial \phi(X,t)}{\partial t} = [A_0 + \alpha A_1(t)]\phi(X,t) \ . \tag{13}$$

It is easy to recognize that while A_0 denotes the constant part A_1 contains the multiplicative fluctuations through the phase space variables of the dynamical system $\{z_i(t)\}$. α is a parameter introduced from outside to keep track of the order of fluctuations in the calculations. At the end we put $\alpha = 1$.

One of the main results for the linear equations of the form (13) with multiplicative noise may now be in order [12]. The average equation of $\langle\phi\rangle$ obeys [$P(X,t) \equiv \langle\phi\rangle$],

$$\dot{P} = \left\{ A_0 + \alpha\langle A_1\rangle + \alpha^2 \int_0^\infty d\tau \langle\langle A_1(t)\exp(\tau A_0) A_1(t-\tau)\rangle\rangle \exp(-\tau A_0) \right\} P(X,t) \ . \tag{14}$$

The above result is based on second order cumulant expansion and is valid when fluctuations are small but rapid and the correlation time τ_c is short but finite or more precisely

$$\langle\langle A_1(t)A_1(t')\rangle\rangle = 0 \quad \text{for} \quad |t-t'| \ > \ \tau_c \ . \tag{15}$$

We have, in general, $\langle A_1\rangle \neq 0$. Here $\langle\langle\cdots\rangle\rangle$ implies $\langle\langle\zeta_i\zeta_j\rangle\rangle = \langle\zeta_i\zeta_j\rangle - \langle\zeta_i\rangle\langle\zeta_j\rangle$.

The Eq.(14) is exact in that limit $\tau_c \to 0$. Making use of relation (12) in (11) we obtain

$$\begin{aligned} \frac{\partial P}{\partial t} = \Big\{ &-\nabla_X \cdot \mathbf{L}^0 - \alpha\langle\nabla_X \cdot \mathbf{L}^1\rangle + \alpha^2 \int_0^\infty d\tau \langle\langle \nabla_X \cdot \mathbf{L}^1(t) \exp(-\tau \nabla_X \cdot \mathbf{L}^0) \\ &\nabla_X \cdot \mathbf{L}^1(t-\tau)\rangle\rangle \exp(\tau \nabla_X \cdot \mathbf{L}^0) \Big\} P \ . \end{aligned} \tag{16}$$

The above equation can be transformed into the following Fokker-Planck equation ($\alpha = 1$) for probability density function $P(X,t)$, (the details are given in the Appendix A);

$$\frac{\partial P(X,t)}{\partial t} = -\nabla_X . FP + \sum_{i,j} \mathcal{D}_{ij} \frac{\partial^2 P}{\partial X_i \partial X_j} \tag{17}$$

where,

$$F = L^0 + \langle L^1 \rangle + Q \tag{18}$$

and Q is a $2N$-dimensional vector whose components are defined by

$$Q_j = -\int_0^\infty \langle\langle R'_j \rangle\rangle d\tau det_1(\tau) det_2(\tau) \quad . \tag{19}$$

Here the determinants det_1, det_2 and R'_j are given by

$$det_1(\tau) = \left| \frac{dX^{-\tau}}{dX} \right| \quad \text{and} \quad det_2(\tau) = \left| \frac{dX}{dX^{-\tau}} \right|$$
$$\text{and} \quad R'_j = \sum_i L^1_i(X,t) \frac{\partial}{\partial X_i} \sum_k L^1_k(X^{-\tau}, t-\tau) \frac{\partial X_j}{\partial X_k^{-\tau}} \quad . \tag{20}$$

It is easy to recognize F as an evolution operator. Because of the dissipative perturbation we note that div $F < 0$.

The diffusion coefficient $\mathcal{D}_{ij}$ in Eq.(17) is defined as

$$\mathcal{D}_{ij} = \int_0^\infty \sum_k \langle\langle L^1_i(X,t) L^1_k(X^{-\tau}, t-\tau) \frac{dX_j}{dX_k^{-\tau}} \rangle\rangle d\tau \quad . \tag{21}$$

We have followed closely van Kampen's approach [12] to generalized Fokker-Planck equation (17). Before concluding this section several critical remarks regarding this derivation need attention:

First, the process $M_1(t)$ determined by $\{z_i\}$ is obtained *exactly* by solving equations of motion (3) for the chaotic motion of the system. It is therefore necessary to emphasize that we have *not assumed* any special property of noise, such as, $M_1(t)$ is Gaussian or δ-correlated. We reiterate Van Kampen's emphasis in this approach.

Second, the only assumption made about the noise is that its correlation time τ_c is short but finite compared to the coarse-grained timescale over which the average quantities evolve.

Or in other words the velocity changes should be small, smooth and uncorreleted after short times. This assumption, however, puts a restriction on the applicability of the present theory to certain class of systems, e. g. , systems subjected to 'hard' collisions such as billiards and molecular systems also in certain non-dissipative Hamiltonian systems like standard map for which the usual assumptions about the rapid decay of correlations and fluctuations are not valid and entropy production does not occur. The special reference may be made in this connection to the work of Zaslavsky and collaborators [19] to demonstrate that in many real systems the decay of correlation exhibits power law dependence, distributions admit of infinite moments and the fluctuations become long-lasting. Since the mathematical difficulties of dealing with finite arbitrary correlation time of noise in a chaotic system is quite formidable we confine ourselves in the present discussion to chaotic systems with short but finite noise correlation time.

Third, we take care of fluctuations upto second order which implies that the deterministic noise is not too strong.

Eq.(17) is the required Fokker-Planck equation in the tangent space $\{X_i\}$. However the important point is to note that the drift and diffusion terms are determined by the phase space $\{z_i\}$ properties of the chaotic system and directly depend on the correlation function of the fluctuations of the second derivatives of the Hamiltonian (5).

III. INFORMATION ENTROPY BALANCE: ENTROPY PRODUCTION

We shall now consider the well-known relation between probability density function $P(X,t)$ and information entropy S as given by

$$S = -\int dX P(X,t) \ln P(X,t) \ . \tag{22}$$

Note that in the above definition of entropy we use $P(X,t)$, the probability distribution function in the tangent space since one is concerned here with the expansion of the phase space in terms of a tangent space and dialation coefficients of the dynamical system for which

the expanding and contracting manifolds can be defined. On the other hand it is worthwhile to recall the dynamic entropy of a dynamical system (Kolmogorov entropy) defined in terms of the properties of evolution in the tangent space. A remark on the connection between entropy and expansion by Sinai [20] is noteworthy in this context ; " It already seems clear that positiveness of the entropy and presence of mixing is related to extreme instability of the motion of the system : trajectories emanating from the nearby points must, generally speaking, diverge with exponential velocity. Thus entropy is characterized here by the speed of approach of the asymptotic trajectories" which is formalized by defining the expansion coefficient as logarithm of the relative increase under the flow of a volume element in the expanding manifold. Our definition of information entropy (22) makes use of the tangent space description of the systems in terms of logarithm of the probability of expansion in the tangent space, keeping in mind that $-\ln P$ is "a measure of unexpectedness of an event (the amount of information) and the information entropy is a mean value of this unexpectedness for the entire system" [21]. The definition (22) is therefore different from Kolmogorov entropy. We emphasize that even in absence of any direct formal connection between $P(X,t)$ and the phase space distribution function it is possible to use the distribution function $P(X,t)$ defined in the tangent space to have an explicit expression for entropy production-like quantity as a function of the properties of phase space variables $\{z_i\}$ of the dynamical system (i. e. , in terms of drift and diffusion coefficients of the Fokker-Planck equation).

The above definition of an information entropy like quantity allows us to have an evolution equation for entropy. To this end we observe from Eqs.(17) and (22) that [22,23]

$$\frac{dS}{dt} = -\int dX \left[-\sum_i \frac{\partial}{\partial X_i}(F_i P) + \sum_i \sum_j \int \mathcal{D}_{ij} \frac{\partial^2 P}{\partial X_i \partial X_j} \right] \ln P \quad . \tag{23}$$

Note that the probability density function P(X,t) is defined in tangent space $\{X_i\}$. $\mathcal{D}$ and F as expressed in Eqs.(21) and (18), respectively, are determined by the correlation functions of fluctuations of second derivative of the Hamiltonian of the system. Eq.(23) therefore suggests that the entropy production-like term originating from Eq.(23) is likely to bear the signature of the chaotic dynamics. The relation is direct and general as it is evident

from the following equation [obtained after partial integration of Eq.(23) with the natural boundary condition on $P(X,t)$ that it vanishes as $|X| \to \infty$ and assuming the X-dependence of $\mathcal{D}_{ij}$ to be weak (as a first approximation)];

$$\frac{dS}{dt} = \int dX P \nabla_X \cdot F + \sum_i \sum_j \mathcal{D}_{ij} \int \frac{1}{P} \frac{\partial P}{\partial X_i} \frac{\partial P}{\partial X_j} dX \quad . \tag{24}$$

The first term in (24) has no definite sign while the second term is positive definite because of positive definiteness of $\mathcal{D}_{ij}$. Therefore the second one can be identified [22] as the entropy production

$$S_{prod} = \sum_i \sum_j \mathcal{D}_{ij} \int \frac{1}{P_s} \frac{\partial P_s}{\partial X_i} \frac{\partial P_s}{\partial X_j} dX \tag{25}$$

in the steady state. The subscript s of P_s refers to steady state. It is evident from Eq.(24) that

$$\begin{aligned} S_{flux} &= \int dX \; P_s(X) \; \nabla_X \cdot F \quad = \overline{\nabla_X \cdot F} \\ S_{prod} &= -S_{flux} \quad . \end{aligned} \tag{26}$$

Note that since the chaotic system is dissipative $\overline{\nabla_X . F}$ is negative [See Eq.(18)].

It is thus evident that the relations (25), (26) illustrate the dynamical origin of entropy production like quantity in a chaotic dissipative system. The dynamical signature is manifested through the drift term F and the chaotic diffusion terms in $\mathcal{D}_{ij}$. It must be emphasized that the notion of diffusion has nothing to do with any external reservoir. Rather it pertains to intrinsic diffusion in phase space of the chaotic system itself.

IV. THE CHAOTIC SYSTEM DRIVEN BY AN EXTERNAL FORCE

We shall now examine the entropy production when the dissipative chaotic system is thrown away from the steady state due to an additional weak applied force. To this end we consider the drift F_1 due to external force so that the total drift F has now two contributions

$$F(X) = F_0(X) + hF_1(X) \quad . \tag{27}$$

When $h = 0$, $P = P_s$. The deviation of P from P_s in presence of nonzero small h can be explicitly taken into account once we make use of the identity for the diffusion term [22]

$$\frac{\partial^2 P}{\partial X_i \partial X_j} = \frac{\partial}{\partial X_i}\left[P\frac{\partial \ln P_s}{\partial X_j}\right] + \frac{\partial}{\partial X_i}\left[P_s\frac{\partial}{\partial X_j}\frac{P}{P_s}\right] \quad . \tag{28}$$

When $P = P_s$ the second term in (28) vanishes. In presence of additional forcing the Eq.(17) becomes,

$$\frac{\partial P}{\partial t} = -\nabla_X.\psi P - h\nabla_X.F_1 P + \sum_i \sum_j \mathcal{D}_{ij}\frac{\partial}{\partial X_i}\left(P_s\frac{\partial}{\partial X_j}\frac{P}{P_s}\right) \tag{29}$$

where ψ is defined as

$$\psi = F_0 - \sum_j \mathcal{D}_{ij}\frac{\partial \ln P_s}{\partial X_j} \quad . \tag{30}$$

Here we have assumed for simplicity that $\mathcal{D}_{ij}$ is not affected by the additional forcing. The leading order influence is taken into account by the additional drift term in Eq.(29).

Under steady state condition ($P = P_s$) and $h = 0$, the second and the third terms in (29) vanish yielding

$$\nabla_X.\psi P_s = 0 \quad . \tag{31}$$

It is immediately apparent that ψP_s refers to a current $\mathcal{J}$ where $\mathcal{J} = \psi P_s$. The steady state condition therefore reduces to an equilibrium condition ($\mathcal{J} = 0$) if

$$\psi = 0 \quad . \tag{32}$$

(In Sec. V we shall consider an explicit example to show that $\psi = 0$). This suggests a formal relation between F_{0i} and $\mathcal{D}_{ij}$ as

$$F_{0i} = \sum_j \mathcal{D}_{ij}\frac{\partial \ln P_s}{\partial X_j} \tag{33}$$

where P_s may now be referred to as the *equilibrium* density function in separation co-ordinate space. F_0 contains dissipation constant γ and the diffusion matrix $\mathcal{D}_{ij}$ is a function of correlation function of fluctuations of the second derivative of the Hamiltonian.

To consider the information entropy balance equation in presence of external forcing we first differentiate Eq.(22) with respect to time and use Eq.(29) to obtain

$$\frac{dS}{dt} = -\frac{d}{dt}\int dX P \ln P_s$$
$$-\int dX \ln\frac{P}{P_s}\left[-\nabla_X\cdot\psi P - h\nabla_X\cdot F_1 P + \sum_i\sum_j \mathcal{D}_{ij}\frac{\partial}{\partial X_i}\left(P_s\frac{\partial}{\partial X_j}\frac{P}{P_s}\right)\right] \quad . \tag{34}$$

It is apparent that as P deviates from P_s, $\frac{P}{P_s}$ differs from unity and the entire second integral within the parenthesis [22] is of the second order. Note that $\ln P_s$ in the first integral in Eq.(34) is a constant of motion and the integral denotes its average. The first term vanishes because it is of higher order as it involves P^2 and others. (Moreover since in the discussion that follows we consider the steady state, this term does not contribute to the subsequent calculations). To compute the contribution ΔS to entropy balance due to the external forcing only we perform integration of the second, third and fourth terms by parts. We thus obtain,

$$\frac{d\Delta S}{dt} = h^2\int dX\delta P\nabla_X\cdot F_1 + h^2\int dX\left(\sum_i F_{1i}\frac{\partial\ln P_s}{\partial X_i}\right)\delta P$$
$$+\sum_i\sum_j \mathcal{D}_{ij}\int dX P(\frac{\partial}{\partial X_i}\ln\frac{P}{P_s})(\frac{\partial}{\partial X_j}\ln\frac{P}{P_s}) \tag{35}$$

where we have put $h\delta P = P - P_s$. In the new steady state (in presence of $h \neq 0$), the entropy production and the flux like terms balance each other as follows:

$$\Delta S_{prod} = -\Delta S_{flux} \tag{36}$$

with

$$\Delta S_{prod} = \sum_{i,j}\mathcal{D}_{ij}\int dX P\left(\frac{\partial}{\partial X_i}\ln\frac{P}{P_S}\right)\left(\frac{\partial}{\partial X_j}\ln\frac{P}{P_S}\right) \tag{37}$$

and

$$\Delta S_{flux} = h^2\int dX\delta P\nabla_X.F_1 + h^2\int dX\left(\sum_i F_{1i}\frac{\partial\ln P_S}{\partial X_i}\right)\delta P \quad . \tag{38}$$

In the following section we shall work out a specific example to provide explicit expressions for the entropy production and some related quantities due to external forcing.

V. APPLICATIONS

A. Entropy production in the steady state

To illustrate the theory developed above, we now choose a driven double-well oscillator system with Hamiltonian

$$H = \frac{p_1^2}{2} + aq_1^4 - bq_1^2 + \epsilon q_1 \cos \Omega t \tag{39}$$

where p_1 and q_1 are the momentum and position variables of the system. a and b are the constants characterizing the potential. ϵ includes the effect of coupling constant and the driving strength of the external field with frequency Ω. This model (39) has been extensively used in recent years for the study of chaotic dynamics [13–15,24].

The dissipative equations of motion for the tangent space variables X_1 and X_2 corresponding to q_1 and p_1 (Eq.8) read as follows:

$$\frac{d}{dt}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \underline{J}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad , \quad \begin{Bmatrix} \Delta q_1 = X_1 \\ \Delta p_1 = X_2 \end{Bmatrix} \quad , \tag{40}$$

where $\underline{J}$ as expressed in our earlier notation

$$z_1 = q_1 \qquad z_2 = p_1$$

is given by

$$\underline{J} = \begin{pmatrix} 0 & E \\ -E & -\gamma E \end{pmatrix} \begin{pmatrix} \frac{\partial^2 H}{\partial z_1 \partial z_1} & \frac{\partial^2 H}{\partial z_1 \partial z_2} \\ \frac{\partial^2 H}{\partial z_2 \partial z_1} & \frac{\partial^2 H}{\partial z_2 \partial z_2} \end{pmatrix} \quad .$$

Therefore $\underline{J}$ reduces to

$$\begin{pmatrix} 0 & 1 \\ \zeta(t) + 2b & -\gamma \end{pmatrix} \quad ,$$

where $\zeta(t) = -12az_1^2$. Thus we have

$$M_0 = 2b \quad M_1 = \zeta(t) \quad .$$

Eq.(40) is thus rewritten as

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = L^0 + L^1 \tag{41}$$

with

$$L^0 = \begin{pmatrix} X_2 \\ 2bX_1 - \gamma X_2 \end{pmatrix} \quad \text{and} \quad L^1 = \begin{pmatrix} 0 \\ \zeta(t)X_1 \end{pmatrix} \ ,$$

where L^0 and L^1 are the constant and the fluctuating parts, respectively. The fluctuations in L^1, i.e., in $\zeta(t)$ is due to stochasticity of the following chaotic dissipative dynamical equations of motion;

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= -az_1^3 + 2bz_1 - \epsilon\cos\Omega t - \gamma z_2 \ . \end{aligned} \tag{42}$$

Now for the constant part and the fluctuating part we write

$$\begin{aligned} L^{01} = X_2 \quad L^{02} = 2bX_1 - \gamma X_2 \\ L^{11} = 0 \quad L^{12} = \zeta(t)X_1 \quad . \end{aligned}$$

We may then apply the result of Eq.(A5).

The mapping $X \to X^t$ is found by solving the 'unperturbed' equations

$$\begin{aligned} \dot{X}_1 &= X_2 \\ \dot{X}_2 &= G_2 - \gamma X_2 \end{aligned}$$

Comparison with Eq.(A7) shows that $G_2(= 2bX_1)$ is free from X_2.

As a short time approximation we consider the variation of X_1 and X_2 during τ_c ;

$$\begin{aligned} X_1^{-\tau} &= -\tau X_2 + X_1 = \bar{G}_1(X_1, X_2) \\ X_2^{-\tau} &= -\tau G_2 + e^{\gamma\tau}X_2 = \bar{G}_2(X_1, X_2) \quad . \end{aligned} \tag{43}$$

So the g-matrix of Eq.(A15) becomes

$$\underline{g} = \begin{pmatrix} 1 & \tau e^{-\gamma\tau} \\ 2b\tau e^{-\gamma\tau} & e^{-\gamma\tau} \end{pmatrix} \quad . \tag{44}$$

The vector R from Eq.(A15) can then be identified as

$$\begin{aligned} R &= \begin{pmatrix} \zeta(t-\tau)(X_1 - \tau X_2)g_{12} \\ \zeta(t-\tau)(X_1 - \tau X_2)g_{22} \end{pmatrix} \\ &= \begin{pmatrix} \zeta(t-\tau)X_1\tau e^{-\gamma\tau} \\ \zeta(t-\tau)(X_1 - \tau X_2)e^{-\gamma\tau} \end{pmatrix} \quad . \quad [\text{ neglecting the terms of } 0(\tau^2) \text{ }] \quad . \end{aligned} \tag{45}$$

Similarly the vector R' is given by

$$R' = \begin{pmatrix} 0 \\ -\zeta(t-\tau)\tau e^{-\gamma\tau} X_1 \zeta(t) \end{pmatrix} \quad . \tag{46}$$

From Eqs.(43) and (44) we have

$$det_1(\tau) det_2(\tau) \simeq 1 \quad . \tag{47}$$

Then the vector Q can be written as

$$Q = \begin{pmatrix} 0 \\ X_1 \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau e^{-\gamma\tau} d\tau \end{pmatrix} \quad . \tag{48}$$

Now the diffusion matrix $\mathcal{D}$ can be constructed as

$$\mathcal{D} = \begin{pmatrix} 0 & 0 \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{pmatrix} \tag{49}$$

where

$$\mathcal{D}_{21} = X_1^2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau e^{-\gamma\tau} d\tau$$

and

$$\mathcal{D}_{22} = X_1^2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle e^{-\gamma\tau} d\tau - X_1(0)X_2(0) \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau e^{-\gamma\tau} d\tau \quad .$$

It is important to mention that the assumption of weak X-dependence of the diffusion coefficient (by freezing its time dependence) is permitted as a first approximation within the perview of the present second order theory for which the strength of noise is not too large. We also emphasize that for actual theoretical estimate of the entropy production in terms of the formulae (26) or (58) explicit evaluation of the diffusion coefficients is not required (see the next section). A straightforward calculation of drift is sufficient for the purpose. This point will be clarified in greater detail in the next Sec.VB.

Then the Fokker-Planck equation (17) for the dissipative driven double-well oscillator assumes the following form:

$$\frac{\partial P}{\partial t} = -X_2 \frac{\partial P}{\partial X_1} - \omega^2 X_1 \frac{\partial P}{\partial X_2} + \gamma \frac{\partial}{\partial X_2}(X_2 P) + \mathcal{D}_{21} \frac{\partial^2 P}{\partial X_2 \partial X_1} + \mathcal{D}_{22} \frac{\partial^2 P}{\partial X_2^2} \tag{50}$$

where

$$\omega^2 = 2b + c + c_2$$

with

$$\begin{aligned} c_2 &= \int_0^\infty \langle\langle \zeta(t)\zeta(t-\tau)\rangle\rangle \tau e^{-\gamma\tau} d\tau \\ c &= \langle \zeta \rangle \quad . \end{aligned} \tag{51}$$

The similarity of the equation (50) to generalized Kramers' equation can not be overlooked. This suggests a clear interplay of chaotic diffusive motion and dissipation in the dynamics.

We now let

$$U = a_s X_1 + X_2 \tag{52}$$

where a_s is a constant to be determined.

Then under steady state condition Eq.(50) reduces to the following form :

$$\frac{\partial}{\partial U}(\lambda_s U) P_s + \mathcal{D}_s \frac{\partial^2 P_s}{\partial U^2} = 0 \quad , \tag{53}$$

where

$$\mathcal{D}_s = \mathcal{D}_{22} + a_s \mathcal{D}_{21} \ \ ,$$

and

$$\lambda_s U = -a_s X_2 - \omega^2 X_1 + \gamma X_2 \ \ .$$

Here λ_s is again a constant to be determined. Putting (52) in $\lambda_s U$ as given above and comparing the coefficients of X_1 and X_2 we obtain

$$\lambda_s a_s = -\omega^2 \quad \text{and} \ \ \lambda_s = -a_s + \gamma \ \ .$$

The physically allowed solutions for a_s and λ_s are as follows ;

$$a_s = \frac{\gamma - \sqrt{\gamma^2 + 4\omega^2}}{2} \quad \text{and} \ \ \lambda_s = \frac{\gamma + \sqrt{\gamma^2 + 4\omega^2}}{2} \ \ . \tag{54}$$

The stationary solution of (53) P_s is given by

$$P_s = N e^{-\frac{\lambda_s U^2}{2\mathcal{D}_s}} \ \ . \tag{55}$$

Here N is the normalization constant. By virtue of (55) ψ corresponding to Eq.(30) is therefore

$$\psi = \lambda_s U - \mathcal{D}_s \frac{\partial \ln P_s}{\partial U} = 0 \ \ . \tag{56}$$

Since ψP_s defines a current, P_s defines a zero current situation or an equilibrium condition.

The equilibrium solution P_s from (55) can now be used to calculate the steady state entropy production as given by Eq.(25). We thus have

$$S_{prod} = \mathcal{D}_s \int_{-\infty}^{\infty} \frac{1}{P_s} \left(\frac{\partial P_s}{\partial U} \right)^2 dU \ \ . \tag{57}$$

Explicit evaluation shows

$$S_{prod} = \lambda_s \ \ , \tag{58}$$

where λ_s is given by Eq.(54).

The above result demonstrates a rather straightforward connection between entropy production-like quantity of a chaotic system and the dynamics through dissipation constant γ, parameters of the Hamiltonian and correlation of fluctuations of the second derivatives of the Hamiltonian in the steady state. It is important to note that since one is concerned here with a few degree-of-freedom system with no explicit reservoir temperature does not appear in the expression for entropy production like term (58). The entropy production in a truly thermodynamic system and in the present case are therefore distinct.

B. Numerical Verifications

To verify the above theoretical analysis in terms of numerical experiments we now concentrate on the following two points. First it is necessary to establish numerically that the dynamical system reaches a steady state, i . e ., the probability density function $P(\{X_i\}, t)$ attains a steady state distribution $P_s(\{X_i\})$ in the long time limit. Second, the entropy production in the steady state calculated by the formula (58) needs to be verified numerically. To address the first issue we now proceed as follows ;

The dissipative chaotic dynamics corresponding to the model Hamiltonian (39) is governed by Eqs.(40) and (42). We choose the following values of the parameters [24] $a = 0.5$, $b = 10$, $\omega = 6.07$ and $\gamma = 0.001$. The coupling-cum-field strength ϵ is varied from set to set. We fix the initial condition $z_1(0) = -3.5$, $z_2(0) = 0$ which ensures strong global chaos [24]. To determine the steady state distribution function, say, $P_s(X_1)$, where $X_1 = \Delta q$ (Eq. 40) from the dynamical point of view we first define d_0 as the separation of the two initially nearby trajectories and $d(t)$ as the corresponding separation at time t. To express $d(t)$ we write $d(t) = [\sum_i^N (X_i)^2 + \sum_{i=N+1}^{2N} (X_i)^2]^{\frac{1}{2}}$. $d(t)$ is determined by solving numerically Eqs.(40) and (42), simultaneously for the initial conditions of z_1 and z_2 corresponding to Eq.(42). To follow the evolution of X_1 numerically in time, i . e . , in going from j-th to (j+1)-th step of iteration, say, X_1 has to be initialized as $X_1^{j0} = \frac{X_1^j}{d_j} d_0$. (The time evolution of the other components of X can be followed similarly). This initialization implies that at each step,

iteration starts with the same magnitude of d_0 but the direction of d_0 for step j+1 is that of $d(t)$ for j-th step (considered in terms of the ratio $\frac{X_1^j}{d_j}$). For a more pictorial illustration we refer to Fig. 1 of Ref. [25]. j-th term of iteration means $t = jT$ ($j = 1, 2, \cdots \infty$), where T is the characteristic time which corresponds to the shortest ensemble averaged period of nonlinear dynamical system.

Having obtained the time series in X_1 (it may be noted that the series in X_i-s are also required for calculation of the largest Lyapunov exponent as defined by

$$\lambda = \lim_{\substack{n \to \infty \\ d_0 \to 0}} \frac{1}{nT} \sum_{j=1}^{n} \ln \frac{d_j}{d_0}$$

for the chaotic system) the stationary probability density function $P_s(X_1)$ is computed as follows : The X_1 axis ranging from -2 to +2 is divided into small intervals ΔX_1 of size 0.025. The time series in X_1 is computed over the time intervals of 1000-10000 times of the time period T. For each time interval ΔX_1 a counter is maintained and is initially set to zero before the simulation is started. The respective counter is incremented whenever X_1 falls within the given interval. Finally the steady state probability distribution function $P_s(X_1)$ is obtained by normalizing the counts. The result is shown in Fig.1 for $\epsilon = 10$. Our numerical analysis shows that the distribution function attains stationarity at around $t = 1000T$, beyond which no perceptible change in the distribution is obtained.

We now turn to the second issue. In what follows we shall be concerned with the steady state entropy production (58) and its numerical verification. This quantity can be calculated in two different ways. First, it may be noted that the determination of S_{prod} (Eq.(57)) rests on two quantities defined in the tangent space; the steady state probability distribution function $P_s(U)$ and the diffusion coefficient $\mathcal{D}_s$ in U-space. Once the procedure for calculation of the distribution function as illustrated above is known from the time series in X_1 or X_2 the evaluation of $P_s(U)$ is quite straightforward since U is expressed as $U = a_s X_1 + X_2$ according to (52). Here a_s is given by (54) with $\omega^2 = 2b + c + c_2$ and the average c and the integral over the correlation function, c_2 are as defined in (51).

To calculate the correlation function $\langle\langle\zeta(t)\zeta(t-\tau)\rangle\rangle$ and the average $\langle\zeta(t)\rangle$ it is necessary to determine long time series in $\zeta(t)$ ($\zeta(t) = -12az_1^2$) by numerically solving the classical equation of motion (42) in phase space followed by averaging over the time series. For further details of numerical analysis of correlation functions we refer to the earlier work [15–17]. The diffusion coefficient $\mathcal{D}_s$ can be numerically determined from the time series in U. Having known $P_s(U)$ and $\mathcal{D}_s$ one can make use of the formula (57) to obtain the entropy production in the steady state. S_{prod} is thus numerically calculated. The second procedure of calculation of S_{prod} is the direct theoretical evaluation of λ_s from the expression (54). Since the value of λ_s rests again on ω^2 and γ and the dependence of ω^2 on the averages and the correlation functions are already known from the numerical analysis of phase space, λ_s can be calculated in the usual way. In Fig.2 we compare the values of steady state entropy production thus obtained by the two different ways for several values of coupling-cum-field strength ϵ. Here it should be noted that the curve connecting the squares (theoretically calculated entropy production in the steady state) corresponds to the negetive of entropy flux (S_{flux}) since for the given example S_{flux} in U space is $-\lambda_s$ ($-\int \lambda_s P_s(U)dU$, from Eq. (26)) for normalised probability distribution function $P_S(U)$. Thus Fig.2 is a numerical proof of $S_{prod} = -S_{flux}$. The agreement is found to be quite satisfactory. We therefore conclude that at least for the model studied here and for the similar class of models the correspondence between the formulae of steady state entropy production and the numerical computation is fairly general.

C. Entropy production in presence of weak forcing

We now introduce an additional weak forcing in the dynamics. This is achieved by subjecting the dissipative chaotic system to a weak magnetic field ($\vec{B}$) through a velocity ($\vec{v}$) dependent force term $\frac{e}{c_l}\vec{v}\times\vec{B}$ where e and c_l are the electric charge and the velocity of light. For simplicity we apply the constant field B_z which is perpendicular to q_1-direction [see Eq.39]. In presence of this force field the motion of the particle will not be restricted to B_z and q_1 only. We shall have to consider the other direction q_2 which is perpendicular to

both q_1 and B_z.

To make the notation consistent with Eq.(8) we would now like to correspond X_1, X_2, X_3 and X_4 to $\Delta q_1, \Delta q_2, \Delta p_1$ and Δp_2, respectively.

The relevant equations of motion are therefore as follows;

$$\begin{aligned}
\dot{X}_1 &= X_3 \ , \\
\dot{X}_2 &= X_4 \ , \\
\dot{X}_3 &= 2bX_1 - \gamma X_3 + \zeta(t)X_1 + h\frac{eX_4}{c_l}B_z \ \ , \\
\dot{X}_4 &= -h\gamma X_4 - h\frac{eX_3}{c_l}B_z \ \ .
\end{aligned} \tag{59}$$

Here e/c_l is the ratio of electric charge to the velocity of light used to put the equation in appropriate dimension.

Then the non-equilibrium situation (due to additional forcing, $h \neq 0$) corresponding to Eq.(59) is governed by

$$\begin{aligned}
\frac{\partial P}{\partial t} = &-h\frac{\partial}{\partial X_2}(X_4 P) + h\frac{\partial}{\partial X_4}(\gamma X_4)P - hB_z'\frac{\partial}{\partial X_3}(X_4 P) \\
&+hB_z'\frac{\partial}{\partial X_4}(X_4 P) + \mathcal{D}_s\frac{\partial}{\partial U}\left(P_s\frac{\partial}{\partial U}\frac{P}{P_s}\right)
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
B_z' &= \frac{e}{c_l}B_z \\
\mathcal{D}_s &= \mathcal{D}_{22} + \mathcal{D}_{21}a_s
\end{aligned}$$

or more explicitly,

$$\begin{aligned}
\frac{\partial P}{\partial t} = &-X_3\frac{\partial P}{\partial X_1} - \omega^2 X_1\frac{\partial P}{\partial X_3} + \gamma\frac{\partial}{\partial X_3}(X_3 P) + h\gamma\frac{\partial}{\partial X_4}(X_4 P) - hX_4\frac{\partial P}{\partial X_2} \\
&-hB_z'\frac{\partial}{\partial X_3}(X_4 P) + hB_z'\frac{\partial}{\partial X_4}(X_3 P) + \mathcal{D}_{21}\frac{\partial^2 P}{\partial X_3 \partial X_1} + \mathcal{D}_{22}\frac{\partial^2 P}{\partial X_3^2} \ \ .
\end{aligned} \tag{61}$$

Proceeding as before we make use of the transformation of variables using

$$U' = a'X_1 + b'X_2 + c'X_3 + X_4 \tag{62}$$

so that Eq.(61) becomes

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial U'}(\lambda' U' P) + \mathcal{D}\frac{\partial^2 P}{\partial U'^2} \tag{63}$$

where

$$\mathcal{D} = \mathcal{D}_{33} c'^2 + \mathcal{D}_{31} a' c' \ . \tag{64}$$

$\mathcal{D}_{33}$ and $\mathcal{D}_{31}$ in four dimension correspond to $\mathcal{D}_{22}$ and $\mathcal{D}_{21}$ in two dimension, respectively and

$$\lambda' U' = -a' X_3 - c'\omega^2 X_1 + \gamma X_3 c' + h\gamma X_4 - h B'_z X_4 c' + h B'_z X_3 - h b' X_4 \ . \tag{65}$$

Using (62) in (65) and comparing the coefficients of X_i-s we obtain

$$\begin{aligned} b' = 0 \qquad & a' = c'\gamma + h B'_z - \lambda' c' \\ & c' = (h\gamma - \lambda')/h B'_z \end{aligned} \tag{66}$$

where λ' is a solution of the cubic algebraic equation

$$\lambda'^3 + \lambda'(h\gamma^2 + h^2 B'^2_z - \omega^2) - \lambda'^2\gamma(1+h) + \omega^2\gamma h = 0 \ . \tag{67}$$

We now seek for a perturbative solution of the algebraic equation (67) which is given by (h as a small parameter) ;

$$\lambda' = \lambda'_0 + \frac{h\left(\lambda'^2_0\gamma - \omega^2\gamma - \lambda'_0\gamma^2 - \lambda'_0 h B'^2_z\right)}{3\lambda'^2_0 - 4\lambda'_0\gamma + h\gamma^2 + h^2 B'^2_z - \omega^2} \tag{68}$$

where λ'_0 is the solution of (67) for $h = 0$;

$$\lambda'_0 = \frac{\gamma + \sqrt{\gamma^2 + 4\omega^2}}{2} \ . \tag{69}$$

This is identical to λ_s (Eq.54). Therefore by virtue of Eqs.(66-69) all the constants in (62), i.e., a', b', c' are now known. The stationary solution of (63) is now given by,

$$P'_s = N' e^{-\frac{\lambda' U'^2}{2\mathcal{D}}} \ , \tag{70}$$

where N' is the normalization constant.

We are now in a position to calculate the steady state entropy flux ΔS_{flux} due to external forcing ($h \neq 0$) from Eq.(38)

$$\Delta S_{flux} = h^2 \int dX \delta P \nabla . F_1 + h^2 \int dX \left(\sum_i F_{1i} \frac{d \ln P_s}{dX_i} \right) \delta P \ . \tag{71}$$

where the components of F_1 can be identified as

$$\begin{aligned} F_{11} = 0 \quad & F_{13} = B'_z X_4 \\ F_{12} = X_4 \quad & F_{14} = -B'_z X_3 - \gamma X_4 \\ \text{and} \quad \nabla_X . F_1 = -\gamma \ , & \end{aligned} \tag{72}$$

and $h\delta P = P'_s - P_s$ denotes the deviation from the initial equilibrium state due to external forcing. For normalized probability functions P'_s and P_s the first integral in (71) vanishes.

Since P_s is given by (55) with U as defined in (52), the expression for (71) reduces to

$$\begin{aligned} \Delta S_{prod} &= -\Delta S_{flux} \\ &= h B'_z \frac{\lambda_s}{\mathcal{D}_s} \int X_4 \left(a_s X_1 + X_3 \right) dX P'_s \ . \end{aligned} \tag{73}$$

We now use the following transformations of variables

$$\begin{aligned} u' &= a' X_1 + c' X_3 + X_4 \quad (\text{since } \ b' = 0) \\ v' &= X_3 \\ w' &= X_4 \\ \text{and } \ dX_1 dX_3 dX_4 &= a' du' dv' dw' \ . \end{aligned} \tag{74}$$

to calculate the integrals,

$$\begin{aligned} \int X_3 X_4 dX P'_s &= \frac{\mathcal{D}}{2\lambda' |c'|} \\ \text{and} \quad \int X_1 X_4 dX P'_s &= \frac{\mathcal{D}}{2\lambda' a'} \end{aligned} \tag{75}$$

which yield

$$\Delta S_{prod} = -\Delta S_{flux}$$
$$= hB'_z \frac{\lambda_s}{\mathcal{D}_s} \frac{\mathcal{D}}{2\lambda'} \left[\frac{1}{|c'|} + \frac{|a_s|}{a'} \right] \quad . \tag{76}$$

For numerical verification of the above theoretical result (76) one can calculate entropy production (ΔS_{prod}) numerically in the steady state in presence of weak forcing from Eq. (37) as in the previous subsection. The Eq. (37) for the present example reduces to the following form in the steady state,

$$\Delta S_{prod} = \mathcal{D}_{33} \int\int\int P'_s (\frac{\partial}{\partial X_3} \ln \frac{P'_s}{P_s})^2 dX_1 dX_3 dX_4$$
$$+\mathcal{D}_{31} \int\int\int P'_s (\frac{\partial}{\partial X_3} \ln \frac{P'_s}{P_s})(\frac{\partial}{\partial X_1} \ln \frac{P'_s}{P_s}) dX_1 dX_3 dX_4 \quad . \tag{77}$$

To calculate numerically ΔS_{prod} $\mathcal{D}_{33}, \mathcal{D}_{31}$ and P_s can be determined by directly using the procedure mentioned in the subsection VB and by simultaneously solving Eqs. (41) and (42). Similarly one can calculate P'_s from Eqs. (42) and (59). Finally making use of all these quantities in Eq. (77) ΔS_{prod} can be obtained. Thus the numerically evaluated ΔS_{prod} should correspond to the results of Eq. (76) since our numerical verification in Fig.2 shows good agreement between numerical and theoretical results, $\mathcal{D}_{33}, \mathcal{D}_{31}$ being very close to $\mathcal{D}_{22}$ and $\mathcal{D}_{21}$, respectively, since h is very small.

In the limit h and γ small the above expression (76) can be simplified further. To this end we first note that

$$|a_s| \sim \omega \quad \lambda_s \sim \omega \quad \lambda' \sim \lambda'_0 \sim \omega$$
$$c' = \frac{a'}{\omega} \quad \text{and} \quad a' = \frac{\omega h}{\gamma} B'_z \quad . \tag{78}$$

This reduces $\mathcal{D}$ further as follows;

$$\mathcal{D} = \mathcal{D}_{33} c'^2 + \mathcal{D}_{31} a' c'$$
$$\simeq \frac{a'^2}{\omega^2} (\mathcal{D}_{33} - \mathcal{D}_{31}\omega) \quad . \tag{79}$$

Thus we have

$$\frac{\mathcal{D}}{\mathcal{D}_s} = \frac{a'^2}{\omega^2} \quad . \tag{80}$$

Making use of (78-80), expression (76) can be approximated as

$$\Delta S_{prod} = hB'_z \frac{a'^2}{\omega^2} \frac{1}{2} \left(\frac{\omega}{a'} + \frac{\omega}{a'} \right)$$
$$= \frac{h^2 e^2}{c_l^2 \gamma} B_z^2 \ . \tag{81}$$

This expression is due to average of the work per unit time of the external force B_z acting on the chaotic system. Note that the quadratic dependence on the magnetic field B_z in Eq. (81) is characteristic of an expression for entropy production in the steady state. Since the system is not thermostated this is independent of temperature. Although the leading order expression (81) is apparently free from diffusion coefficients, a close look into the more exact expression (76) reveals that their influence is quite significant in the higher order.

VI. CONCLUSIONS

Ever since the development of the theory of chaos, the dynamical variables in the strong chaotic regime have been interpreted as stochastic variables. One of the earliest well-known examples in this connection was set by demonstrating [11] the linear divergence of mean-square momentum in time in standard map, mimicking the Brownian motion. In the present paper we have tried to relate this chaotic diffusion to thermodynamic-like quantities by establishing a generalized Fokker-Planck equation pertaining to the tangent space. The explicit dependence of drift and diffusion terms on the dynamical characteristics of the phase space of the system is demonstrated.

The main conclusions of our study are;

(i) We analyze the nature of chaotic diffusion in terms of the properties of the phase space of chaotic systems. The drift and diffusion terms are dependent on the correlation of fluctuations of the linear stability matrix of the equation of motion. Since the later is the key point for understanding the stability of motion in a dynamical system, we emphasize that the thermodynamic-like quantities as discussed here have a deeper root in the intrinsic nature of motion of a few-degree-of-freedom system.

(ii) We identify the information entropy flux and production-like terms in the steady state which explicitly reveal their connection to dynamics through drift and diffusion term, in presence and absence of the external force field.
(iii) The connection between the thermodynamically-inspired quantities and chaos are fairly general for the N-degree-of-freedom systems.

The theory developed in this paper is based on the derivation of the Fokker-Planck equation for chaotic systems pertaining to the processes with correlation time which is short but finite (i. e. , for the systems with hard chaos). The suitable generalization of the approach to more general cases, where one encounters long correlation time is worthwhile for further investigation in this direction.

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APPENDIX A: THE DERIVATION OF THE FOKKER-PLANCK EQUATION

We first note that the operator $\exp(-\tau\nabla_X \cdot \mathbf{L}^0)$ provides the solution of the equation [Eq.(13), $\alpha = 0$]

$$\frac{\partial f(X,t)}{\partial t} = -\nabla_X \cdot \mathbf{L}^0 f(X,t) \tag{A1}$$

f signifies the "unperturbed" part of P which can be found explicitly in terms of characteristic curves. The equation

$$\dot{X} = \mathbf{L}^0(X) \tag{A2}$$

determines for a fixed t a mapping from $X(\tau = 0)$ to $X(\tau)$, i. e., $X \to X^\tau$ with inverse $(X^\tau)^{-\tau} = X$. The solution of (A1) is

$$f(X,t) = f(X^{-t},0)\left|\frac{dX^{-t}}{dX}\right| = \exp\left[-t\nabla_X \cdot \mathbf{F}_0\right] f(X,0). \tag{A3}$$

$\left|\frac{d(X^{-t})}{d(X)}\right|$ being a Jacobian determinant. The effect of $\exp(-t\nabla_X \cdot \mathbf{L}^0)$ on $f(X)$ is as

$$\exp(-t\nabla_X \cdot \mathbf{L}^0) f(X,0) = f(X^{-t},0)\left|\frac{dX^{-t}}{dX}\right| \quad . \tag{A4}$$

This simplification in Eq.(16) yields

$$\begin{aligned}\frac{\partial P}{\partial t} = \Bigg\{ & -\nabla_X \cdot \mathbf{L}^0 - \alpha\langle \nabla_X \cdot \mathbf{L}^1 \rangle + \alpha^2 \int_0^\infty d\tau \left|\frac{dX^{-\tau}}{dX}\right| \\ & \langle\langle \nabla_X \cdot \mathbf{L}^1(X,t) \nabla_{X^{-\tau}} \cdot \mathbf{L}^1(\mathbf{x}^{-\tau}, t-\tau)\rangle\rangle \left|\frac{dX}{dX^{-\tau}}\right| \Bigg\} P \quad .\end{aligned} \tag{A5}$$

Now to express the Jacobian, $X^{-\tau}$ and $\nabla_{X^{-\tau}}$ in terms of ∇_X and X we solve Eq.(A2) for short time (this is consistent with the assumption that the fluctuations are rapid [12]). Using Eqs.(4-6) we may rewrite "unperturbed" Eq.(A2) as

$$\frac{d}{dt}\begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} \tag{A6}$$

$$\text{and} \quad \frac{d}{dt}\begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} = -\gamma \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} + \begin{pmatrix} G_{N+1}(X) \\ \vdots \\ G_{2N}(X) \end{pmatrix} \tag{A7}$$

Here $G_{N+1}(X)\cdots G_{2N}(X)$ are the functions of $\{X_i\}$ with $i = 1\cdots N$ only. This allows us to rewrite the solution of (A6) and (A7) as,

$$\begin{pmatrix} X_1^{-\tau} \\ \vdots \\ X_N^{-\tau} \end{pmatrix} = -\tau \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} + \begin{pmatrix} X_1 \\ \vdots \\ X_N \end{pmatrix} = \begin{pmatrix} \bar{G}_1(X) \\ \vdots \\ \bar{G}_N(X) \end{pmatrix} \tag{A8}$$

and

$$\begin{pmatrix} X_{N+1}^{-\tau} \\ \vdots \\ X_{2N}^{-\tau} \end{pmatrix} = e^{\gamma\tau} \begin{pmatrix} X_{N+1} \\ \vdots \\ X_{2N} \end{pmatrix} - \tau \begin{pmatrix} G_{N+1}(X) \\ \vdots \\ G_{2N}(X) \end{pmatrix} = \begin{pmatrix} \bar{G}_{N+1}(X) \\ \vdots \\ \bar{G}_{2N}(X) \end{pmatrix} \tag{A9}$$

Here the terms of $O(\tau^2)$ are neglected. Since the vector $X^{-\tau}$ is expressible as a function of X we write

$$X^{-\tau} = \bar{G}(X) \ , \tag{A10}$$

and the following simplification holds good;

$$\begin{aligned} L^1(X^{-\tau}, t-\tau).\nabla_{X^{-\tau}} &= L^1(\bar{G}(X), t-\tau).\nabla_{X^{-\tau}} \\ &= \sum_k L_k^1(\bar{G}(X), t-\tau)\frac{\partial}{\partial X_k^{-\tau}} \\ &= \sum_j \sum_k L_k^1(\bar{G}(X), t-\tau) g_{jk} \frac{\partial}{\partial X_j} \qquad ; j,k = 1\cdots 2N \end{aligned} \tag{A11}$$

where

$$g_{jk} = \frac{\partial X_j}{\partial X_k^{-\tau}} \tag{A12}$$

In view of Eqs.(A8) and (A9) we note:

$$\text{if} \quad j = k \ \text{ then } \ g_{jk} = 1, \ \ k = 1\cdots N$$

$$= e^{-\gamma\tau}, \ \ k = N+1\cdots 2N$$

$$\text{if} \quad j \neq k \ \text{ then } \ g_{jk} \propto -\tau e^{-\gamma\tau}$$

$$\text{or } 0$$

Thus g_{jk} is a function of τ only.

Let

$$R_j = \sum_k L_k^1(\bar{G}(X), t-\tau) g_{jk} \tag{A13}$$

From Eqs.(8), (9) and (A10) we write

$$L_i^1(X^{-\tau}, t-\tau) = L_i^1(\bar{G}(X), t-\tau) = 0 \quad \text{for } i = 1 \cdots N \tag{A14}$$

So the conditions (A13), (A14) and (A8) imply that

$$\begin{aligned} R_j(X, t-\tau) &= R_j(X_1 \cdots X_N, t-\tau) \quad \text{for } j = 1 \cdots N \\ R_j(X, t-\tau) &= R_j(X_1 \cdots X_{2N}, t-\tau) \quad \text{for } j = N+1 \cdots 2N \end{aligned} \tag{A15}$$

We next carry out the following simplifications of α^2-term in Eq.(A5). We make use of the relation (10) to obtain

$$\begin{aligned} L^1(X,t).\nabla \sum_j R_j \frac{\partial}{\partial X_j} P(X,t) &= \sum_i L_i^1(X,t) \frac{\partial}{\partial X_i} \sum_j R_j \frac{\partial}{\partial X_j} P(X,t) \\ &= \sum_{i,j} L_i^1(X,t) R_j \frac{\partial^2}{\partial X_i \partial X_j} P(X,t) \\ &\quad + \sum_j R_j' \frac{\partial}{\partial X_j} P(X,t) \end{aligned} \tag{A16}$$

where

$$R_j' = \sum_i L_i^1(X,t) \frac{\partial}{\partial X_i} R_j \tag{A17}$$

The conditions (A14) and (A15) imply that

$$\begin{aligned} R_j' &= 0 \quad \text{for } j = 1 \cdots N \\ R_j' &= R_j'(X_1 \cdots X_N, t-\tau) \neq 0; \quad \text{for } j = N+1 \cdots 2N \end{aligned} \tag{A18}$$

By (A18) one has

$$R'.\nabla_X P(X,t) = \nabla_X . R' P(X,t) \tag{A19}$$

Making use of Eqs.(10), (A11), (A16) and (A19) in Eq.(A5) we obtain the Fokker-Planck equation (17).

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Figure Captions

Fig.1. A plot of numerically calculated stationary distribution function $P_s(X_1)$ as a function of X_1 for the set of parameter values described in Sec.V.

Fig.2. The steady entropy production calculated numerically (circle) and theoretically using Eq.(58) (square) for different values of driven field strength ϵ for the model described in Sec.V.

Ref. : A//114898/PAP

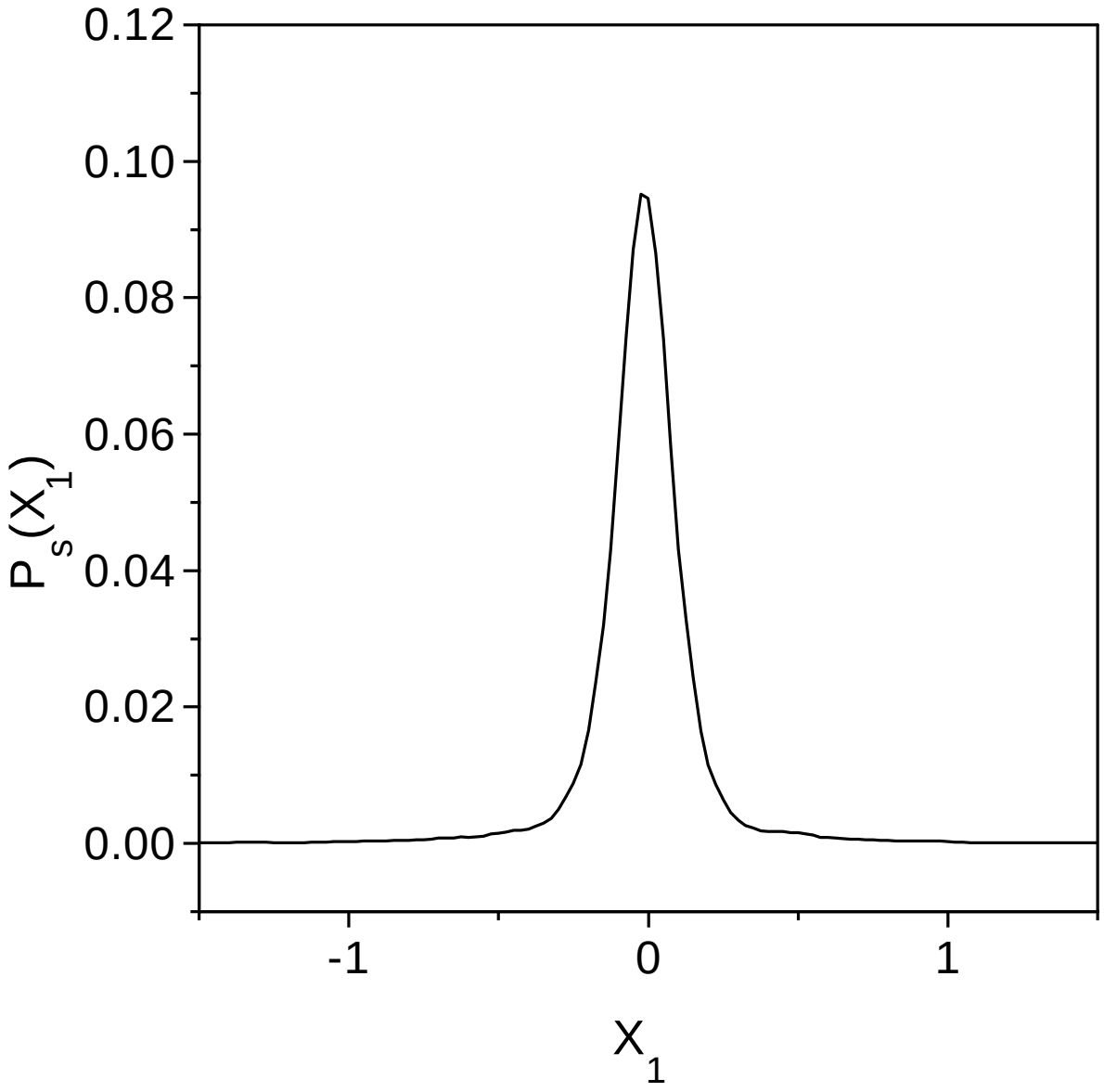

Fig.1

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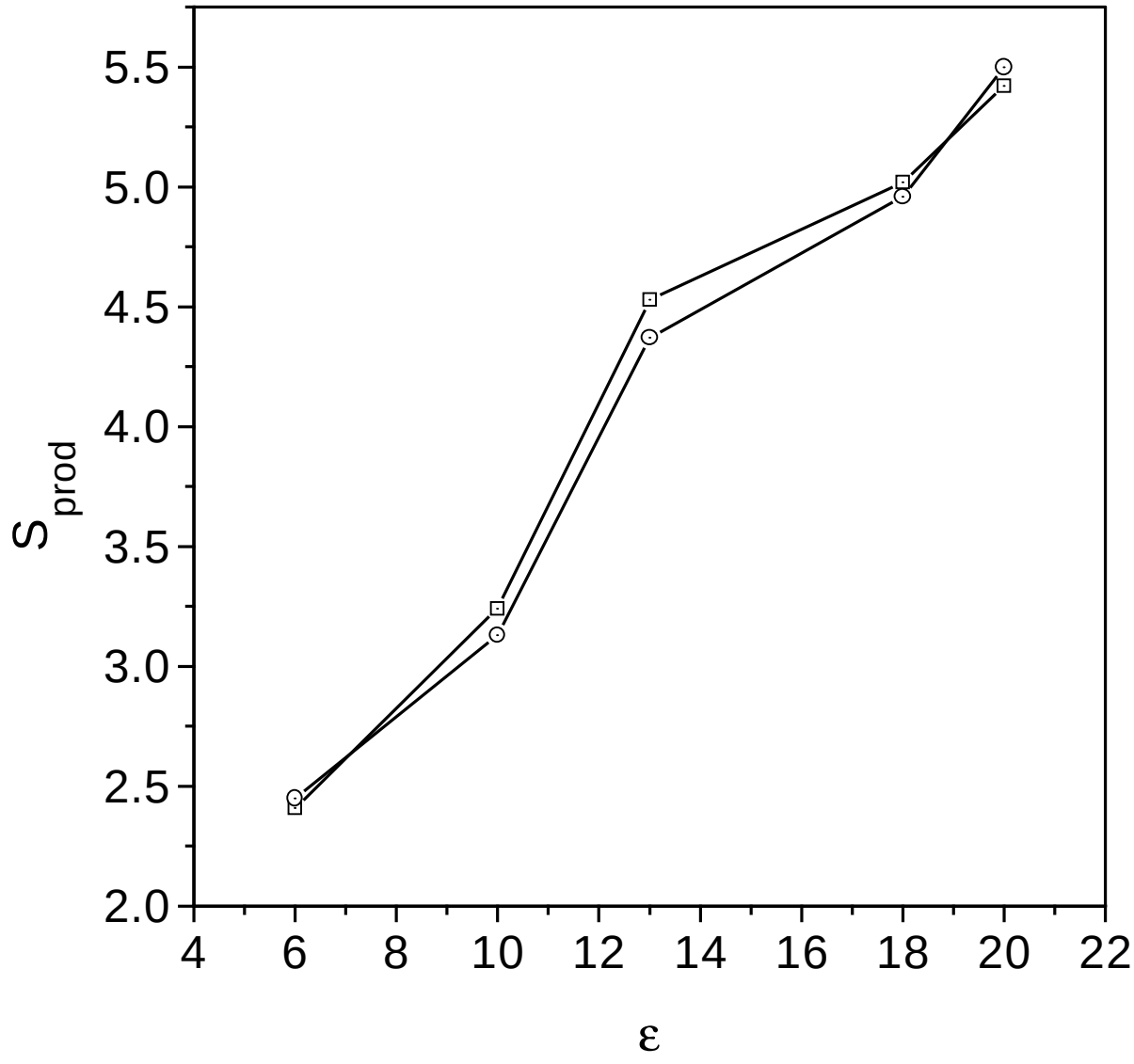

Fig.2